Solutions to deferred material Text and Tests 5

Inferential Statistics (Chapter 5)

Exercise 5.1

1. (i) When a large number of samples of size \( n \) are taken from a population, then the distribution of \( \bar{x} \), the sample mean, is known as the "sampling distribution" of the mean.

(ii) As the sample size increases, the standard deviation of the sampling distribution of the sample means will decrease.

(iii) If the mean of the underlying population is \( \mu \), the mean of the sampling distribution of the means is \( \mu \).

(iv) If the standard deviation of a population is \( \sigma \) and samples of size \( n \) are taken from it, then the standard deviation of the distribution of the sample means is \( \frac{\sigma}{\sqrt{n}} \).

2. \( A \) represents the distribution of the sample means.

   [Note, same mean, smaller standard deviation].

3. (i) The curve will have a Normal distribution shape based on the "Central Limit Theorem".

   (ii) Because the sample size is greater than 30.

   (iii) The mean = 12 and the standard deviation = \( \frac{2}{\sqrt{36}} = \frac{2}{6} = \frac{1}{3} \)

4. (i) \((4, 6), (4, 8), (4, 10), (6, 8), (6, 10), (8, 10)\)

   (ii) \( \frac{4+6}{2} = 5, \frac{4+8}{2} = 6, \frac{4+10}{2} = 7, \frac{6+8}{2} = 7, \frac{6+10}{2} = 8, \frac{8+10}{2} = 9. \)

   (iii) They are a statistic obtained from the samples

   (iv) The mean of the population = \( \frac{4+6+8+10}{4} = 7, \)

      The mean of the samples of size 2 = \( \frac{5+6+7+8+9}{6} = \frac{42}{6} = 7 \)

5. A "parameter" is a numerical property of the whole population.

   A "statistic" is numerical property of a sample.

6. (i) Distribution A is **positively** skewed (as most of the data is to the left)

   (ii) Distribution B is a Normal distribution.

   (iii) Because the sample size is \( \geq 30 \) the Central Limit theorem applies

7. The sample mean is normally distributed as \( n \geq 30 \)

   Standard unit \( z = \frac{\bar{x} - \mu}{\sigma} = \frac{13 - 12}{3/\sqrt{36}} = 2 \)

   \( P(\bar{x} > 13) = P(z > 2) \)

   \[ = 1 - P(z \leq 2) \]

   \[ = 1 - 0.9772 = 0.0228 \]
8. The distribution is normal with $\mu = 60$ and $\sigma = 4, n = 15$

\[
\text{Standard unit } z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{58 - 60}{\frac{4}{\sqrt{15}}} = \frac{-2}{1.03} = -1.94
\]

\[
P(\bar{x} < 58) = P(z < -1.94) = 1 - P(z \leq 1.94) = 1 - 0.9738 = 0.0262
\]

9. The sample mean is normally distributed as $n \geq 30$

\[(i) \text{ Standard unit } z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{177 - 176}{\frac{11}{\sqrt{36}}} = \frac{1}{1.23} = 0.81
\]

\[
P(\bar{x} > 177) = P(z > 0.81)
\]

\[
= 1 - P(z \leq 0.81)
\]

\[
= 1 - 0.791 = 0.209
\]

\[(ii) \text{ Standard unit } z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{174.8 - 176}{\frac{11}{\sqrt{36}}} = \frac{-1.2}{1.23} = -0.98
\]

\[
P(\bar{x} < 174.8) = P(z < -0.98)
\]

\[
= 1 - P(z \leq 0.98)
\]

\[
= 1 - 0.8365 = 0.1635
\]

10. $\mu = 4.2$ hours and $\sigma = 1.8$ hours, $n = 15$

\[(i) \text{ Standard error } = \frac{\sigma}{\sqrt{n}} = \frac{1.8}{\sqrt{15}} = 0.3
\]

\[(ii) \text{ Greater than 4.8 hours } \Rightarrow \bar{x} > 4.8 \text{ hours}
\]

\[
\text{Standard unit } z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{4.8 - 4.2}{\frac{1.8}{\sqrt{15}}} = \frac{0.6}{0.3} = 2
\]

\[
P(\bar{x} > 4.8) = P(z > 2)
\]

\[
= 1 - P(z \leq 2)
\]

\[
= 1 - 0.9772 = 0.0228
\]

\[(iii) \text{ From 4.1 to 4.5 hours}
\]

For $\bar{x} = 4.1$ hours, standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{4.1 - 4.2}{\frac{1.8}{\sqrt{15}}} = \frac{-0.1}{0.3} = -0.333$

For $\bar{x} = 4.5$ hours, standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{4.5 - 4.2}{\frac{1.8}{\sqrt{15}}} = \frac{0.3}{0.3} = 1$

\[
P(4.1 \leq \bar{x} \leq 4.5) = P(-0.333 \leq z \leq 1)
\]

\[
= P(z \leq 1) - P(z > -0.333)
\]

\[
= P(z \leq 1) - [1 - P(z \leq 0.333)]
\]

\[
= 0.8413 - [1 - 0.6293]
\]

\[
= 0.4706
\]

11. $\mu = 5.8$ and $\sigma = 1.2$, $n = 900$

For $\bar{x} = 5.85$, standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{5.85 - 5.8}{\frac{1.2}{\sqrt{900}}} = \frac{0.05}{0.04} = 1.25$

\[
P(\bar{x} \leq 5.85) = P(z \leq 1.25)
\]
12. $\mu = 8$ years $[= 96$ months$]$ and $\sigma = 6$ months, $n = 144$

For $\bar{x} = 8$ years and 1 month $= 97$ months, standard unit $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{97 - 96}{6 / \sqrt{144}} = \frac{1}{0.5} = 2.0$

\[ P(\bar{x} > 97) = P(z > 2) \]
\[ = 1 - P(z \leq 2) \]
\[ = 1 - 0.9772 \]
\[ = 0.0228 \]

∴ out of the 40 samples taken we would expect $0.0228 \times 40 = 0.912 = 1$ sample to have a mean lifetime of 8 years and 1 month

13. $\mu = 200$ and $\sigma = 10, n = 10$

For $\bar{x} = 198$, standard unit $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{198 - 200}{10 / \sqrt{10}} = -2 / 3.16 = -0.63$

For $\bar{x} = 205$, standard unit $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{205 - 200}{10 / \sqrt{10}} = 5 / 3.16 = 1.58$

\[ P(\bar{x} < 198, \bar{x} > 205) = P(z < -0.63, z > 1.58) \]
\[ = P(z < -0.63) + P(z > 1.58) \]
\[ = [1 - P(z \leq 0.63)] + [1 - P(z \leq 1.58)] \]
\[ = [1 - 0.7357] + [1 - 0.9429] \]
\[ = 0.3214 \]

14. Since both distributions have the same mean $C = 80$

Since the standard deviation, $\sigma = 8$, the point $D = \mu + 2\sigma = 80 + 2(8) = 96$

For distribution $B$ the standard deviation $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{8}{\sqrt{10}} = \frac{4}{3}$

∴ the point $E = \mu - \sigma_{\bar{x}} = 80 - \frac{4}{3} = 78\frac{2}{3}$

15. $\mu = 75$ and $\sigma = 9$, $P(\bar{x} > 73) = 0.8708$

If $P(\bar{x} > 73) = 0.8708 \implies z = 1.13$ using page 37 of the tables.

\[ z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = 1.13 \]

∴ $73 - 75 = 1.13 \left( \frac{9}{\sqrt{n}} \right)$

\[ -2\sqrt{n} = 1.13(9) \]

Squaring both sides
\[ 4n = 103.4289 \]
\[ n = 25.86 = 26 \]

16. $\mu = 30$ and $\sigma = \sqrt{5}, n = 40$

(i) For $\bar{x} = 30.5$, standard unit $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{30.5 - 30}{\sqrt{5} / \sqrt{40}} = \frac{0.5}{0.3535} = 1.41$

\[ P(\bar{x} > 30.5) = P(z > 1.41) \]
\[ = 1 - P(z \leq 1.41) \]
\[ = 1 - 0.9207 \]
\[ = 0.0793 \]
(ii) $\mu = 30$ and $\sigma = \sqrt{5}$, $P(\bar{x} > 30.4) = 0.01$

If $P(\bar{x} > 30.4) = 0.01 \Rightarrow [1 - P(\bar{x} \leq 30.4)] = 0.01$

$\therefore P(\bar{x} \leq 30.4) = 0.99 \Rightarrow z = 2.33$ using page 37 of the tables.

\[
\begin{align*}
   z &= \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = 2.33 \\
   \therefore 30.4 - 30 &= 2.33 \left(\frac{\sqrt{5}}{\sqrt{n}}\right) \\
   0.4\sqrt{n} &= 2.33(\sqrt{5})
\end{align*}
\]

Squaring both sides $0.16n = 27.144$

$n = 169.6 = 170$

17. $\mu = 52g$ and $\sigma = 4g$

(i) $P(x > 60g)$

For $x = 60$, standard unit $z = \frac{x - \mu}{\sigma} = \frac{60 - 52}{4} = \frac{8}{4} = 2$

$P(x > 60g) = P(z > 2) = [1 - P(z \leq 2)]$

$= 1 - 0.9772$

$= 0.0228$

(ii) $P(50 \leq \bar{x} \leq 55g)$, $n = 5$

For $\bar{x} = 50$, standard unit $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{50 - 52}{\frac{4}{\sqrt{5}}} = \frac{-2}{1.788} = -1.12$

For $\bar{x} = 55$, standard unit $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{55 - 52}{\frac{4}{\sqrt{5}}} = \frac{3}{1.788} = 1.68$

$P(50 \leq \bar{x} \leq 55) = P(-1.12 \leq z \leq 1.68)$

$= P(z \leq 1.68) - P(z \geq -1.12)$

$= P(z \leq 1.68) - [1 - P(z \leq 1.12)]$

$= 0.9535 - [1 - 0.8686]$

$= 0.822$

(iii) $P(52.1 < \bar{x} \leq 52.2g)$, $n = 90$

For $\bar{x} = 52.1$, standard unit $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{52.1 - 52}{\frac{4}{\sqrt{9}}} = \frac{0.1}{0.4216} = 0.24$

For $\bar{x} = 52.2$, standard unit $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{52.2 - 52}{\frac{4}{\sqrt{9}}} = \frac{0.2}{0.4216} = 0.47$

$P(52.1 < \bar{x} \leq 52.2) = P(0.24 < z \leq 0.47)$

$= P(z \leq 0.47) - P(z \leq 0.24)$

$= 0.6806 - 0.5948$

$= 0.086$

Since $n \geq 30$, in part(iii) the sample means will approximate to a normal distribution regardless of population distribution therefore the answer to (iii) will be unchanged.
Solutions to deferred material Text and Tests 5

Inferential Statistics (Chapter 5)

Exercise 5.2

1. \( \bar{x} = 63 \) and \( \sigma = 12, n = 800 \)

   The 95% confidence interval \([CI]\) for \( \mu \) is \( \bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \)

   \[ \therefore \quad CI = 63 \pm 1.96 \frac{12}{\sqrt{800}} \]
   
   \[ = 63 \pm 0.83 \]  
   
   \[ = 62.17, \, 63.83 \]

   \( \Rightarrow 62.17 < \mu < 63.83 \)

2. \( \bar{x} = 284 \text{kg} \) and \( \sigma = 42 \text{kg}, \, n = 280 \)

   The 95% confidence interval \([CI]\) for \( \mu \) is \( \bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \)

   \[ \therefore \quad CI = 284 \pm 1.96 \frac{42}{\sqrt{280}} \]
   
   \[ = 284 \pm 4.92 \]  
   
   \[ = 279.08, \, 288.92 \]

   \( \Rightarrow 279.1 \text{kg} < \mu < 288.9 \text{kg} \)

3. \( \bar{x} = 227 \text{g} \) and \( \sigma = 7.5 \text{g}, \, n = 70 \)

   (i) The 95% confidence interval \([CI]\) for \( \mu \) is \( \bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \)

   \[ \therefore \quad CI = 227 \pm 1.96 \frac{7.5}{\sqrt{70}} \]
   
   \[ = 227 \pm 1.76 \]  
   
   \[ = 225.2, \, 228.8 \]

   \( \Rightarrow 225.2 \text{g} < \mu < 228.8 \text{g} \)

   (ii) A probability of 95% in the interval 225.2g < \( \mu \) < 228.8g

   \( \Rightarrow \) a probability of 5% outside this interval.

4. \( \bar{x} = 62.7 \text{marks} \) and \( \sigma = 9.2 \text{marks}, \, n = 100 \)

   The 95% confidence interval \([CI]\) for \( \mu \) is \( \bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \)

   \[ \therefore \quad CI = 62.7 \pm 1.96 \frac{9.2}{\sqrt{100}} \]
   
   \[ = 62.7 \pm 1.8 \]  
   
   \[ = 60.9, \, 64.5 \]

   \( \Rightarrow 60.9 \text{ marks} < \mu < 64.5 \text{ marks} \)
5. $\bar{x} = 5.12\text{mg}$ and $\sigma = 0.04\text{mg}, \ n = 12$

(i) The 95% confidence interval [CI] for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

$\therefore CI = 5.12 \pm 1.96 \frac{0.04}{\sqrt{12}}$

$= 5.12 \pm 0.0226$

$= 5.097, 5.142$

$\Rightarrow 5.097\text{mg} < \mu < 5.142\text{mg}$

(ii) 5.10 mg and 5.14 mg

(iii) A 95% confidence interval means that the "mean" lies in the interval 5.10 mg to 5.14 mg 95 times out of a 100.

6. $\bar{x} = €280$ and $\sigma = €105, n = 400$

The 95% confidence interval [CI] for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

$\therefore CI = 280 \pm 1.96 \frac{105}{\sqrt{400}}$

$= 280 \pm 10.29$

$= 269.71, 299.29$

$\Rightarrow €269.71 < \mu < €290.29$

7. $\bar{x} = 29.2\text{cm}$ and $\sigma = 1.47\text{cm}, \ n = 180$

(i) The 95% confidence interval [CI] for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

$\therefore CI = 29.2 \pm 1.96 \frac{1.47}{\sqrt{180}}$

$= 29.2 \pm 0.215$

$= 28.985, 29.415$

$\Rightarrow 28.99\text{cm} < \mu < 29.41\text{cm}$

(ii) Because the sample size, $n = 180 \geq 30$, is sufficiently large to apply the Central limit Theorem.

8. $\bar{x} = 0.932\text{g}$ and $\sigma = 0.1\text{g}, \ n = 64$

(i) Standard error on the mean $= \frac{\sigma}{\sqrt{n}} = \frac{0.1}{\sqrt{64}}$

$= 0.0125\text{g}$

(ii) 0.932g the same as the sample.

(iii) The 95% confidence interval [CI] for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

$\therefore CI = 0.932 \pm 1.96 (0.0125)$

$= 0.932 \pm 0.0245$

$= 0.9075\text{g}, 0.9565\text{g}$

$\Rightarrow 0.9075\text{g} < \mu < 0.9565\text{g}$
(iv) The 95% confidence interval would change to

\[ CI = 0.932 \pm 1.96 \times (0.01) \]
\[ = 0.932 \pm 0.0196 \]
\[ = 0.9124g, 0.9516g \]
\[ \Rightarrow 0.9124g < \mu < 0.9516g \]

(v) Increasing the sample size reduces the standard error and hence the confidence interval gets smaller.

9. $\bar{x} = 4.6\text{years}$ and $\sigma = 2.5\text{years}$, $n = 240$

(i) The 95% confidence interval $[CI]$ for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

\[ \therefore CI = 4.6 \pm 1.96 \frac{2.5}{\sqrt{240}} \]
\[ = 4.6 \pm 0.316 \]
\[ = 4.284, 4.916 \]
\[ \Rightarrow 4.28 \text{ years} < \mu < 4.91 \text{ years} \]

(ii) $\pm 1.96 \frac{\sigma}{\sqrt{n}} = \pm 0.2 \text{ years}$

\[ \Rightarrow \pm 1.96 \frac{2.5}{\sqrt{n}} = \pm 0.2 \]
\[ \Rightarrow \left( \pm 1.96 \frac{2.5}{\sqrt{n}} \right)^2 = (\pm 0.2)^2 \]
\[ \Rightarrow \frac{24.01}{n} = 0.04 \]
\[ \Rightarrow n = 600.25 \]
\[ \therefore 601 \text{ cars needed as a sample.} \]

10. $\bar{x} = 748g$ and $\sigma = 3.6g$, $n = 150$

(i) The 95% confidence interval $[CI]$ for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

\[ \therefore CI = 748 \pm 1.96 \frac{3.6}{\sqrt{150}} \]
\[ = 748 \pm 0.5761 \]
\[ = 747.42, 748.58 \]
\[ \Rightarrow 747.42g < \mu < 748.58g \]

(ii) $\pm 1.96 \frac{\sigma}{\sqrt{n}} = \pm 1.5g$

\[ \Rightarrow \pm 1.96 \frac{3.6}{\sqrt{n}} = \pm 1.5 \]
\[ \Rightarrow \left( \pm 1.96 \frac{3.6}{\sqrt{n}} \right)^2 = (\pm 1.5)^2 \]
\[ \Rightarrow \frac{49.787}{n} = 2.25 \]
\[ \Rightarrow n = 22.127 \]
\[ \therefore 23 \text{ boxes are needed as a sample.} \]
11. $\bar{x} = 69$ beats and $\sigma = 4$ beats, $n = 80$

(i) The 95% confidence interval [CI] for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

\[
\therefore CI = 69 \pm 1.96 \frac{4}{\sqrt{80}}
\]

\[
= 69 \pm 0.8765
\]

\[
= 68.12, 69.88
\]

\[ \Rightarrow 68.12 \text{ beats} < \mu < 69.88 \text{ beats} \]

(ii) $\pm 1.96 \frac{\sigma}{\sqrt{n}} = \pm 1.5$ beats

\[ \Rightarrow \pm 1.96 \frac{4}{\sqrt{80}} = \pm 1.5 \]

\[ \Rightarrow (\pm 1.96 \frac{4}{\sqrt{80}})^2 = (\pm 1.5)^2 \]

\[ \Rightarrow \frac{61.4656}{n} = 2.25 \]

\[ \Rightarrow n = 27.318 \]

\[ \therefore 28 \text{ people needed.} \]

12. $\mu = 48.6g$ and $\sigma = 8.5g$, $n = 50$

(i) $P(\bar{x} < 49 g)$

For $\bar{x} = 49g$, standard unit $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{49 - 48.6}{\frac{8.5}{\sqrt{50}}} = \frac{0.4}{1.202} = 0.3327$

\[ P(\bar{x} < 49) = P(z \leq 0.3327) \]

\[ = 0.629 \]

(ii) $\bar{x} = 48.6g$ and $\sigma = 8.5g$, $n = 50$

The 95% confidence interval [CI] for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

\[ \therefore CI = 48.6 \pm 1.96 \frac{8.5}{\sqrt{50}} \]

\[ = 48.6 \pm 2.356 \]

\[ = 46.2, 51.0 \]

\[ \Rightarrow 46.2g < \mu < 51.0g \]

(iii) $\pm 1.96 \frac{\sigma}{\sqrt{n}} = \pm 2$

\[ \Rightarrow \pm 1.96 \frac{8.5}{\sqrt{50}} = \pm 2 \]

\[ \Rightarrow (\pm 1.96 \frac{8.5}{\sqrt{50}})^2 = (\pm 2)^2 \]

\[ \Rightarrow \frac{277.555}{n} = 4 \]

\[ \Rightarrow n = 69.388 \]

\[ \therefore 70 \text{ pebbles would be needed as a sample.} \]
13. Confidence interval = (54.09, 60.71), \( n = 80 \)

(i) \( \bar{x} = \frac{54.09 + 60.71}{2} = 57.4 \)

(ii) Interval width = 60.71 – 57.4 = 3.31

\[ \pm 1.96 \frac{\sigma}{\sqrt{80}} = \pm 3.31 \]

\[ \pm 0.219\sigma = \pm 3.31 \]

\[ \sigma = 15.1 \text{ marks} \]
Exercise 5.3

1. \( n = 300 \), the sample proportion \( \hat{p} = \frac{45}{300} = 0.15 \)

The 95% confidence interval \([CI]\) for a proportion \( p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}}\)

\[ \therefore CI = 0.15 \pm 1.96 \sqrt{\frac{0.15(1-0.15)}{300}} \]

\[ = 0.15 \pm 1.96(0.0206) \]

\[ = 0.15 \pm 0.0404 \]

\[ = 0.1096, 0.1904 \]

\( \Rightarrow 0.1096 < p < 0.1904 \)

2. \( n = 200 \), the sample proportion \( \hat{p} = \frac{72}{200} = 0.36 \)

The 95% confidence interval \([CI]\) for a proportion \( p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}}\)

\[ \therefore CI = 0.36 \pm 1.96 \sqrt{\frac{0.36(1-0.36)}{200}} \]

\[ = 0.36 \pm 1.96(0.03394) \]

\[ = 0.36 \pm 0.06652 \]

\[ = 0.29348, 0.42652 \]

\( \Rightarrow 0.293 < p < 0.427 \)

3. \( n = 235 \), the sample proportion \( \hat{p} = \frac{75}{235} = 0.31915 \)

The 95% confidence interval \([CI]\) for a proportion \( p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}}\)

\[ \therefore CI = 0.31915 \pm 1.96 \sqrt{\frac{0.31915(1-0.31915)}{235}} \]

\[ = 0.31915 \pm 1.96(0.03040) \]

\[ = 0.31915 \pm 0.05959 \]

\[ = 0.25956, 0.37874 \]

\( \Rightarrow 0.260 < p < 0.379 \)
4. $n = 50$, the sample proportion $\hat{p} = \frac{12}{50} = 0.24$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}}$

$\therefore CI = 0.24 \pm 1.96 \sqrt{\frac{0.24(1-0.24)}{50}}$

$= 0.24 \pm 1.96(0.06039)$

$= 0.24 \pm 0.11838$

$= 0.12162, 0.35838$

$\Rightarrow 0.122 < p < 0.358$

5. $n = 400$, the sample proportion $\hat{p} = \frac{136}{400} = 0.34$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}}$

$\therefore CI = 0.34 \pm 1.96 \sqrt{\frac{0.34(1-0.34)}{400}}$

$= 0.34 \pm 1.96(0.02368)$

$= 0.34 \pm 0.04642$

$= 0.29358, 0.38642$

$\Rightarrow 0.294 < p < 0.386$

6. $n = 120$, the sample proportion (fiction) $\hat{p} = \frac{88}{120} = 0.73333$

the sample proportion (paperback) $\hat{p} = \frac{74}{88} = 0.84090$

The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}}$

(i) $\therefore CI = 0.73333 \pm 1.96 \sqrt{\frac{0.73333(1-0.73333)}{120}}$

$= 0.73333 \pm 1.96(0.04036)$

$= 0.73333 \pm 0.07912$

$= 0.65421, 0.81245$

$\Rightarrow 0.654 < p < 0.812$

(ii) $\therefore CI = 0.84090 \pm 1.96 \sqrt{\frac{0.84090(1-0.84090)}{88}}$

$= 0.84090 \pm 1.96(0.03899)$

$= 0.84090 \pm 0.07642$

$= 0.76448, 0.91732$

$\Rightarrow 0.764 < p < 0.917$
7. \( n = 400 \)

(i) the sample proportion \( \hat{p} = \frac{136}{400} = 0.34\% = 34\% \)

(ii) The 95% confidence interval \( [CI] \) for a proportion \( p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}} \)

\[ \therefore CI = 0.34 \pm 1.96 \sqrt{\frac{0.34(1-0.34)}{400}} \]
\[ = 0.34 \pm 1.96(0.02368) \]
\[ = 0.34 \pm 0.04642 \]
\[ = 0.29358, 0.38642 \]
\[ \Rightarrow 0.294 < p < 0.386 \]
\[ \Rightarrow 29.4\% < p < 38.6\% \]

The true proportion lies in this interval 95 times out of 100

(iii) \( 2\% = 0.02 \)

\[ \Rightarrow \pm 1.96 \sqrt{\frac{0.34(1-0.34)}{n}} = \pm 0.02 \]
\[ \Rightarrow \left( \pm 1.96 \sqrt{\frac{0.34(1-0.34)}{n}} \right)^2 = (\pm 0.02)^2 \]
\[ \Rightarrow \frac{0.86205}{n} = 0.0004 \]
\[ \Rightarrow n = 2155.13 \]

\[ \therefore \text{2156 shops would be needed as a sample.} \]

8. \( n = 1200, \text{ the sample proportion } \hat{p} = \frac{324}{1200} = 0.27 \)

The 95% confidence interval \( [CI] \) for a proportion \( p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}} \)

\[ \therefore CI = 0.27 \pm 1.96 \sqrt{\frac{0.27(1-0.27)}{1200}} \]
\[ = 0.27 \pm 1.96(0.01281) \]
\[ = 0.27 \pm 0.025119 \]
\[ = 0.24488, 0.295111 \]
\[ \Rightarrow 0.245 < p < 0.295 \]

9. \( n = 100, \text{ the sample proportion } \hat{p} = \frac{15}{100} = 0.15 \)

The 95% confidence interval \( [CI] \) for a proportion \( p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}} \)

(i) \[ \therefore CI = 0.15 \pm 1.96 \sqrt{\frac{0.15(1-0.15)}{100}} \]
\[
= 0.15 \pm 1.96(0.03570) \\
= 0.15 \pm 0.06998 \\
= 0.0800, 0.21998 \\
\Rightarrow 0.080 < p < 0.220
\]

(ii) 1.5% = 0.015

\[
\Rightarrow \pm 1.96 \sqrt{\frac{0.15(1-0.15)}{n}} = \pm 0.015
\]

\[
\Rightarrow \left( \pm 1.96 \sqrt{\frac{0.15(1-0.15)}{n}} \right)^2 = (\pm 0.015)^2
\]

\[
\Rightarrow \frac{.48980}{n} = 0.000225
\]

\[
\Rightarrow n = 2176.91
\]

\[
\therefore 2180 \text{ people would be needed as a sample.}
\]
Solutions to deferred material Text and Tests 5

Inferential Statistics (Chapter 5)

Exercise 5.4

1. (i) \( H_0 \): the mean \( \mu \) is 50  
   \( H_1 \): the mean is not 50  
   (ii) 5% level of significance \( \Rightarrow -1.96 < z < 1.96 \)  
   (iii) \( \mu = 50 \), \( \bar{x} = 52.4 \), \( \sigma = 14.3 \) and \( n = 100 \)  
   (iv) Standard unit \( z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{52.4 - 50}{\frac{14.3}{\sqrt{100}}} = \frac{2.4}{1.43} = 1.67 \)  

   Since 1.67 < 1.96, the test statistic is not in the critical region and hence we accept the null hypothesis, the mean is 50.  
   There is no evidence to suggest that the true mean is different from the assumed mean.

2. (i) \( H_0 \): the students do not differ from the normal  
   \( H_1 \): the students do differ from the normal  
   (ii) 5% level of significance \( \Rightarrow -1.96 < z < 1.96 \)  
   (iii) \( \mu = 70 \), \( \bar{x} = 68 \), \( \sigma = 6 \) and \( n = 64 \)  
   (iv) Standard unit \( z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{68 - 70}{\frac{6}{\sqrt{64}}} = \frac{-2}{0.75} = -2.67 \)  

   Since -2.67 < -1.96, the test statistic is in the critical region and hence we reject the null hypothesis that the students do not differ from the normal.  
   Yes there is evidence to suggest that they differ from the normal.

3. (i) \( H_0 \): the mean age of patients is 45 years  
   (ii) \( H_1 \): the mean age of patients is not 45 years  
   (iii) The test statistic is \( z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{48.4 - 45}{\frac{18}{\sqrt{100}}} = \frac{3.4}{1.8} = 1.89 \)  
   (iv) 5% level of significance \( \Rightarrow -1.96 < z < 1.96 \)  

   Since 1.89 < 1.96, the test statistic is not in the critical region and hence we accept the null hypothesis that the mean age of patients is 45 years.  
   No there is no evidence to suggest that the mean age is not 45 years.
4. (i) $H_0$: the mean length has not changed  
(ii) $H_1$: the mean length has changed  
(iii) The test statistic is $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{211.5 - 210}{0.6} = 2.5$  
(iv) 5% level of significance $\Rightarrow -1.96 < z < 1.96$  
Since $2.15 > 1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the mean length has not changed  
Yes there is evidence to suggest that the mean length has changed.

5. (i) $H_0$: the mean lifespan is 258 days  
$H_1$: the mean lifespan is not 258 days  
(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$  
(iii) $\mu = 258, \bar{x} = 269, \sigma = 45$ and $n = 64$  
(iv) Standard unit $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{269 - 258}{45 / \sqrt{64}} = \frac{11}{5.625} = 1.9555$  
Since $1.9555 < 1.96$, the test statistic is not in the critical region and hence we accept the null hypothesis that the mean lifespan is 258 days  
There is no evidence to suggest that the drug has altered the mean lifespan.

6. (i) $H_0$: the mean number of children is 3.8  
$H_1$: the mean number of children is not 3.8  
(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$  
(iii) $\mu = 3.8, \bar{x} = \frac{144}{40} = 3.6, \sigma = 0.6$ and $n = 40$  
(iv) Standard unit $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{3.6 - 3.8}{0.6 / \sqrt{40}} = \frac{-0.2}{0.09486} = -2.1$  
Since $-2.11 < -1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the mean number of children is 3.8  
There is evidence to suggest that the mean number of children has changed.

7. (i) $H_0$: the mean mark of students in this town is 48.7  
$H_1$: the mean mark of students in this town is not 48.7  
(ii) 5% level of significance $\Rightarrow -1.96 < z < 1.96$  
(iii) $\mu = 48.7, \bar{x} = 46.5, \sigma = 9.5$ and $n = 120$  
(iv) Standard unit $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{46.5 - 48.7}{9.5 / \sqrt{120}} = \frac{-2.2}{0.8672} = -2.54$
Since $-2.54 < -1.96$, the test statistic is in the critical region and hence we reject the null hypothesis that the mean mark of students in this town is 48.7. There is evidence to suggest that the mean mark has changed.

8. $p-value = 2 \times P(z > |z_1|)$
   (i) $z_1 = 1.73, \quad p-value = 2 \times P(z > |1.73|)$
   \[= 2 \times [1 - P(z \leq 1.73)]\]
   \[= 2 \times [1 - 0.9582]\]
   \[= 0.0836\]
   
   (ii) $z_1 = -1.91, \quad p-value = 2 \times P(z > |-1.91|)$
   \[= 2 \times [1 - P(z \leq 1.91)]\]
   \[= 2 \times [1 - 0.9719]\]
   \[= 0.0562\]
   
   (iii) $z_1 = -1.65, \quad p-value = 2 \times P(z > |-1.65|)$
   \[= 2 \times [1 - P(z \leq 1.65)]\]
   \[= 2 \times [1 - 0.9505]\]
   \[= 0.099\]
   
   (iv) $z_1 = -2.06, \quad p-value = 2 \times P(z > |-2.06|)$
   \[= 2 \times [1 - P(z \leq 2.06)]\]
   \[= 2 \times [1 - 0.9803]\]
   \[= 0.0394\]

9. $\mu = 85$ hours, $\bar{x} = 86.5$ hours, $\sigma = 12$ hours and $n = 200$
   
   (i) The sample statistic is $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{86.5 - 85}{12 / \sqrt{200}} = \frac{1.5}{0.8485} = 1.77$
   
   (ii) $p-value = 2 \times P(z > |z_1|)$
   \[= 2 \times P(z > |1.77|)\]
   \[= 2 \times [1 - P(z \leq 1.77)]\]
   \[= 2 \times [1 - 0.9616]\]
   \[= 0.0768\]
   
   (iii) $\Rightarrow p > 0.05$ we accept $H_0$, the result is not significant at the 5% level of significance.

10. $\mu = 70, \bar{x} = 68.5, \sigma = 6$ and $n = 36$
    
   (i) The sample statistic is $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{68.5 - 70}{6 / \sqrt{36}} = \frac{-1.5}{1} = -1.5$
    
   (ii) $p-value = 2 \times P(z > |z_1|)$
    \[= 2 \times P(z > |-1.5|)\]
\[= 2 \times [1 - P(z \leq |1.5|)]\]
\[= 2 \times [1 - 0.9332]\]
\[= 0.1336\]

(iii) \(\Rightarrow p > 0.05\) we accept \(H_0\), the result is not significant at the 5% level of significance.

11. \(\mu = 12\) minutes, \(\bar{x} = 12.3\) minutes, \(\sigma = 1.2\) minutes and \(n = 36\)

(i) The test statistic is \(z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{12.3 - 12}{\frac{1.2}{\sqrt{36}}} = \frac{0.3}{0.2} = 1.5\)

(ii) 5% level of significance \(\Rightarrow -1.96 < z < 1.96\)

Since \(1.5 < 1.96\), the test statistic is not in the critical region and hence we accept the null hypothesis that the average time is 12 minutes.

(iii) \(p - value = 2 \times P(z > |z_1|)\)

\[= 2 \times P(z > |1.5|)\]
\[= 2 \times [1 - P(z \leq |1.5|)]\]
\[= 2 \times [1 - 0.9332]\]
\[= 0.1336\]

(iv) \(\Rightarrow p > 0.05\) we accept \(H_0\), the result is not significant at the 5% level of significance so we reach the same conclusion

12. \(\mu = 420\,cm, \bar{x} = 423\,cm, \sigma = 12\,cm\) and \(n = 100\)

(i) The sample statistic is \(z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{423 - 420}{\frac{12}{\sqrt{100}}} = \frac{3}{1.2} = 2.5\)

(ii) \(p - value = 2 \times P(z > |z_1|)\)

\[= 2 \times P(z > |2.5|)\]
\[= 2 \times [1 - P(z \leq |2.5|)]\]
\[= 2 \times [1 - 0.9938]\]
\[= 0.0125\]

(iii) \(\Rightarrow p < 0.05\) we reject \(H_0\), the result is significant at the 5% level of significance so we reach the conclusion that there is a change in the mean length of the bars.

13. \(\mu = 5\,mm, \bar{x} = 5.008\,mm, \sigma = 0.072\,mm\) and \(n = 400\)

Standard error on the mean \(= \frac{\sigma}{\sqrt{n}} = \frac{0.072}{\sqrt{400}} = 0.0036\)

The 95% confidence interval \([CI]\) for \(\mu\) is \(\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}\)

\[
\therefore CI = 5.00 \pm 1.96 \left(\frac{0.072}{\sqrt{400}}\right)
\]
\[= 5.00 \pm 0.007056\]
\[ = 4.9929, \ 5.007056 \]
\[ \Rightarrow 4.993\text{mm} < \mu < \ 5.007\text{mm} \]

5% level of significance \[\Rightarrow -1.96 < z < 1.96\]

The test statistic is \[ z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{5.008 - 5}{\frac{0.072}{\sqrt{400}}} = \frac{0.008}{0.0036} = 2.22 \]

Since 2.22 > 1.96, the test statistic is in the critical region and hence we reject the null hypothesis that the mean length is 5mm, the sample does differ significantly from the stated mean.
Inferential Statistics (Chapter 5)

Test yourself 5

A questions

1. $\mu = 25\ kg$, $\sigma = \sqrt{5}\ kg$ and $n = 50$

   From 24.5kg to 25.5kg

   For $\bar{x} = 24.5$, standard unit $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{24.5 - 25}{\sqrt{5}/\sqrt{50}} = -0.5 \div 0.31622 = -1.58$

   For $\bar{x} = 25.5$, standard unit $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{25.5 - 25}{\sqrt{5}/\sqrt{50}} = 0.5 \div 0.31622 = 1.58$

   $P(24.5 \leq \bar{x} \leq 25.5) = P(-0.158 \leq z \leq 1.58)$

   $= P(z \leq 1.58) - P(z > 1.58)$

   $= P(z \leq 1.58) - [1 - P(z \leq 1.58)]$

   $= 0.9394 - [1 - 0.9394]$

   $= 0.8788$

   $= 0.89$

2. $\mu = 2.85$, $\sigma = 0.07$ and $n = 20$

   Mean of sample $\bar{x} = \mu = 2.85$

   Standard error on the mean $= \frac{\sigma}{\sqrt{n}} = \frac{0.07}{\sqrt{20}} = 0.016$

3. $\bar{x} = 26.2\ beats$ and $\sigma_{\bar{x}} = 5.15\ beats$, $n = 32$

   The 95% confidence interval [CI] for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma_{\bar{x}}}{\sqrt{n}}$

   $\therefore CI = 26.2 \pm 1.96 \frac{5.15}{\sqrt{32}}$

   $= 26.2 \pm 1.7843$

   $= 24.4157, 27.9843$

   $\Rightarrow 24.42\ beats < \mu < 27.98\ beats$

4. $\bar{x} = 266ml$, $\sigma = 20ml$ and $n = 40$

   The 95% confidence interval [CI] for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

   $\therefore CI = 266 \pm 1.96 \frac{20}{\sqrt{40}}$

   $= 266 \pm 6.198$

   $= 259.802, 272.198$

   $\Rightarrow 259.80ml < \mu < 272.20ml$
5. \( n = 150 \), the sample proportion \( \hat{p} = \frac{90}{150} = 0.6 \)

The 95% confidence interval [CI] for a proportion \( p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}} \)

\[ \therefore CI = 0.6 \pm 1.96 \sqrt{\frac{0.6(1-0.6)}{150}} \]

\[ = 0.6 \pm 1.96(0.04) \]

\[ = 0.6 \pm 0.0784 \]

\[ = 0.5216, 0.6784 \]

\[ \Rightarrow 0.522 < p < 0.678 \]

6. \( n = 100 \), the sample proportion \( \hat{p} = 0.55 \)

The 95% confidence interval [CI] for a proportion \( p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}} \)

\[ \therefore CI = 0.55 \pm 1.96 \sqrt{\frac{0.55(1-0.55)}{100}} \]

\[ = 0.55 \pm 1.96(0.04975) \]

\[ = 0.55 \pm 0.09750 \]

\[ = 0.4525, 0.6475 \]

\[ \Rightarrow 0.453 < p < 0.648 \]

7. (i) \( H_0 \): the height of the Irish students does not differ from the height of the German students

\( H_1 \): the height of the Irish students does differ from the height of the German students

(ii) 5% level of significance \( \Rightarrow -1.96 < z < 1.96 \)

(iii) \( \mu = 176cm, \bar{x} = 179cm, \sigma = 11cm \) and \( n = 60 \)

(iv) Standard unit \( z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{179 - 176}{11/\sqrt{60}} = \frac{3}{1.42} = 2.11 \)

Since \( 2.11 > 1.96 \), the test statistic is in the critical region and hence we reject the null hypothesis that the heights are the same.

Yes there is evidence to suggest that the average German student is taller than the average Irish student.

8. (i) \( H_0 \): the mean quantity of honey has not changed

\( H_1 \): the mean quantity of honey has changed

(ii) \( \mu = 460.3g, \bar{x} = 461.2g, \sigma = 3.2g \) and \( n = 60 \)

Standard unit \( z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{461.2 - 460.3}{3.2/\sqrt{60}} = \frac{0.9}{0.4131} = 2.18 \)

(iii) 5% level of significance \( \Rightarrow -1.96 < z < 1.96 \)
Since 2.18 > 1.96, the test statistic is in the critical region and hence we reject the null hypothesis that the quantity of honey has not changed.

Yes there is evidence to suggest that the sample mean is different from the population mean.

9. (i) A Normal distribution. Central Limit Theorem
(ii) Because the sample size \( n > 30 \)
(iii) \( \mu = 96 \text{ hrs} \), \( \sigma = 6 \text{ hrs} \) and \( n = 36 \)

Greater than 98 hours \( \implies \bar{x} > 98 \text{ hours} \)

Standard unit \( z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{98 - 96}{\frac{6}{\sqrt{36}}} = \frac{2}{1} = 2 \)

\( P(\bar{x} > 98) = P(z > 2) \)

\( = 1 - P(z \leq 2) \)

\( = 1 - 0.9772 = 0.0228 \)

\( P(\bar{x} > 98)(40) = 0.0228(40) = 0.912 = 1 \)

10.
(i)

(ii) \( z < -1.96, z > 1.96 \)
(iii) The sample statistic is \( z_1 = 1.6 \)

(ii) \( p-value = 2 \times P(z > |z_1|) \)

\( = 2 \times P(z > |1.6|) \)

\( = 2 \times [1 - P(z \leq |1.6|)] \)

\( = 2 \times [1 - 0.9452] \)

\( = 0.1096 \)

Test yourself 5

B questions

1. \( \mu = 74 \text{ and } \sigma = 6 \), \( P(\bar{x} > 72) = 0.854 \)

If \( P(\bar{x} > 72) = 0.854 \) \( \implies z = 1.05 \) using page 37 of the tables.

\( z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = 1.05 \)
\[ \therefore 72 - 74 = 1.05 \left( \frac{6}{\sqrt{n}} \right) \]
\[ -2\sqrt{n} = 1.05(6) \]
(Squaring both sides) \[ 4n = 39.69 \]
\[ n = 9.92 \approx 10 \]

2. \( \mu = 12.1 \text{kg} \) and \( \sigma = 0.4 \text{kg} \)

For \( x = 12 \), standard unit \( z = \frac{x - \mu}{\sigma} = \frac{12 - 12.1}{0.4} = \frac{-0.1}{0.4} = -0.25 \)

\[ P(x \leq 12|k|) = P(z \leq -0.25) = [1 - P(z \leq 0.25)] \]
\[ = 1 - 0.5987 \]
\[ = 0.401 \]

3. \( \bar{x} = 31.4 \text{kg} \) and \( \sigma = 2.4 \text{kg}, \ n = 36 \)

The 95\% confidence interval [CI] for \( \mu \) is \( \bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \)

\[ \therefore CI = 31.4 \pm 1.96 \frac{2.4}{\sqrt{36}} \]
\[ = 31.4 \pm 0.784 \]
\[ = 30.616, 32.184 \]

\[ \Rightarrow 30.6 \text{kg} < \mu < 32.2 \text{kg} \]

4. (i) Since \( n \geq 30 \), the central Limit theorem can be applied.

(ii) The 95\% confidence interval [CI] for a proportion \( p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}} \)

\[ p = 20\% = 0.2 \]

\[ \therefore CI = 0.2 \pm 1.96 \sqrt{\frac{0.2(1-0.2)}{30}} \]
\[ = 0.2 \pm 1.96(0.073) \]
\[ = 0.2 \pm 0.1431 \]
\[ = 0.0569, 0.3431 \]

\[ \Rightarrow 0.057 < p < 0.343 \]

\[ \Rightarrow 5.7\% < p < 34.3\% \]

5. (i) If 100 samples of the same size are taken, then the true population mean (or proportion) will lie in the given interval on 95 occasions out of 100.

(ii) \( \bar{x} = 13.52 \text{km/l} \) and \( \sigma = 2.23 \text{km/l}, \ n = 150 \)

The 95\% confidence interval [CI] for \( \mu \) is \( \bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \)

\[ \therefore CI = 13.52 \pm 1.96 \frac{2.23}{\sqrt{150}} \]
\[ = 13.52 \pm 0.35687 \]
6.  

(i) **$H_0$**: the mean response time is unchanged, $\mu = 1.2s$

(ii) **$H_1$**: the mean response time changes, $\mu \neq 1.2s$

(iii) $z < -1.96, z > 1.96$

\[ \mu = 1.2s, \bar{x} = 1.05s \text{ and } \sigma = 0.5s, \ n = 100 \]

(iv) The test statistic is

\[ z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{1.05 - 1.2}{\frac{0.5}{\sqrt{100}}} = -0.15 \times 0.05 = -3 \]

Since $-3 < -1.96$, the test statistic is in the critical region and hence we reject the null hypothesis.

Yes the drug has an effect on the response time.

\[ p-value = 2 \times P(z > |z_1|) \]
\[ = 2 \times P(z > |3|) \]
\[ = 2 \times [1 - P(z \leq |3|)] \]
\[ = 2 \times [1 - 0.9987] \]
\[ = 0.0026 \]

Since $p < 0.05$ we reject the null hypothesis that $\mu = 1.2s$

7. $n = 72$, the sample proportion $\hat{p} = \frac{50}{72} = 0.6944$

(i) margin of error $= \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{72}} = 0.118$

(ii) $n = 72$, the sample proportion $\hat{p} = \frac{50}{72} = 0.69$

(iii) The 95% confidence interval $[\text{CI}]$ for a proportion $p = \hat{p} \pm \frac{1}{\sqrt{n}}$

\[ p = \frac{50}{72} \pm \frac{1}{\sqrt{72}} \]
\[ 0.577 < p < 0.812 \]

(iv) Since $80\% = 0.8$ is within the confidence interval we accept the school’s claim.

8.  

(i) **$H_0$**: the mean weight has not changed, $\mu = 25kg$

(ii) **$H_1$**: the mean weight has changed, $\mu \neq 25kg$

\[ \mu = 25kg, \ \hat{x} = 24.5kg, \sigma = 1.5kg \text{ and } n = 50 \]
Sample statistic, \( z = \frac{x - \mu}{\sigma/\sqrt{n}} = \frac{24.5 - 25}{1.9 / \sqrt{50}} = -0.5 / 0.212 = -2.36 \)

(iii) \( p-value = 2 \times P(z > |z_1|) \)
\[ = 2 \times P(z > |-2.36|) \]
\[ = 2 \times [1 - P(z \leq |2.36|)] \]
\[ = 2 \times [1 - 0.9909] \]
\[ = 0.0182 \]

(iv) At the 5% level of significance since \( p < 0.05 \) we reject the null hypothesis that \( \mu = 25 \text{kg} \), the wholesaler’s suspicion is justified.

(v) The \( p-value \) is the smallest level of significance at which the null hypothesis could be rejected.

9. (i) Mean = 68kg,
\[
\text{Standard error} = \frac{\sigma}{\sqrt{n}} = \frac{3}{\sqrt{80}} = 0.65 kg
\]

(ii) Standard unit \( z = \frac{x - \mu}{\sigma/\sqrt{n}} = \frac{67.5 - 68}{0.65} = -0.769 \)
\[
P(\bar{x} < 667.5) = P(z < -0.769) = 1 - P(z \leq 0.77) = 1 - 0.7794 = 0.2206
\]
No of samples = 0.2206(80) = 17.64 = 17

Test yourself 5

C questions

1. (i) \( H_0 \): the mean weight has not changed, \( \mu = 500 \text{g} \)
\( H_1 \): the mean weight has changed, \( \mu \neq 500 \text{g} \)

(ii) \( \mu = 500 \text{g} \), \( \bar{x} = 505 \text{g} \), \( \sigma = 18 \text{g} \) and \( n = 36 \)

Sample statistic, \( z = \frac{x - \mu}{\sigma/\sqrt{n}} = \frac{505 - 500}{18 / \sqrt{36}} = \frac{5}{3} = -1.67 \)

(iii) \( p-value = 2 \times P(z > |z_1|) \)
\[ = 2 \times P(z > |-1.67|) \]
\[ = 2 \times [1 - P(z \leq |1.67|)] \]
\[ = 2 \times [1 - 0.9525] \]
\[ = 0.095 \]
(iv) Since $p > 0.05$ we accept the null hypothesis that $\mu = 500g$. The result is not significant, the mean weight has not changed.

2. (i) $n = 80$, the sample proportion $\hat{p} = \frac{28}{80} = 0.35$

The standard error $= \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.35(1-0.35)}{80}}$

$= 0.0533$

(ii) The 95% confidence interval [CI] for a proportion $p = \hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}}$

$\therefore CI = 0.35 \pm 1.96 \sqrt{\frac{0.35(1-0.35)}{80}}$

$= 0.35 \pm 1.96(0.0533)$

$= 0.35 \pm 0.1045$

$= 0.245, 0.455$

$\Rightarrow 0.245 < p < 0.455$

3. (i) Mean weight $= \bar{x} = \frac{79.93 + 82.87}{2} = 81.4g$

(ii) The 95% confidence interval [CI] for $\mu$ is $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

$\Rightarrow \bar{x} + 1.96 \frac{\sigma}{\sqrt{400}} = 82.87 and \bar{x} - 1.96 \frac{\sigma}{\sqrt{400}} = 79.93$

$\Rightarrow 81.4 + 1.96 \frac{\sigma}{20} = 82.87$

$\Rightarrow 1.96 \frac{\sigma}{20} = 1.47$

$\sigma = 15g$

4. (i) $x < 475g, \mu = 500g, \sigma = 20g$

Standard unit $z = \frac{\bar{x} - \mu}{\sigma} = \frac{475 - 500}{20} = \frac{-25}{20} = -1.25$

$P(x < 475) = P(z < -1.25)$

$= 1 - P(z \leq 1.25)$

$= 1 - 0.8944 = 0.1056$

$= 10.6\%$

$x > 530g, \mu = 500g, \sigma = 20g$

Standard unit $z = \frac{\bar{x} - \mu}{\sigma} = \frac{530 - 500}{20} = \frac{30}{20} = 1.5$

$P(x > 530) = P(z > 1.5)$

$= 1 - P(z \leq 1.5)$

$= 1 - 0.9332 = 0.068$

$= 6.7\%$
(ii) \( \mu = 500g, \bar{x} = 495g, \sigma = 20g \) and \( n = 40 \)

Sample statistic, \( z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{495 - 500}{\frac{20}{\sqrt{40}}} = -\frac{5}{3.1622} = -1.58 \)

(iii) \( p - value = 2 \times P(z > |z_1|) \)

\[ = 2 \times P(z > -1.58) \]

\[ = 2 \times [1 - P(z \leq 1.58)] \]

\[ = 2 \times [1 - 0.9429] \]

\[ = 0.114 \]

(iv) Since \( p > 0.05 \) we accept the null hypothesis that \( \mu = 500g \). The result is not significant, the mean weight has not changed.

5. (a) Mean = \( \mu \), the standard deviation \( \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \)

(i) When \( n \) is large (\( n > 30 \)) the distribution is Normal.

(ii) When the population distribution is normal the distribution of the sample means is Normal. The Central Limit Theorem can be applied to (i) if \( n > 30 \). If the underlying population is normal the distribution of the sample means is always normal.

(b) (i) \( \mu = 37e, \sigma = 8.5 \) and \( n = 100 \)

Greater than 37.5 \( \Rightarrow \bar{x} > 37.5 \)

Standard unit \( z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{37.5 - 37}{\frac{8.5}{\sqrt{100}}} = \frac{0.5}{0.85} = 0.5882 \)

\[ P(\bar{x} > 37.5) = P(z > 0.59) \]

\[ = 1 - P(z \leq 0.59) \]

\[ = 1 - 0.7224 = 0.278 \]

(ii) \( P(\bar{x} > 37.5) < 0.06 \)

\[ P(z > z_1) = [1 - P(z \leq z_1)] = 0.06 \]

\[ P(z \leq z_1) = 0.94 \]

\[ z_1 = 1.55 \]

\[ z_1 = 1.55 = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{37.5 - 37}{\frac{8.5}{\sqrt{n}}} \]

\[ 1.55 \times \frac{8.5}{\sqrt{n}} = 0.5 \]

\[ \sqrt{n} = 26.35 \]

\[ n = 694.3 \approx 695 \]

6. (i) The lower quartile is €12.80 \( \Rightarrow \) 75% earn more than this amount

\[ P(x > 12.8) = 0.75 \]

(ii) 4 out of six earn more than €12.80 \( \Rightarrow \) \( 0.75 \times 0.75 \times 0.75 \times 0.75 \times 0.25 \times 0.25 \times \frac{6!}{4!2!} \)

\[ = 0.2966 \]
(iii) The distribution of the sample means will be normal with a mean of €22.05 and the standard error
\[ \sigma_x = \frac{\sigma}{\sqrt{n}} = \frac{10.64}{\sqrt{200}} = 0.7524 \]

(iv) \( P(\bar{x} > €23) \)

\[ \text{Standard unit } z = \frac{\bar{x} - \mu}{\sigma_x} = \frac{23 - 22.05}{0.7524} = \frac{0.95}{0.7524} = 1.2626 = 1.26 \]

\[ P(\bar{x} > €23) = P(z > 1.26) = [1 - P(z \leq 1.26)] \]
\[ = 1 - 0.8962 \]
\[ = 0.1038 \]

Number of samples \( = 0.1038(1000) = 103.8 \approx 104 \)

7. \( \mu = 3.05 kg, \sigma = 0.08 kg \)

(i) \( x = 3.11 kg \)

\[ \text{Standard unit } z = \frac{x - \mu}{\sigma} = \frac{3.11 - 3.05}{0.08} = \frac{0.06}{0.08} = 0.75 \]

\[ P(x < 3.11) = P(z \leq 0.75) = 0.7734 = 77.34\% \]

(ii) \( P(3.00 < x < 3.15) \)

\[ \text{For } x = 3.00, \text{ standard unit } z = \frac{x - \mu}{\sigma} = \frac{3.00 - 3.05}{0.08} = \frac{-0.05}{0.08} = -0.625 \]

\[ \text{For } x = 3.15, \text{ standard unit } z = \frac{x - \mu}{\sigma} = \frac{3.15 - 3.05}{0.08} = \frac{0.1}{0.08} = 1.25 \]

\[ P(3.00 < x < 3.15) = P(-0.625 < z < 1.25) \]
\[ = P(z \leq 1.25) - P(z < -0.625) \]
\[ = P(z \leq 1.25) - P(z > 0.625) \]
\[ = P(z \leq 1.25) - [1 - P(z \leq 0.625)] \]
\[ = 0.8944 - [1 - 0.7357] \]
\[ = 0.1637 \]

8. (a) \( \mu = 65 min, \sigma = 60 min \)

(i) \( x = 185 \)

\[ \text{Standard unit } z = \frac{x - \mu}{\sigma} = \frac{185 - 65}{60} = \frac{120}{60} = 2 \]

\[ P(x > 185) = P(z > 2) = [1 - P(z \leq 2)] \]
\[ = 1 - 0.9772 \]
\[ = 0.0228 \]

(ii) \( P(50 < x < 125) \)

\[ \text{For } x = 50, \text{ standard unit } z = \frac{x - \mu}{\sigma} = \frac{50 - 65}{60} = \frac{-15}{60} = -0.25 \]
For \( x = 125 \), standard unit \( z = \frac{x - \mu}{\sigma} = \frac{125 - 65}{60} = \frac{60}{60} = 1 \)

\[
P(3.00 < x < 3.15) = P(-0.25 < z < 1) = P(z \leq 1) - P(z < -0.25) = P(z \leq 1) - P(z > 0.25) = P(z \leq 1) - [1 - P(z \leq 0.25)] = 0.8413 - [1 - 0.5987] = 0.44
\]

(iii) \( P(\bar{x} < 70) \) from a sample of 90.

Standard unit \( z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{70 - 65}{\frac{60}{\sqrt{90}}} = \frac{5}{6.3245} = 0.7906 \)

\[
P(\bar{x} < 70) = P(z \leq 0.7906) = 0.785
\]

(b) (i) The standard deviation is so big that there are only \( \frac{65}{60} = 1.083 \) standard deviations above zero.

\[
P(z < -1.083) = P(z > 1.083) = [1 - P(z \leq 1.083)] = 1 - 0.8599 = 0.14 \text{ at time } = 0 \text{ minutes}
\]

This means that there is a probability of 0.14 of negative times, which are impossible.

(ii) A large sample of 90 \( \Rightarrow \) the mean is approximately normally distributed.

\[ 
9. \mu = 60g, \sigma = 15g 
\]

(i) \( x = 45g \)

Standard unit \( z = \frac{x - \mu}{\sigma} = \frac{45 - 60}{15} = -1 \)

\[
P(x < 45g) = P(z \leq -1) = P(z > 1) = [1 - P(z \leq 1)] = 1 - 0.8413 = 0.1587
\]

(ii) \( P(\bar{x} < 58g) \) from a sample of 50.

Standard unit \( z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{58 - 60}{\frac{15}{\sqrt{50}}} = \frac{-2}{2.1213} = -0.94 \)

\[
P(\bar{x} < 58) = P(z < -0.94) = P(z > 0.94) = [1 - P(z \leq 0.94)] = 1 - 0.8264 = 0.1736
\]

(iii) \( P(x < 45g(small)) = 0.1587 \)

\[ 
\Rightarrow P(x \ medium \ or \ large) = 1 - 0.1587 = 0.8413
\]

Equal probabilities \( \Rightarrow \frac{0.8413}{2} = 0.4206 \) in each group
$$\Rightarrow P(x \text{ small or medium}) = 0.1587 + 0.4206 = 0.5793$$

$$P(z) = 0.5793 \implies z = 0.2$$

Standard unit $z = \frac{x - \mu}{\sigma} = \frac{x - 60}{15} = 0.2$

$$\Rightarrow x = 0.2(15) + 60 = 63g$$