

SOLUTIONS

PROJECT MATHS

Text & Tests

7

**LEAVING CERTIFICATE
HIGHER LEVEL
STRAND 5**

FUNCTIONS & CALCULUS

**FULLY WORKED
SOLUTIONS
TO ALL QUESTIONS**

**O. D. MORRIS
PAUL COOKE**



The Celtic Press



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Preface

This book contains fully worked solutions to all questions contained in **Text & Tests 7** for Leaving Certificate Higher Level Maths.

While there is generally more than one way to solve any problem, only one approach is given in these solutions.

To access these solutions online, contact Celtic Press by email at info@celticpress.ie or by phoning (01) 4530328.

I would like to thank P.J. O'Reilly for producing these solutions.

O.D. Morris

Paul Cooke

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Chapter 1: Functions

Exercise 1.1

Q1. Not a function because the input 2 has two different outputs, ie. 5 and 10.

Q2. (i) Is a function

(ii) Not a function because the input -2 has two different outputs, ie. 1 and 5.

(iii) Is a function

Q3. (i) Yes

(ii) No, as inputs a, c each have two different outputs.

(iii) No, as input 9 has two different outputs, ie. 14 and 20.

(iv) Yes

Q4. Rule: $y = 2x - 4$

$$x = -1 \Rightarrow y = 2(-1) - 4 = -6$$

$$x = 0 \Rightarrow y = 2(0) - 4 = -4$$

$$x = 1 \Rightarrow y = 2(1) - 4 = -2$$

$$x = 2 \Rightarrow y = 2(2) - 4 = 0$$

$$x = 3 \Rightarrow y = 2(3) - 4 = 2 \Rightarrow \text{couples} = (-1, -6), (0, -4), (1, -2), (2, 0), (3, 2)$$

$$\Rightarrow \text{Range} = \{-6, -4, -2, 0, 2\}$$

Q5. $f(x) = 3x - 2$

(i) $f(2) = 3(2) - 2 = 4$

(ii) $f(-3) = 3(-3) - 2 = -11$

(iii) $f(k) = 3k - 2$

(iv) $f(2k - 1) = 3(2k - 1) - 2 = 6k - 3 - 2 = 6k - 5$

Q6. $g(x) = (x - 2)^2$

(i) $g(4) = (4 - 2)^2 = 2^2 = 4$

(ii) $g(-4) = (-4 - 2)^2 = (-6)^2 = 36$

(iii) $g(8) = (8 - 2)^2 = (6)^2 = 36$

(iv) $g(a) = (a - 2)^2 = a^2 - 4a + 4$

Q7. $f(x) = 3x - 4$
 $f(k) = 3k - 4$
 $f(2k) = 3(2k) - 4 = 6k - 4$
hence, $f(k) + f(2k) = 0$
 $\Rightarrow 3k - 4 + 6k - 4 = 0$
 $\Rightarrow 9k - 8 = 0 \Rightarrow 9k = 8 \Rightarrow k = \frac{8}{9}$

Q8. $f(x) = 4x$ and $g(x) = x + 1$
 $f(3) = 4(3) = 12$ $g(3) = 3 + 1 = 4$
hence, $g(3) + k[f(3)] = 8$
 $\Rightarrow 4 + k(12) = 8$
 $\Rightarrow 12k = 4 \Rightarrow k = \frac{4}{12} = \frac{1}{3}$

Q9. $f(x) = 2x^2 - 1$ and $g(x) = x + 2$

(i) Solve $f(x) = 3 \Rightarrow 2x^2 - 1 = 3$
 $\Rightarrow 2x^2 - 4 = 0$
 $\Rightarrow x^2 - 2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

(ii) Solve $g(x) = f(3)$
 $\Rightarrow x + 2 = 2(3)^2 - 1 = 17 \Rightarrow x = 17 - 2 = 15$

(iii) $f(x) = g(x)$
 $\Rightarrow 2x^2 - 1 = x + 2$
 $\Rightarrow 2x^2 - x - 3 = 0$
 $\Rightarrow (2x - 3)(x + 1) = 0$
 $\Rightarrow 2x - 3 = 0$ OR $x + 1 = 0$
 $\Rightarrow 2x = 3$ OR $x = -1$
 $\Rightarrow x = 1\frac{1}{2}, -1$

Q10. $f(x) = 1 + \frac{2}{x}$

(i) $f(-4) = 1 + \frac{2}{-4} = 1 - \frac{1}{2} = \frac{1}{2}$
 $f\left(\frac{1}{5}\right) = 1 + \frac{2}{\frac{1}{5}} = 1 + 10 = 11$

(ii) $f(x) = 2$
 $\Rightarrow 1 + \frac{2}{x} = 2$
 $\Rightarrow \frac{2}{x} = 1 \Rightarrow x = 2$

$$\begin{aligned}
 \text{(iii)} \quad k f(2) &= f\left(\frac{1}{2}\right) \\
 \Rightarrow k \left[1 + \frac{2}{2}\right] &= 1 + \frac{2}{\frac{1}{2}} \\
 \Rightarrow 2k &= 1 + 4 = 5 \Rightarrow k = \frac{5}{2}
 \end{aligned}$$

Q11. $g(x) = 1 - 4x$

(i) $g(k+1) = 1 - 4(k+1) = 1 - 4k - 4 = -4k - 3$

(ii) Solve $g(k+1) = g(-3)$

$$\Rightarrow -4k - 3 = 1 - 4(-3) = 13$$

$$\Rightarrow -4k = 13 + 3 = 16 \Rightarrow k = \frac{16}{-4} = -4$$

Q12. $g(x) = 3x - 2$

(i) $g(-x) = 6$

$$\Rightarrow 3(-x) - 2 = 6 \Rightarrow -3x = 8 \Rightarrow x = \frac{8}{-3} = -\frac{8}{3}$$

(ii) $g(2x) = 4$

$$\Rightarrow 3(2x) - 2 = 4 \Rightarrow 6x - 2 = 4 \Rightarrow 6x = 6 \Rightarrow x = 1$$

(iii) $\frac{1}{g(x)} = 6 \Rightarrow \frac{1}{3x-2} = \frac{6}{1}$

$$\Rightarrow 18x - 12 = 1 \Rightarrow 18x = 13 \Rightarrow x = \frac{13}{18}$$

Q13. (i) $f(x) = x^2 - 2x \Rightarrow x^2 - 2x = 3$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x+1)(x-3) = 0$$

$$\Rightarrow x = -1, x = 3$$

(ii) $g(x) = x^2 - x - 6 \Rightarrow g(x) = 0$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x+2)(x-3) = 0$$

$$\Rightarrow x = -2, 3$$

(iii) $h(x) = x + \frac{1}{x} = 2$

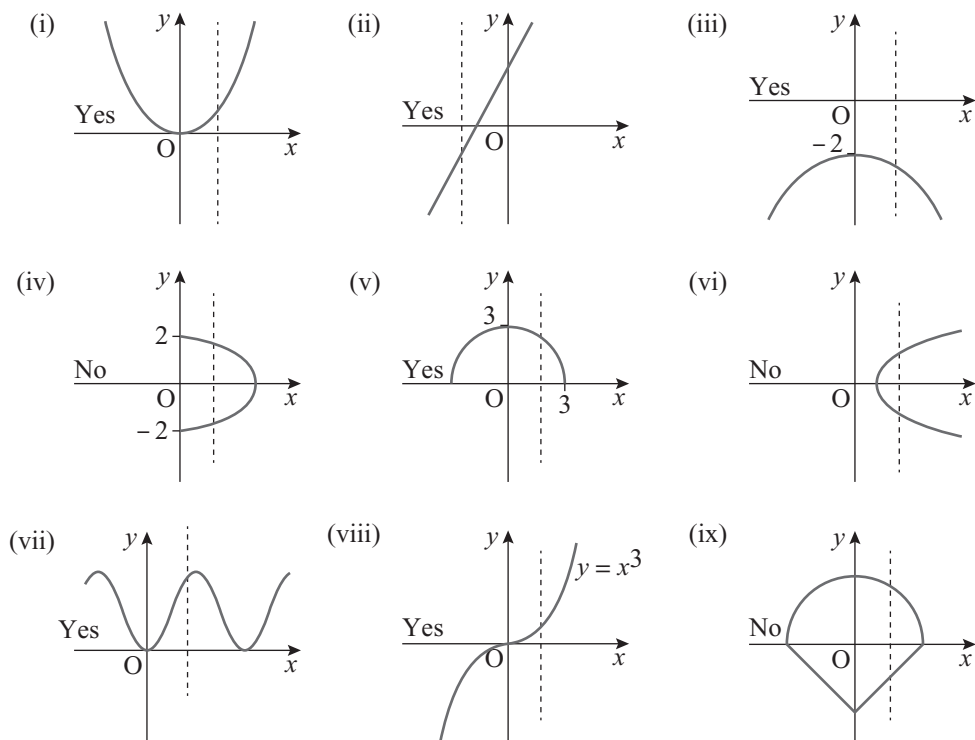
$$\Rightarrow x.x + \frac{1}{x}.x = 2.x$$

$$\Rightarrow x^2 + 1 = 2x$$

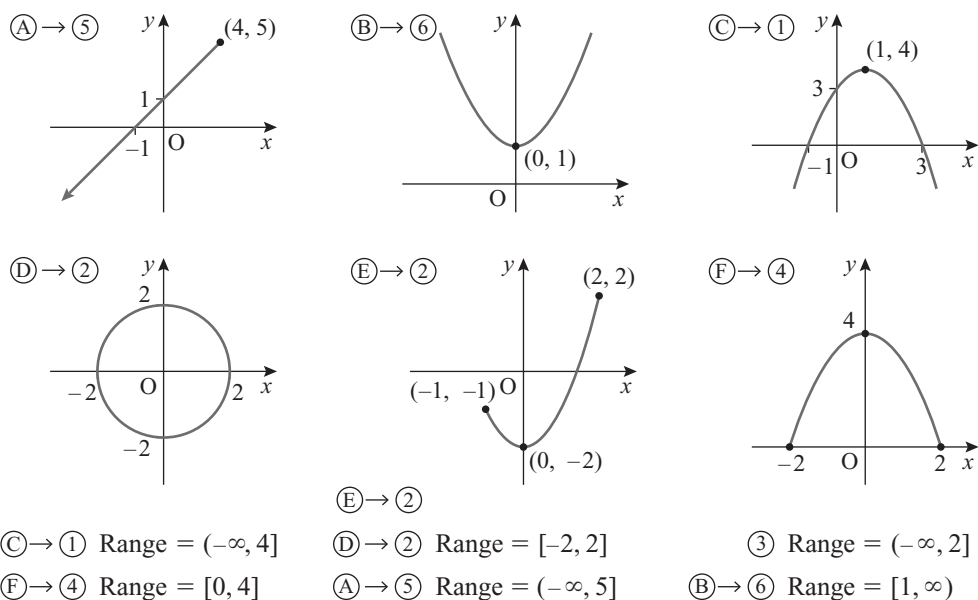
$$\Rightarrow x^2 - 2x + 1 = 0$$

$$\Rightarrow (x-1)(x-1) = 0 \Rightarrow x = 1$$

Q14. Use the vertical line test to determine if each of the following is the graph of a function where $x \in \mathbb{R}$.



Q15. The graphs and the ranges of six relations are given below. Connect each graph to its correct range.



- Q16.** (i) Domain = \mathbb{R} ; Range = $[-2, \infty]$
(ii) Domain = $[-\infty, 2]$; Range = \mathbb{R}
(iii) Domain = $(-2, 3)$; Range = $(0, 9)$
(iv) Domain = $(-3, 1)$; Range = $(-6, 2)$
(v) Domain = $(-4, 0)$; Range = $(0, 4)$
(vi) Domain = \mathbb{R} ; Range = $[-\infty, 4]$

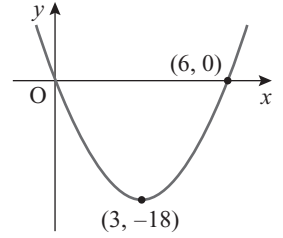
Q17. (i), (iii), (iv) and (vi) are functions

Q18. $f(x) = kx(x-6) = k(x^2 - 6x)$

Point $(3, -18) \Rightarrow f(3) = k[(3)^2 - 6(3)] = -18$

$$\Rightarrow k[9 - 18] = -18$$

$$\Rightarrow -9k = -18 \Rightarrow k = \frac{-18}{-9} = 2$$



Q19. $f(x) = x^2 + px + q$

$$\Rightarrow f(3) = (3)^2 + p(3) + q = 4$$

$$\Rightarrow 9 + 3p + q = 4 \Rightarrow 3p + q = -5$$

and $f(-1) = (-1)^2 + p(-1) + q = 4$

$$\Rightarrow 1 - p + q = 4 \Rightarrow -p + q = 3$$

hence, $3p + q = -5$

and $\underline{p - q = -3}$

add $\Rightarrow 4p = -8 \Rightarrow p = -2$

$$\Rightarrow 3(-2) + q = -5$$

$$\Rightarrow -6 + q = -5 \Rightarrow q = 1$$

Solve $x^2 - 2x + 1 = 0$

$$\Rightarrow (x-1)(x-1) = 0 \Rightarrow x = 1$$

Q20. $f(x) = x^2 + bx + c$

(i) $(-3, 0) \Rightarrow f(-3) = (-3)^2 + b(-3) + c = 0$
 $\Rightarrow 9 - 3b + c = 0 \Rightarrow -3b + c = -9$

(ii) $(0, -3) \Rightarrow f(0) = (0) + b(0) + c = -3$
 $\Rightarrow c = -3$

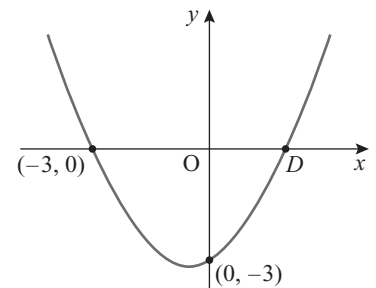
$$\Rightarrow -3b - 3 = -9$$

$$\Rightarrow -3b = -6 \Rightarrow b = \frac{-6}{-3} = 2$$

(iii) $f(x) = x^2 + 2x - 3 = 0$

$$\Rightarrow (x+3)(x-1) = 0$$

$$\Rightarrow x = -3, x = 1 \Rightarrow D = (1, 0)$$



Exercise 1.2

Q1. $f(x) = x^2 + 1$ and $g(x) = 2x - 1$

(i) $f(3) = (3)^2 + 1 = 10$

(ii) $g f(3) = g(10) = 2(10) - 1 = 19$

$$(iii) \quad g(3) = 2(3) - 1 = 5$$

$$(iv) \quad f g(3) = f(5) = (5)^2 + 1 = 26$$

$$(v) \quad f^2(3) = f(f(3)) = f(10) = (10)^2 + 1 = 101$$

$$(vi) \quad g^2(3) = g(g(3)) = g(5) = 2(5) - 1 = 9$$

$$(vii) \quad g f(-4) = g[(-4)^2 + 1] = g(17) = 2(17) - 1 = 33$$

$$(viii) \quad f g\left(\frac{1}{2}\right) = f\left[2\left(\frac{1}{2}\right) - 1\right] = f(0) = (0)^2 + 1 = 1$$

Q2. $f(x) = 2x + 1$ and $g(x) = 4x - 3$

$$(i) \quad f(3) = 2(3) + 1 = 7$$

$$(ii) \quad g f(3) = g[7] = 4(7) - 3 = 25$$

$$(iii) \quad f g(-2) = f[4(-2) - 3] = f(-11) = 2(-11) + 1 = -21$$

$$(iv) \quad g f(x) = g(2x + 1) = 4(2x + 1) - 3 = 8x + 4 - 3 = 8x + 1$$

$$\text{Solve } f g(x) = 19.$$

$$\Rightarrow f(4x - 3) = 19$$

$$\Rightarrow 2(4x - 3) + 1 = 19$$

$$\Rightarrow 8x - 6 + 1 = 19$$

$$\Rightarrow 8x - 5 = 19$$

$$\Rightarrow 8x = 24 \quad \Rightarrow x = \frac{24}{8} = 3$$

Q3. $f(x) = 2x - 1$ and $g(x) = x^2 + 2$

$$(i) \quad f g(-2) = f[(-2)^2 + 2] = f(6) = 2(6) - 1 = 11$$

$$(ii) \quad g f\left(\frac{1}{2}\right) = g\left[2\left(\frac{1}{2}\right) - 1\right] = g(0) = (0)^2 + 2 = 2$$

$$(iii) \quad f g(x) = f(x^2 + 2) = 2(x^2 + 2) - 1 = 2x^2 + 4 - 1 = 2x^2 + 3$$

$$(iv) \quad g f(x) = g(2x - 1) = (2x - 1)^2 + 2 = 4x^2 - 4x + 1 + 2 = 4x^2 - 4x + 3$$

$$\text{Solve } g f(x) = f g(x).$$

$$\Rightarrow 4x^2 - 4x + 3 = 2x^2 + 3$$

$$\Rightarrow 2x^2 - 4x = 0$$

$$\Rightarrow x^2 - 2x = 0$$

$$\Rightarrow x(x - 2) = 0 \quad \Rightarrow x = 0, 2$$

Q4. $f(x) = 2^{x-1}$ and $g(x) = 3 + 4x$

(i) $fg(x) = f(3 + 4x) = 2^{3+4x-1} = 2^{4x+2}$

(ii) $gf(x) = g(2^{x-1}) = 3 + 4(2^{x-1}) = 3 + 2^2 \cdot 2^{x-1}$
 $= 3 + 2^{2+x-1}$
 $= 3 + 2^{x+1}$

Q5. $f(x) = 3x^2$ and $g(x) = 2x + 1$

$$fg(x) = f(2x + 1) = 3(2x + 1)^2 = 3(4x^2 + 4x + 1) = 12x^2 + 12x + 3$$

$$\Rightarrow fg(a) = 12a^2 + 12a + 3 = g(1) = 2(1) + 1 = 3$$

$$\Rightarrow 12a^2 + 12a + 3 = 3$$

$$\Rightarrow 12a^2 + 12a = 0$$

$$\Rightarrow a^2 + a = 0$$

$$\Rightarrow a(a + 1) = 0 \Rightarrow a = 0, -1$$

Q6. $f(x) = 2x + 3$ and $g(x) = 2x - 3$

(i) $fg(x) = f(2x - 3) = 2(2x - 3) + 3$
 $= 4x - 6 + 3 = 4x - 3$

$$gf(x) = g(2x + 3) = 2(2x + 3) - 3$$

 $= 4x + 6 - 3 = 4x + 3$

(ii) $fg(x) \times gf(x) = (4x - 3)(4x + 3)$
 $= 16x^2 + 12x - 12x - 9$
 $= 16x^2 - 9$

$$\Rightarrow \text{least value} = 16(0)^2 - 9 = 0 - 9 = -9$$

Q7. $f(x) = 2x + 1$ and $g(x) = 3x + c$

(i) $gf(x) = g(2x + 1) = 3(2x + 1) + c = 6x + 3 + c$

$$fg(x) = f(3x + c) = 2(3x + c) + 1 = 6x + 2c + 1$$

$$gf(x) = fg(x)$$

$$\Rightarrow 6x + 3 + c = 6x + 2c + 1$$

$$\Rightarrow -c = -2 \Rightarrow c = 2$$

(ii) $f^2(m) = f(f(m)) = f(2m + 1) = 2(2m + 1) + 1 = 4m + 2 + 1 = 4m + 3$

hence, $f^2(m) = m$

$$\Rightarrow 4m + 3 = m$$

$$\Rightarrow 3m = -3 \Rightarrow m = -1$$

Q8. $f(x) = s + tx$ $g(x) = x^2 - 4$ and $h(x) = 3x + 1$

$$\begin{aligned}\Rightarrow h g f(x) &= h g(s + tx) \\ &= h[(s + tx)^2 - 4] \\ &= h[s^2 + 2stx + t^2x^2 - 4] \\ &= 3(s^2 + 2stx + t^2x^2 - 4) + 1 \\ &= 3s^2 + 6stx + 3t^2x^2 - 12 + 1 \\ &= 3t^2x^2 + 6stx + 3s^2 - 11\end{aligned}$$

Solve $h g f(x) = 4(3x^2 + 3x - 2)$.

$$\begin{aligned}\Rightarrow 3t^2x^2 + 6stx + 3s^2 - 11 &= 12x^2 + 12x - 8 \\ \Rightarrow 3t^2 &= 12 \quad \text{and} \quad 6st = 12 \\ \Rightarrow t^2 &= 4 \quad \Rightarrow st = 2 \\ \Rightarrow t = 2 \text{ as } t \in N &\Rightarrow s(2) = 2 \\ &\Rightarrow 2s = 2 \Rightarrow s = 1\end{aligned}$$

Q9. $f(x) = \cos x$ and $g(x) = x + \frac{\pi}{6}$

$$\begin{aligned}\Rightarrow f g\left(\frac{\pi}{6}\right) &= f\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = f\left(\frac{2\pi}{6}\right) = f\left(\frac{\pi}{3}\right) \\ \Rightarrow f\left(\frac{\pi}{3}\right) &= \cos \frac{\pi}{3} = \frac{1}{2}\end{aligned}$$

Q10. $f(x) = x^2 - x + 10$ $g(x) = 5 - x$ and $h(x) = \log_2 x$

(i) $h f(x) = h(x^2 - x + 10) = \log_2(x^2 - x + 10)$
 $h g(x) = h(5 - x) = \log_2(5 - x)$

(ii) Solve $h f(x) - h g(x) = 3$.

$$\begin{aligned}\Rightarrow \log_2(x^2 - x + 10) - \log_2(5 - x) &= 3 = \log_2 8 \\ \Rightarrow \log_2 \frac{x^2 - x + 10}{5 - x} &= \log_2 8 \\ \Rightarrow \frac{x^2 - x + 10}{5 - x} &= \frac{8}{1} \Rightarrow x^2 - x + 10 = 40 - 8x \\ &\Rightarrow x^2 + 7x - 30 = 0 \\ &\Rightarrow (x + 10)(x - 3) = 0 \\ &\Rightarrow x = -10, 3 \Rightarrow x = 3 \text{ as } x > 0\end{aligned}$$

Q11. $f(x) = 2x + 3$

(i) $f^2(x) = f f(x) = f(2x + 3) = 2(2x + 3) + 3$
 $= 4x + 6 + 3 = 4x + 9$
 $= 2^2x + 3(2^2 - 1)$

$$\begin{aligned}
 \text{(ii)} \quad f^3(x) &= f f f(x) = f(4x+9) = 2(4x+9)+3 \\
 &= 8x+18+3 = 8x+21 \\
 &= 2^3x+3(2^3-1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad f^4(x) &= f f f f(x) = f(8x+21) = 2(8x+21)+3 \\
 &= 16x+42+3 = 16x+45 \\
 &= 2^4x+3(2^4-1)
 \end{aligned}$$

$$f^n(x) = 2^n x + 3(2^n - 1)$$

Q12. $f(x) = x^2 + 1$ and $g(x) = 1 - 2x$

$$g f(x) = g(x^2 + 1) = 1 - 2(x^2 + 1) = 1 - 2x^2 - 2 = -2x^2 - 1$$

$$f g(x) = f(1 - 2x) = (1 - 2x)^2 + 1 = 1 - 4x + 4x^2 + 1 = 4x^2 - 4x + 2$$

$\Rightarrow g f(x) \neq f g(x)$; composition of functions is not commutative.

Q13. $f(x) = \frac{1}{2} \left(\frac{1}{x} + 1 \right)$ and $g(x) = \frac{1}{2x-1}$

$$f g(x) = f \left(\frac{1}{2x-1} \right) = \frac{1}{2} \left(\frac{\frac{1}{2x-1}}{\frac{1}{2x-1}} + 1 \right) = \frac{1}{2} (2x - 1 + 1) = \frac{1}{2} (2x) = x$$

$$g f(x) = g \left[\frac{1}{2} \left(\frac{1}{x} + 1 \right) \right] = \frac{1}{2 \cdot \frac{1}{2} \left(\frac{1}{x} + 1 \right) - 1}$$

$$= \frac{1}{\frac{1}{x} + 1 - 1} = \frac{1}{\frac{1}{x}} = x$$

\Rightarrow Yes, $f g(x) = g f(x)$ as both are equal to x .

Q14. $p(x) = (3x-4)^3$ and $f(x) = 3x$, $g(x) = x-4$, $h(x) = x^3$

$$h g f(x) = h g(3x) = h(3x-4) = (3x-4)^3 = p(x)$$

Q15. (i) $h(x) = (3x-1)^2 = f g(x)$

$$\Rightarrow f(x) = x^2 \quad \text{and} \quad g(x) = 3x-1$$

(ii) $h(x) = \frac{1}{5x+3} = g f(x)$

$$\Rightarrow f(x) = 5x+3 \quad \text{and} \quad g(x) = \frac{1}{x}$$

(iii) $h(x) = \sin^2(3x) = f g k(x)$

$$\Rightarrow f(x) = x^2, \quad g(x) = \sin x \quad \text{and} \quad k(x) = 3x$$

(iv) $b(x) = \cos(\sqrt{2x}) = h g f(x)$

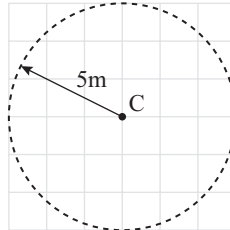
$$\Rightarrow f(x) = 2x, \quad g(x) = \sqrt{x} \quad \text{and} \quad h(x) = \cos x$$

Q16. $f(x) = 2^{x-1}$ and $g(x) = 3 + 4x$
 $\Rightarrow fg(x) = f(3 + 4x) = 2^{3+4x-1} = 2^{4x+2}$
hence, solve $fg(x) = 64$
 $\Rightarrow 2^{4x+2} = 64 = 2^6$
 $\Rightarrow 4x + 2 = 6 \Rightarrow 4x = 4 \Rightarrow x = 1$

Q17. $f(r) = \frac{5}{4}t$

(i) $f(A) = \pi r^2$

(ii) $f(A) = \pi \left(\frac{5}{4}t \right)^2$



Q18. (i) $f(x) = \text{€ } 0.04x \Rightarrow$ this function represents 4% of sales

(ii) $g(x) = \text{€}(x - 4000) \Rightarrow$ value of sales in excess of € 4000

$fg(x) = f(x - 4000) = 0.04(x - 4000) =$ average weekly commission

$fg(8000) = 0.04(8000 - 4000) = 0.04(4000) = \text{€ } 160$

Exercise 1.3

Q1. (a) (i) f is a function; only one arrow from each element in A.

(ii) f is not injective; Two elements in A map to the same element in B.

(iii) f is not surjective; an element in B is not in the range of f .

(b) (i) g is a function; only one arrow from each element in A.

(ii) g is injective; each element in A maps to only one element in B.

(iii) g is surjective; each element in B is the image of some element in A.

There is an exact one-to-one correspondence between the elements in A and B; hence bijective.

Q2. (i) Yes, h is a function; only one arrow from each element in A.

(ii) No, h is not injective; Two elements in A map to the same element in B.

(iii) No, h is not surjective; an element in B is not in the range of h .

(iv) Not both injective and surjective.

Q3. (a) Surjective

(b) Injective

(c) Bijective

Q4. (i) Yes

(ii) No

(iii) Yes

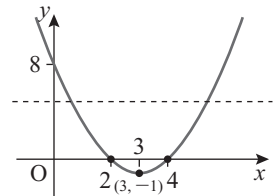
Not a one-to-one correspondence

- Q5.** (i) (a) , (b) , (d) , (e) and (f) are functions
(ii) Only (b) and (e) are injective functions

- Q6.** Injective because any horizontal line will intersect the curve at most once.
Surjective because any horizontal line will intersect the curve at least once.

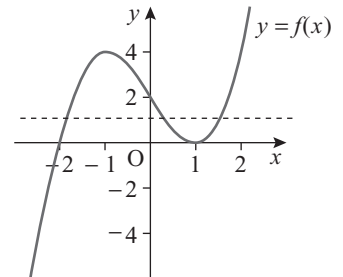
- Q7.** (i) Yes
(ii) Yes

- Q8.** (i) $y \geq -1$
(ii) \mathbb{R}
(iii) Range not equal to codomain
(iv) Codomain : $y \geq -1$
(v) Horizontal line intersects the curve more than once.
(vi) $x \geq 3$ or $x \leq 3$



- Q9.** (i) No; a horizontal line will intersect the curve more than once.
(ii) Yes; as range and codomain are equal
(iii) $x \geq 2$ or $x \leq 2$

- Q10.** (i) Not injective
(ii) Is surjective
No; not both injective and surjective



- Q11.** A vertical line will intersect the curve more than once.
 $y \geq 0$ or $y \leq 0$

- Q12.** (i) N
(ii) N
(iii) all even numbers
(iv) Codomain and range not equal
(v) Yes
(vi) Codomain should be the set of even positive numbers.

- Q13.** (i) No; a horizontal line will intersect the graph more than once.
(ii) Yes; a horizontal line will intersect the graph at least once.
(iii) $\pi \leq x \leq 3\pi$

- Q14.** (i) Yes, as a horizontal line will intersect the graph at least once.
(ii) No, as a horizontal line will intersect the graph more than once.
(iii) $x \geq 0, y \geq 0$

- Q15.** (i) $y > 0$
(ii) Yes, as a horizontal line will intersect the graph at most once.
(iii) Yes, as a horizontal line will intersect the graph at least once.
(iv) Because it is both injective and surjective.

Exercise 1.4

Q1. $y = x - 4 \Rightarrow x = y + 4$
 $\therefore f^{-1}(x) = x + 4$

Q2. $y = 2x - 3 \Rightarrow 2x = y + 3$
 $\Rightarrow x = \frac{y+3}{2}$
 $\therefore f^{-1}(x) = \frac{x+3}{2}$

Q3. $y = 5x + 3 \Rightarrow 5x = y - 3$
 $\Rightarrow x = \frac{y-3}{5}$
 $\therefore f^{-1}(x) = \frac{x-3}{5}$

Q4. $y = 3x \Rightarrow x = \frac{y}{3}$
 $\therefore f^{-1}(x) = \frac{x}{3}$

Q5. $y = \frac{2x}{5} \Rightarrow 2x = 5y$
 $\Rightarrow x = \frac{5y}{2}$
 $\therefore f^{-1}(x) = \frac{5x}{2}$

$$\begin{aligned}
\text{Q6. } y &= \frac{4x-3}{2} \Rightarrow 4x-3=2y \\
&\Rightarrow 4x=2y+3 \\
&\Rightarrow x=\frac{2y+3}{4} \\
\therefore f^{-1}(x) &= \frac{2x+3}{4}
\end{aligned}$$

$$\begin{aligned}
\text{Q7. } y &= \frac{x-6}{x} \Rightarrow xy=x-6 \\
&\Rightarrow xy-x=-6 \\
&\Rightarrow x(y-1)=-6 \\
&\Rightarrow x=\frac{-6}{y-1} \\
\therefore f^{-1}(x) &= \frac{-6}{x-1}
\end{aligned}$$

$$\begin{aligned}
\text{Q8. } y &= \frac{3x}{x-1} \Rightarrow xy-y=3x \\
&\Rightarrow xy-3x=y \\
&\Rightarrow x(y-3)=y \\
&\Rightarrow x=\frac{y}{y-3} \\
\therefore f^{-1}(x) &= \frac{x}{x-3}
\end{aligned}$$

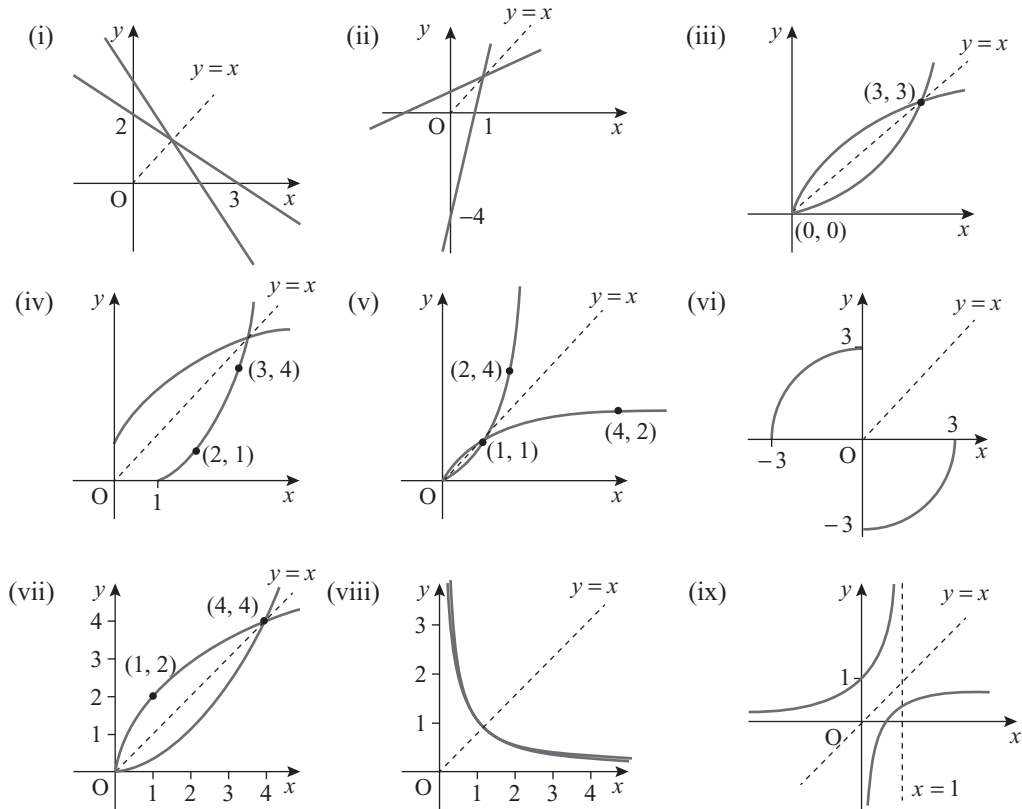
$$\begin{aligned}
\text{Q9. } y &= \frac{10-2x}{3} \Rightarrow 10-2x=3y \\
&\Rightarrow -2x=3y-10 \\
&\Rightarrow 2x=10-3y \\
&\Rightarrow x=\frac{10-3y}{2} \\
\therefore f^{-1}(x) &= \frac{10-3x}{2}
\end{aligned}$$

$$\begin{aligned}
\text{Q10. } y &= 4x+5 \Rightarrow 4x=y-5 \\
&\Rightarrow x=\frac{y-5}{4} \\
\therefore f^{-1}(x) &= \frac{x-5}{4} \\
ff^{-1}(x) &= f\left(\frac{x-5}{4}\right) = 4\left(\frac{x-5}{4}\right) + 5 = x-5+5 = x \\
f^{-1}f(x) &= f^{-1}(4x+5) = \frac{4x+5-5}{4} = \frac{4x}{4} = x \\
\text{Hence, } f^{-1}f(x) &= ff^{-1}(x)
\end{aligned}$$

Q11. $y = \frac{x}{3} - 2 \Rightarrow \frac{x}{3} = y + 2$
 $\Rightarrow x = 3(y + 2)$
 $\therefore f^{-1}(x) = 3(x + 2)$

Hence, $ff^{-1}(x) = f[3(x + 2)] = \frac{3(x + 2)}{3} - 2 = x + 2 - 2 = x$

Q12.



Q13. Line ℓ : $(-4, 0)$ and $(2, 2)$
 (x_1, y_1) and (x_2, y_2)

(i) $\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{2 + 4} = \frac{2}{6} = \frac{1}{3}$

Equation of line ℓ : $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 0 = \frac{1}{3}(x + 4)$$

$$\Rightarrow 3y = x + 4$$

$$\Rightarrow x - 3y + 4 = 0$$

Line m : $(0, -4)$ and $(2, 2)$
 x_1, y_1 and x_2, y_2

$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 + 4}{2 - 0} = \frac{6}{2} = 3$

Equation of line m : $y - y_1 = m(x - x_1)$

$$\Rightarrow y + 4 = 3(x - 0)$$

$$\Rightarrow y + 4 = 3x$$

$$\Rightarrow 3x - y - 4 = 0$$

$$\begin{aligned}
 \text{(ii) Line } \ell: \quad 3y &= x+4 & \text{Line } m: 3x-y-4 &= 0 \\
 \Rightarrow y &= \frac{x+4}{3} & \Rightarrow f^{-1}(x) = y = 3x-4 \\
 \Rightarrow f(x) &= \frac{x+4}{3}
 \end{aligned}$$

$$\text{Hence, } f f^{-1}(x) = f(3x-4) = \frac{3x-4+4}{3} = \frac{3x}{3} = x$$

\Rightarrow equation of ℓ is the inverse of the equation of m .

Q14. $g(x) = \frac{1}{x-2}$ and $f(x) = \frac{1+kx}{x}$

$$\begin{aligned}
 \Rightarrow g f(x) = x &\Rightarrow g f(x) = g\left(\frac{1+kx}{x}\right) = \frac{1}{\frac{1+kx}{x}-2} \\
 &= \frac{1}{\frac{1+kx-2x}{x}} = x \\
 &\Rightarrow \frac{x}{1+kx-2x} = x \\
 &\Rightarrow \frac{1}{1+kx-2x} = 1 \\
 &\Rightarrow 1+kx-2x = 1 \\
 &\Rightarrow x(k-2) = 0 \\
 &\Rightarrow k = 2
 \end{aligned}$$

Q15. $f(x) = 2x-3$ and $g(x) = x-4$

(i) $g f(x) = g(2x-3) = 2x-3-4 = 2x-7$

$$\begin{aligned}
 y = 2x-7 &\Rightarrow 2x = y+7 \\
 \Rightarrow x &= \frac{y+7}{2} \\
 \Rightarrow [g f(x)]^{-1} &= \frac{x+7}{2}
 \end{aligned}$$

(ii) $y = 2x-3$ and $y = x-4$

$$\begin{aligned}
 \Rightarrow 2x &= y+3 & \Rightarrow x &= y+4 \\
 \Rightarrow x &= \frac{y+3}{2} & \Rightarrow g^{-1}(x) &= x+4 \\
 \Rightarrow f^{-1}(x) &= \frac{x+3}{2} \\
 \Rightarrow f^{-1} g^{-1}(x) &= f^{-1}(x+4) \\
 &= \frac{x+4+3}{2} = \frac{x+7}{2} = [g f(x)]^{-1} \quad \therefore \text{YES}
 \end{aligned}$$

Q16. $f(x) = \frac{x+3}{2} = y \Rightarrow f(0) = \frac{0+3}{2} = 1\frac{1}{2} \Rightarrow \text{Points}\left(0, 1\frac{1}{2}\right), (5, 4)$

$$\Rightarrow x+3 = 2y \quad f(5) = \frac{5+3}{2} = 4$$

$$\Rightarrow x = 2y - 3$$

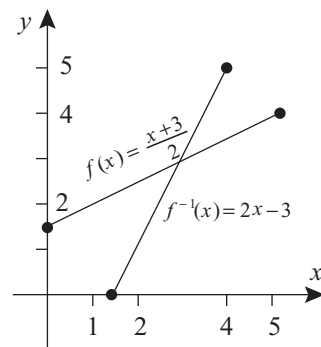
$$\Rightarrow f^{-1}(x) = 2x - 3$$

$$\text{Points}\left(1\frac{1}{2}, 0\right), (4, 5)$$

$$\text{Domain of } f^{-1}(x) = \left\{1\frac{1}{2}, 4\right\}$$

$$= \text{Range of } f$$

$$\text{Range of } f^{-1}(x) = \{0, 5\}$$



Q17. (i) $f(x) = y = x^2 + 4x - 6$

$$\Rightarrow y = x^2 + 4x + 4 - 6 - 4$$

$$\Rightarrow y = (x+2)^2 - 10$$

$$\Rightarrow (x+2)^2 = y+10$$

$$\Rightarrow x+2 = \sqrt{y+10}$$

$$\Rightarrow x = -2 + \sqrt{y+10}$$

$$\therefore f^{-1}(x) = -2 + \sqrt{x+10}, \quad x \geq -10$$

(ii) $f(x) = y = x^2 - 2x - 5$

$$\Rightarrow y = x^2 - 2x + 1 - 5 - 1$$

$$\Rightarrow y = (x-1)^2 - 6$$

$$\Rightarrow (x-1)^2 = y+6$$

$$\Rightarrow x-1 = \sqrt{y+6}$$

$$\Rightarrow x = 1 + \sqrt{y+6}$$

$$\therefore f^{-1}(x) = 1 + \sqrt{x+6}, \quad x \geq -6$$

(iii) $f(x) = y = x^2 - 8x - 3$

$$\Rightarrow y = x^2 - 8x + 16 - 3 - 16$$

$$\Rightarrow y = (x-4)^2 - 19$$

$$\Rightarrow (x-4)^2 = y+19$$

$$\Rightarrow x-4 = \sqrt{y+19}$$

$$\Rightarrow x = 4 + \sqrt{y+19}$$

$$\therefore f^{-1}(x) = 4 + \sqrt{x+19}, \quad x \geq -19$$

$$\begin{aligned}
 \text{(iv)} \quad f(x) = y &= x^2 + 8x + 20 \\
 \Rightarrow y &= x^2 + 8x + 16 + 4 \\
 \Rightarrow y &= (x + 4)^2 + 4 \\
 \Rightarrow (x + 4)^2 &= y - 4 \\
 \Rightarrow x + 4 &= \sqrt{y - 4} \\
 \Rightarrow x &= -4 + \sqrt{y - 4} \\
 \therefore f^{-1}(x) &= -4 + \sqrt{x - 4}, \quad x \geq 4
 \end{aligned}$$

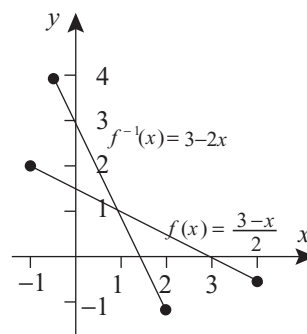
$$\begin{aligned}
 \text{Q18.} \quad f(x) = y &= \frac{3-x}{2} \Rightarrow 2y = 3-x \\
 &\Rightarrow x = 3-2y \\
 \therefore f^{-1}(x) &= 3-2x
 \end{aligned}$$

$$2 \text{ points on } f(x) = (-1, 2) \text{ and } \left(4, -\frac{1}{2}\right)$$

$$2 \text{ points on } f^{-1}(x) = (2, -1) \text{ and } \left(-\frac{1}{2}, 4\right)$$

$$\text{Domain of } f^{-1}(x) = \left\{-\frac{1}{2}, 2\right\}$$

$$\text{Range of } f^{-1}(x) = \{-1, 4\}$$



$$\text{Q19.} \quad A \leq 3 \quad [\text{OR } (-\infty, 3)]$$

$$\text{Q20.} \quad b = 0$$

$$g(x) = y = 1 - x^2$$

$$\Rightarrow x^2 = 1 - y$$

$$\Rightarrow x = \sqrt{1 - y}$$

$$\therefore g^{-1}(x) = \sqrt{1 - x}, \quad x \leq 1$$

$$\text{Domain of } g^{-1}(x) = \{1, -3\} \text{ or } \{-3, 1\}$$

$$\text{Range of } g^{-1}(x) = \{0, 2\}$$

Exercise 1.5

$$\text{Q1. (i)} \quad x = 0$$

$$\text{(ii)} \quad x = 3$$

$$\text{(iii)} \quad x = -2$$

$$\text{(iv)} \quad x = 2$$

$$\text{(v)} \quad x = -3$$

$$\text{(vi)} \quad x = \frac{-\pi}{2} \quad \text{and} \quad x = \frac{\pi}{2}$$

Q2. (i) $x = 0$

(ii) $\frac{2}{0}$ is not defined

Q3. $\tan \frac{\pi}{2} = \frac{k}{0}$ which is not defined

Q4. (i) $x = 4$

(ii) $x = -5$ or $x = 5$

(iii) $x^2 - 3x - 4 = 0 \Rightarrow (x+1)(x-4) = 0$
 $\Rightarrow x = -1$ OR $x = 4$

Q5. (i) $\lim_{x \rightarrow 2} \frac{x+3}{x+2} = \frac{2+3}{2+2} = \frac{5}{4}$

(ii) $\lim_{x \rightarrow 0} (x^2 + 3x - 4) = (0)^2 + 3(0) - 4 = -4$

(iii) $\lim_{x \rightarrow 3} \frac{x^2 - x - 3}{x+1} = \frac{(3)^2 - (3) - 3}{3+1} = \frac{9-3-3}{4} = \frac{3}{4}$

Q6. (i) $\lim_{x \rightarrow 0} \frac{x+2}{x-2} = \frac{0+2}{0-2} = -1$

(ii) $\lim_{x \rightarrow 0} \frac{6x-3}{2+x} = \frac{6(0)-3}{2+0} = \frac{-3}{2}$

(iii) $\lim_{h \rightarrow 2} \frac{h^2 + 2h - 6}{h+1} = \frac{(2)^2 + 2(2) - 6}{2+1} = \frac{4+4-6}{3} = \frac{2}{3}$

Q7. (i) $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x+1) \cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \rightarrow 1} (x+1) = 1+1 = 2$

(ii) $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x+2) \cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x \rightarrow 2} (x+2) = 2+2 = 4$

(iii) $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} = \lim_{x \rightarrow 5} \frac{(x+5) \cancel{(x-5)}}{\cancel{(x-5)}} = \lim_{x \rightarrow 5} (x+5) = 5+5 = 10$

(iv) $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x - 1} = \lim_{x \rightarrow 1} \frac{(x-2) \cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \rightarrow 1} (x-2) = 1-2 = -1$

(v) $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \rightarrow 1} \frac{(x+2) \cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \rightarrow 1} (x+2) = 1+2 = 3$

$$(vi) \lim_{x \rightarrow -3} \frac{x+3}{x^2-x-12} = \lim_{x \rightarrow -3} \frac{\cancel{(x+3)}}{(x-4)\cancel{(x+3)}} = \lim_{x \rightarrow -3} \frac{1}{x-4} = \frac{1}{-3-4} = -\frac{1}{7}$$

Q8. $f(x) = \frac{x^2-9}{x-3}$

x	2.5	2.9	2.999	2.9999	3.0000	3.0001	3.001	3.1	3.5
$f(x)$	5.5	5.9	5.999	5.9999	Undefined	6.0001	6.001	6.1	6.5

Hence, limit = 6

Q9. (i) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

(ii) $\lim_{x \rightarrow \infty} \frac{4}{3x} = 0$

(iii) $\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$

(iv) $\lim_{x \rightarrow \infty} \frac{k}{x^3} = 0$

Q10. (i) $\lim_{x \rightarrow \infty} \frac{3x-2}{2x+3} = \lim_{x \rightarrow \infty} \frac{3-\frac{2}{x}}{2+\frac{3}{x}} = \frac{3-0}{2+0} = \frac{3}{2}$

(ii) $\lim_{x \rightarrow \infty} \frac{4x-3}{7x-6} = \lim_{x \rightarrow \infty} \frac{4-\frac{3}{x}}{7-\frac{6}{x}} = \frac{4-0}{7-0} = \frac{4}{7}$

(iii) $\lim_{x \rightarrow \infty} \frac{1-3x}{4x+2} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}-3}{4+\frac{2}{x}} = \frac{0-3}{4+0} = -\frac{3}{4}$

Q11. (i) $\lim_{n \rightarrow \infty} \frac{n^2+4}{3n^2-4n} = \lim_{n \rightarrow \infty} \frac{1+\frac{4}{n^2}}{3-\frac{4}{n}} = \frac{1+0}{3-0} = \frac{1}{3}$

(ii) $\lim_{n \rightarrow \infty} \frac{5n^2-3}{2n^2-6n+5} = \lim_{n \rightarrow \infty} \frac{5-\frac{3}{n^2}}{2-\frac{6}{n}+\frac{5}{n^2}} = \frac{5-0}{2-0+0} = \frac{5}{2}$

(iii) $\lim_{n \rightarrow \infty} \frac{2n^2-3n+2}{6n^2+5n-6} = \lim_{n \rightarrow \infty} \frac{2-\frac{3}{n}+\frac{2}{n^2}}{6+\frac{5}{n}-\frac{6}{n^2}} = \frac{2-0+0}{6+0-0} = \frac{2}{6} = \frac{1}{3}$

Q12. (i) $f(x) = 2x - 3$
 $f(x+h) = 2(x+h) - 3 = 2x + 2h - 3$
 $\Rightarrow f(x+h) - f(x) = 2x + 2h - 3 - (2x - 3)$
 $\quad = 2x + 2h - 3 - 2x + 3 = 2h$
 $\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2h}{h} = 2$
 $\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} 2 = 2$

(ii) $f(x) = x^2$
 $\Rightarrow f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$
 $\Rightarrow f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2$
 $\quad = 2xh + h^2$
 $\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$
 $\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x + 0 = 2x$

(iii) $f(x) = x^2 + 5$
 $\Rightarrow f(x+h) = (x+h)^2 + 5 = x^2 + 2xh + h^2 + 5$
 $\Rightarrow f(x+h) - f(x) = x^2 + 2xh + h^2 + 5 - (x^2 + 5)$
 $\quad = x^2 + 2xh + h^2 + 5 - x^2 - 5$
 $\quad = 2xh + h^2$
 $\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$
 $\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x$

Q13. $\lim_{x \rightarrow 3} \frac{x-3}{x^3-27} = \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}}{\cancel{(x-3)}(x^2+3x+9)}$
 $\quad = \lim_{x \rightarrow 3} \frac{1}{x^2+3x+9} = \frac{1}{(3)^2+3(3)+9} = \frac{1}{27}$

Q14. $f(n) = \left(1 + \frac{1}{n}\right)^n$

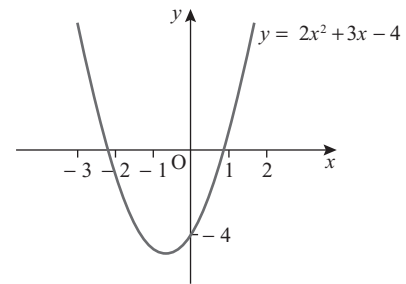
n	1	2	5	10	100	1000	10000
$f(n)$	2	2.25	2.488	2.594	2.705	2.717	2.718

Hence, approximate value for $e = 2.718$

Exercise 1.6

Q1. Graph the function $f(x) = 2x^2 + 3x - 4$ in the domain $-3 \leq x \leq 2$

x	$2x^2 + 3x - 4$	y
-3	$18 - 9 - 4$	5
-2	$8 - 6 - 4$	-2
-1	$2 - 3 - 4$	-5
0	$0 + 0 - 4$	-4
1	$2 + 3 - 4$	1
2	$8 + 6 - 4$	10

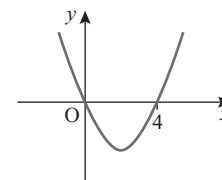


Q2. (i) $f(x) = x^2 - 4x$

$$\Rightarrow y = x(x - 4) = 0$$

$$\Rightarrow x = 0, x = 4$$

Points on x -axis $(0, 0)$, $(4, 0)$



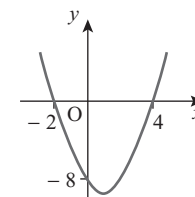
(ii) $f(x) = x^2 - 2x - 8$

$$\Rightarrow y = (x + 2)(x - 4) = 0$$

$$\Rightarrow x = -2, x = 4$$

Points on x -axis $(-2, 0)$, $(4, 0)$

Point on y -axis $(0, -8)$



(iii) $f(x) = -x^2 + 2x + 3$

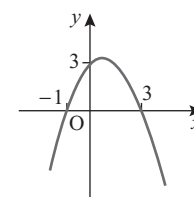
$$\Rightarrow y = -(x^2 - 2x - 3) = 0$$

$$= -(x + 1)(x - 3) = 0$$

$$\Rightarrow x = -1, x = 3$$

Points on x -axis $(-1, 0)$, $(3, 0)$

Point on y -axis $(0, 3)$



Q3. (i) $x^2 - 4x + 2$

$$= x^2 - 4x + 4 + 2 - 4$$

$$= (x - 2)^2 - 2$$

(ii) $x^2 - 12x + 36 = (x - 6)^2$

(iii) $-x^2 + 8x - 12$

$$= -(x^2 - 8x + 12)$$

$$= -(x^2 - 8x + 16 + 12 - 16)$$

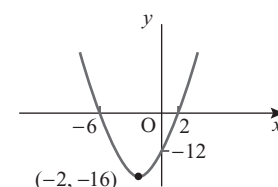
$$= -[(x - 4)^2 - 4] = -(x - 4)^2 + 4$$

Q4. $y = x^2 + 4x - 12$

$$= x^2 + 4x + 4 - 12 - 4$$

$$= (x + 2)^2 - 16$$

$(x + 2)^2 - 16$; Intersects x -axis at $(-6, 0)$ and $(2, 0)$; Turning point = $(-2, -16)$



Q5.

$$y = x^2 + 4x - 5$$

$$\text{on } x\text{-axis} \Rightarrow y = 0$$

$$\Rightarrow x^2 + 4x - 5 = 0$$

$$\Rightarrow (x+5)(x-1) = 0$$

$$\Rightarrow x = -5, x = 1$$

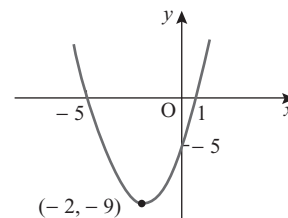
$$\text{on } x\text{-axis} \Rightarrow (-5, 0) \text{ and } (1, 0)$$

$$\text{on } y\text{-axis} \Rightarrow (0, -5)$$

$$y = x^2 + 4x - 5 = x^2 + 4x + 4 - 5 - 4$$

$$= (x+2)^2 - 9$$

$$\Rightarrow \text{Turning point} = (-2, -9)$$

**Q6.**

$$x^2 + 3x - 10 = x^2 + 3x + \frac{9}{4} - 10 - \frac{9}{4}$$

$$= \left(x + \frac{3}{2}\right)^2 - \frac{49}{4}$$

$$\Rightarrow \text{Turning point} = \left(-\frac{3}{2}, -\frac{49}{4}\right)$$

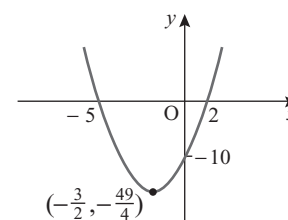
$$\text{On } x\text{-axis, } y = 0 \Rightarrow x^2 + 3x - 10 = 0$$

$$\Rightarrow (x+5)(x-2) = 0$$

$$\Rightarrow x = -5, x = 2$$

$$\text{Points on } x\text{-axis } (-5, 0), (2, 0)$$

$$\text{Point on } y\text{-axis } (0, -10)$$

**Q7.**

$$y = ax^2 + c$$

$$\text{Point } (-1, 4) \Rightarrow 4 = a(-1)^2 + c \Rightarrow a + c = 4$$

$$\text{Point } (0, 8) \Rightarrow 8 = a(0) + c \Rightarrow c = 8$$

$$\Rightarrow a + 8 = 4 \Rightarrow a = -4$$

Q8. (i) $y = x^2$

$$\text{(ii) } y = a(x-1)(x-3) = a(x^2 - 4x + 3)$$

$$\text{point } (0, 3) \Rightarrow 3 = a(0 - 0 + 3)$$

$$\Rightarrow 3 = 3a \Rightarrow a = 1$$

$$\Rightarrow \text{Equation : } y = x^2 - 4x + 3$$

$$\text{(iii) } y = a(x+3)(x-1) = a(x^2 + 2x - 3)$$

$$\text{Point } (-1, 5) \Rightarrow 5 = a[(-1)^2 + 2(-1) - 3]$$

$$\Rightarrow 5 = a(1 - 2 - 3)$$

$$\Rightarrow 5 = -4a \Rightarrow a = -\frac{5}{4}$$

$$\Rightarrow \text{Equation: } y = -\frac{5}{4}(x+3)(x-1)$$

Q9. $y = (x - 4)^2 - 3 \Rightarrow$ Turning point $= (4, -3)$ minimum
 \Rightarrow Graph **(B)**

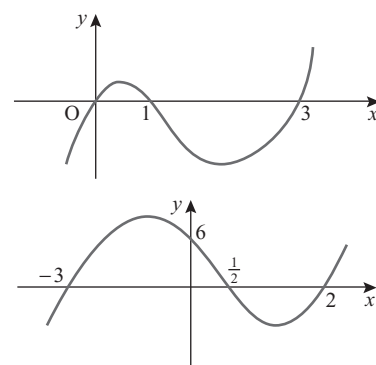
Q10. $y = 3 - (x - 4)^2 \Rightarrow$ Turning point $= (4, 3)$ maximum
 \Rightarrow Graph **(D)**

Q11. Turning point $= (-1, 3) \Rightarrow y = k(x + 1)^2 + 3$
 Point $(0, 4) \Rightarrow 4 = k(0 + 1)^2 + 3$
 $\Rightarrow 1 = k.(1) \Rightarrow k = 1$
 \Rightarrow Equation: $y = (x + 1)^2 + 3$ or $y = x^2 + 2x + 4$

Q12. (a) (i) $y = (x + 1)(x + 2)(x - 3) = 0$
 $\Rightarrow x = -1, -2, 3$
 Intersects x -axis at $(-1, 0), (-2, 0)$ and $(3, 0)$
 (ii) y -axis $\Rightarrow x = 0 \Rightarrow y = (0 + 1)(0 + 2)(0 - 3) = -6 \Rightarrow$ point $(0, -6)$
 (b) (i) $y = x(x - 6)(x + 3) = 0$
 $\Rightarrow x = 0, 6, -3$
 Intersects x -axis at $(0, 0), (6, 0), (-3, 0)$
 (ii) y -axis $\Rightarrow x = 0 \Rightarrow y = 0(0 - 6)(0 + 3) = 0 \Rightarrow$ point $(0, 0)$
 (c) (i) $y = (x - 1)(x + 2)^2 = 0$
 $\Rightarrow x = 1, x = -2$
 Intersects x -axis at $(1, 0)$ and $(-2, 0)$
 (ii) on y -axis $\Rightarrow x = 0 \Rightarrow y = (0 - 1)(0 + 2)^2 = -4 \Rightarrow$ point $(0, -4)$
 (d) (i) $y = x(x^2 - 9) = x(x + 3)(x - 3) = 0$
 $\Rightarrow x = 0, -3, 3$
 Intersects x -axis at $(0, 0), (-3, 0), (3, 0)$
 (ii) On y -axis $\Rightarrow x = 0 \Rightarrow y = 0 [(0)^2 - 9] = 0 \Rightarrow$ point $(0, 0)$

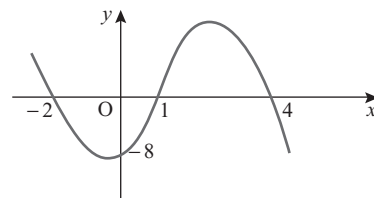
Q13. (i) $y = x(x - 1)(x - 3)$
 Intersects x -axis at $(0, 0), (1, 0)$ and $(3, 0)$; y -axis at $(0, 0)$

(ii) $y = (x - 2)(x + 3)(2x - 1)$
 Intersects x -axis at $(-3, 0), \left(\frac{1}{2}, 0\right)$ and $(2, 0)$; y -axis at $(0, 6)$



(iii) $y = -(x-1)(x+2)(x-4)$

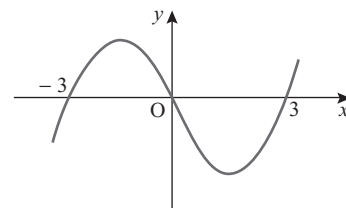
Intersects x -axis at $(-2, 0)$, $(1, 0)$ and $(4, 0)$; y -axis at $(0, -8)$



(iv) $y = x^3 - 9x = x(x^2 - 9)$

$$= x(x+3)(x-3)$$

Intersects x -axis at $(-3, 0)$, $(0, 0)$ and $(3, 0)$; y -axis at $(0, 0)$



Q14. (i) Graph Ⓑ

(ii) Graph Ⓒ

(iii) Graph Ⓑ

(iv) Graph Ⓑ

Q15. $y = x^3 - x^2$ and graph C

$y = 1 - x^2$ and graph A

$y = x - x^2$ and graph B

$y = \frac{-3}{4}x + 3$ and graph F

$y = x^2 + 3x$ and graph E

$y = 9x - x^3$ and graph D

Q16. (i) $f(3) = -27$

(ii) Maximum Turning point = $(-1, 5)$

(iii) Roots $x = -2.8$, $x = 1.8$, $x = 3.9$

(iv) $f(x)$ is decreasing $\Rightarrow -1 < x < 3$

(v) Line $y = 10$ intersects the graph at one point only.

(vi) Line $y = -10$ intersects the graph at three points.

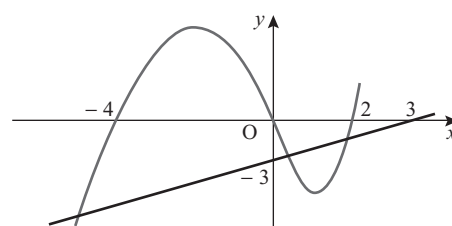
(vii) $f(x) = k$ has three real roots

$$\Rightarrow -27 < k < 5$$

Q17. $y = x(x-2)(x+4)$

Points on x -axis = $(-4, 0)$, $(0, 0)$, $(2, 0)$

$y = x - 3 \Rightarrow 2$ points $(0, -3)$, $(3, 0)$



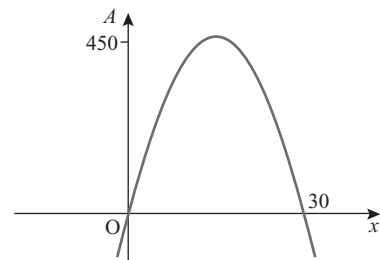
3 intersection points

Q18. $y = k(x+2)(x-1)(x-5)$
 point $(0, -4) \Rightarrow -4 = k(0+2)(0-1)(0-5)$
 $\Rightarrow 10k = -4 \Rightarrow k = \frac{-4}{10} = \frac{-2}{5}$
 $\Rightarrow y = \frac{-2}{5}(x+2)(x-1)(x-5)$

Q19. (i) Length = $\ell \Rightarrow$ perimeter = $x + \ell + x = 60$ m
 $\Rightarrow \ell = (60 - 2x)$ m
 Area $A = x(60 - 2x) \text{ m}^2$

(ii) Points on x -axis = $(0, 0), (30, 0)$

(iii) Maximum occurs at $x = 15$
 $\Rightarrow A = 15[60 - 2(15)] = 450 \text{ m}^2$



Q20. (i) Graph intersects the x -axis at $x = -1$ and it touches the x -axis at $x = 2$

The equation has the form $y = k(x+1)(x-2)^2$

Put $(3, 2)$ into the equation: $2 = k(3+1)(1-2)^2$

$$\Rightarrow 2 = 4k$$

$$\Rightarrow k = \frac{1}{2}$$

The equation of the function is $y = \frac{1}{2}(x+1)(x-2)^2$

(ii) Not injective because a horizontal line will intersect the graph more than once.

(iii) Yes, as every horizontal line will intersect the graph at least once.

(iv) Not injective \Rightarrow not bijective

Q21. (i) $f(x) = x^2 - 6x + 18$
 $= x^2 - 6x + 9 + 9$
 $= (x-3)^2 + 9$

(ii) $P = (0, 18)$, minimum point $Q = (3, 9)$

(iii) $y = x^2 - 6x + 18 \cap y = 41$

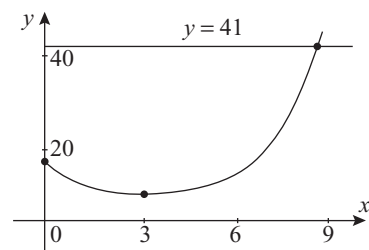
$$\Rightarrow x^2 - 6x + 18 = 41$$

$$\Rightarrow x^2 - 6x - 23 = 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 4(1)(-23)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{128}}{2} = \frac{6 \pm 8\sqrt{2}}{2} = 3 \pm 4\sqrt{2}$$

$$\Rightarrow x = 3 + 4\sqrt{2} \text{ (as } x \geq 0)$$



Exercise 1.7

- Q1.** (i) Graph (A) and $f(x) = a^x$, $a > 1$
 Graph (B) and $f(x) = a^x$, $0 < a < 1$

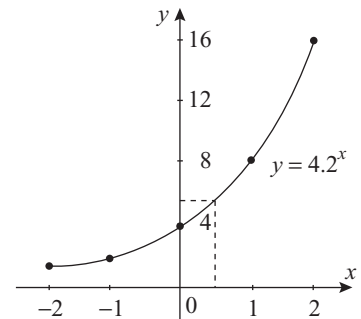
(ii) At $P \Rightarrow x = 0 \Rightarrow f(x) = a^0 = 1 \Rightarrow P = (0, 1)$

(iii) (A): $y = 0$; (B): $y = 0$

Q2.

x	-2	-1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$4 \cdot 2^x$	1	2	4	8	16

$$f(0.5) = 5.65 = 5.7$$



- Q3.** A is $f(x) = 3 \cdot 3^x$
 B is $f(x) = 3^x$
 C is $f(x) = 2^x$

Q4. (i) Point $(0, 5) \Rightarrow f(0) = k \cdot 2^0 = 5$ OR $f(0) = k \cdot 3^0 = 5$
 $\Rightarrow k \cdot 1 = 5 \Rightarrow k = 5$
 $\Rightarrow k = 5 \Rightarrow k = 5$

(ii) $f(x) = 5 \cdot 2^x$ OR $f(x) = 5 \cdot 3^x$
 Point $(2, 20) \Rightarrow f(2) = 5 \cdot 2^2 = 20$ (valid) $f(2) = 5 \cdot 3^2 = 45$ (not valid)
 Hence, function is $y = 5 \cdot 2^x$

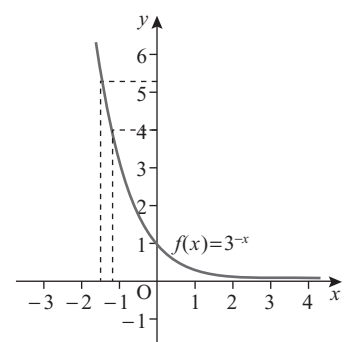
Q5. $y = a(2^x)$
 Point $(1, 3) \Rightarrow 3 = a(2^1) \Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$

Q6.

x	-2	-1	0	1	2	3
$f(x) = 3^{-x}$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

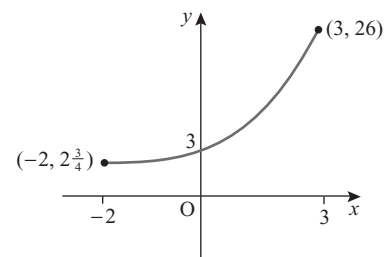
(i) $f(-1.5) = 5.2$

(ii) $f(x) = 4 \Rightarrow x = -1.25$



Q7.

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$3 \cdot 2^x$	$\frac{3}{4}$	$1\frac{1}{2}$	3	6	12	24
$3 \cdot 2^{x+2}$	$2\frac{3}{4}$	$3\frac{1}{2}$	5	8	14	26



$$\text{Range} = \left[2\frac{3}{4}, 26 \right]$$

Q8.

$$y = ae^x + b$$

$$\begin{aligned} \text{Point } (0, 0) &\Rightarrow 0 = ae^0 + b = a \cdot 1 + b \\ &\Rightarrow a + b = 0 \end{aligned}$$

$$\begin{aligned} \text{Point } (1, 14) &\Rightarrow 14 = ae^1 + b = ae + b \\ &\Rightarrow ae + b = 14 \\ &\quad \underline{-a - b = 0} \\ &\Rightarrow ae - a = 14 \end{aligned}$$

$$\Rightarrow a(e - 1) = 14 \quad \Rightarrow a = \frac{14}{e - 1}$$

$$a + b = 0 \Rightarrow b = -a \Rightarrow b = \frac{-14}{e - 1}$$

Q9. (i) $y = 3 \times 4^x$

$$\text{Point } (a, 6) \Rightarrow 6 = 3 \times 4^a \Rightarrow 4^a = \frac{6}{3} = 2$$

$$\Rightarrow 4^a = (2^2)^a = 2^{2a} = 2^1$$

$$\Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

(ii) Point $\left(-\frac{1}{2}, b\right)$ lies on $y = 3 \times 4^x$

$$\Rightarrow b = 3 \times 4^{-\frac{1}{2}} = 3 \times \frac{1}{2} = \frac{3}{2}$$

Q10.

$$y = 3^x \text{ and } y = x + 3$$

$$x = 0 \Rightarrow y = 3^0 = 1 \quad \text{Point } (0, 1)$$

$$x = 1 \Rightarrow y = 3^1 = 3 \quad \text{Point } (1, 3)$$

$$x = 2 \Rightarrow y = 3^2 = 9 \quad \text{Point } (2, 9)$$

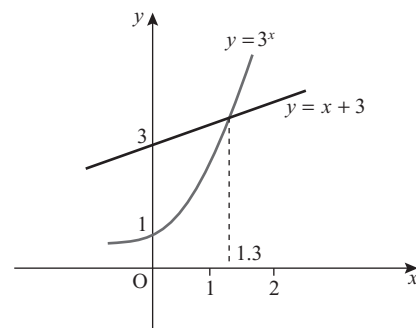
$$\text{line: } y = x + 3$$

$$x = 0 \Rightarrow y = 0 + 3 = 3 \quad \text{Point } (0, 3)$$

$$x = 1 \Rightarrow y = 1 + 3 = 4 \quad \text{Point } (1, 4)$$

$$x = 2 \Rightarrow y = 2 + 3 = 5 \quad \text{Point } (2, 5)$$

$$\Rightarrow \text{Solution for } 3^x = x + 3 \Rightarrow x = 1.3$$

**Q11.**

$$f(x) = \log_2 x$$

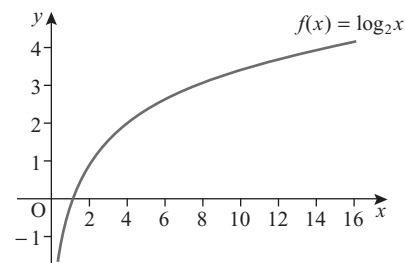
$$x = 1 \Rightarrow f(1) = \log_2 1 = 0, \text{ Point } (1, 0)$$

$$x = 2 \Rightarrow f(2) = \log_2 2 = 1, \text{ Point } (2, 1)$$

$$x = 4 \Rightarrow f(4) = \log_2 4 = 2, \text{ Point } (4, 2)$$

$$x = 8 \Rightarrow f(8) = \log_2 8 = 3, \text{ Point } (8, 3)$$

$$x = 16 \Rightarrow f(16) = \log_2 16 = 4, \text{ Point } (16, 4)$$

**Q12.**

$$y = a \log_2 x + b$$

$$\text{Point } (8, 10) \Rightarrow 10 = a \log_2 8 + b = a(3) + b = 3a + b$$

$$\text{Point } (32, 14) \Rightarrow 14 = a \log_2 32 + b = a(5) + b = 5a + b$$

$$\text{hence } 5a + b = 14$$

$$\text{and } \underline{-3a - b = -10}$$

$$2a = 4 \Rightarrow a = 2$$

$$3a + b = 10 \Rightarrow 3(2) + b = 10$$

$$\Rightarrow 6 + b = 10 \Rightarrow b = 4$$

Q13.

$$y = \log_a x$$

$$\text{Point } (3, 1) \Rightarrow 1 = \log_a 3$$

$$\Rightarrow a^1 = 3 \Rightarrow a = 3$$

Q14.

$$(c): f(x) = \log_3(x - 3)$$

$$\text{Test } (4, 0) \Rightarrow f(4) = \log_3(4 - 3) = \log_3 1 = 0 \text{ True}$$

$$\text{Test } (6, 1) \Rightarrow f(6) = \log_3(6 - 3) = \log_3 3 = 1 \text{ True}$$

Q15.

$$\textcircled{B}: y = \log_5(x - 2)$$

$$\text{Test } (3, 0) \Rightarrow \log_5(3 - 2) = \log_5 1 = 0 \text{ True}$$

$$\text{Test } (7, 1) \Rightarrow \log_5(7 - 2) = \log_5 5 = 1 \text{ True}$$

Q16.

$$y = \log_3(x - 4)$$

$$\text{Point } (q, 2) \Rightarrow \log_3(q - 4) = 2 \Rightarrow q - 4 = 3^2$$

$$\Rightarrow q - 4 = 9 \Rightarrow q = 13$$

Q17. $T = T_0 e^{\frac{t}{20}}$

(i) $t = 10 \Rightarrow 165 = T_0 e^{\frac{10}{20}}$
 $\Rightarrow T_0 e^{\frac{1}{2}} = 165$
 $\Rightarrow T_0 = \frac{165}{e^{\frac{1}{2}}} = \frac{165}{1.6487} = 100.07 = 100$

(ii) $T = 100 e^{\frac{t}{20}}$
 $t = 24 \Rightarrow T = 100 e^{\frac{24}{20}} = 100 e^{1.2} = 100(3.32) = 332^\circ \text{C}$

Q18. $A_t = A_o e^{-0.002t}$

(i) $600 = A_o e^{-0.002(1000)}$
 $\Rightarrow 600 = A_o e^{-2}$
 $\Rightarrow A_o = \frac{600}{e^{-2}} = 4433.43 = 4433$

(ii) $2216.5 = 4433 e^{-0.002t}$
 $\Rightarrow \frac{2216.5}{4433} = e^{-0.002t}$
 $\Rightarrow 0.5 = e^{-0.002t}$
 $\Rightarrow \ln(0.5) = \ln e^{-0.002t}$
 $\Rightarrow -0.693147 = -0.002t(\ln e) = -0.002t$
 $\Rightarrow t = \frac{0.693147}{0.002} = 346.57 = 347 \text{ years}$

Q19. $N = 200 - A e^{-\frac{t}{20}}$

(i) $t = 10 \text{ years} \Rightarrow 91 = 200 - A e^{-\frac{10}{20}}$
 $\Rightarrow A e^{-0.5} = 109$
 $\Rightarrow A = \frac{109}{e^{-0.5}} = 179.7 = 180$

(ii) $t = 0 \Rightarrow N = 200 - 180 e^{-\frac{0}{20}} = 200 - 180 e^0 = 200 - 180 = 20$

(iii) $N = 200 - 180 e^{-\frac{t}{20}}$
 \Rightarrow as time increases, x -axis ($y = 0$) is an asymptote for $e^{-\frac{t}{20}}$
 $\Rightarrow N = 200 - 180(0) = 200$

Q20. $m = m_0 e^{-kt}$

$$m = \frac{9}{10} m_0 \text{ when } t = 10 \Rightarrow 0.9m_0 = m_0 e^{-k(10)}$$

$$\Rightarrow 0.9 = e^{-10k}$$

$$\Rightarrow \ln(0.9) = \ln e^{-10k}$$

$$\Rightarrow -0.10536 = -10k(\ln e) = -10k$$

$$\Rightarrow 10k = 0.10536$$

$$\Rightarrow k = \frac{0.10536}{10} = 0.010536 = 0.0105$$

$$m = \frac{1}{2} m_0 \Rightarrow 0.5m_0 = m_0 e^{-0.0105t}$$

$$\Rightarrow 0.5 = e^{-0.0105t}$$

$$\Rightarrow \ln(0.5) = \ln e^{-0.0105t}$$

$$\Rightarrow -0.693147 = -0.0105t(\ln e)$$

$$\Rightarrow 0.0105t = 0.693147$$

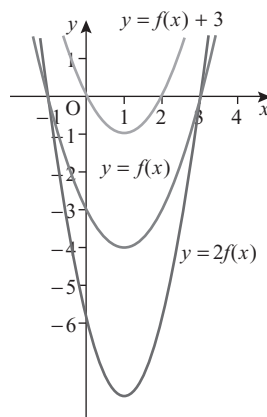
$$\Rightarrow t = \frac{0.693147}{0.0105} = 66 \text{ years}$$

Exercise 1.8

Q1. $y = f(x)$ [Shown]

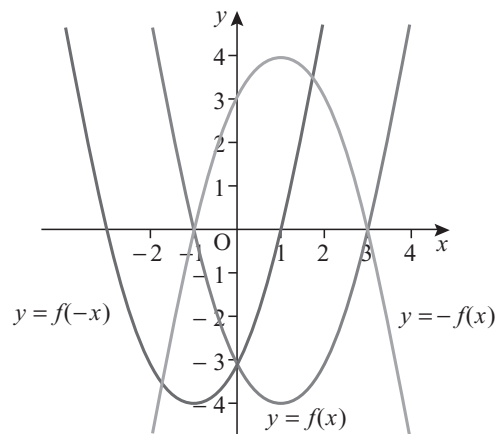
(i) $y = f(x) + 3$ [Shown]

(ii) $y = 2f(x)$ [Shown]



(iii) $y = -f(x)$ [Shown below]

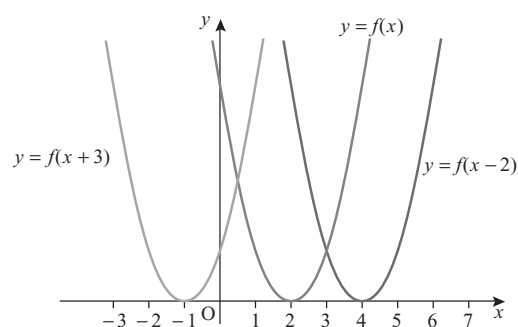
(iv) $y = f(-x)$ [Shown below]



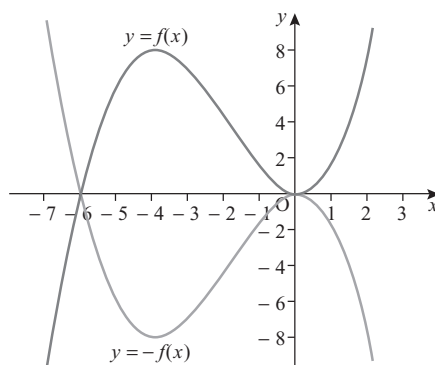
Q2. $g(x) = -f(x)$

$h(x) = f(x) + 3$

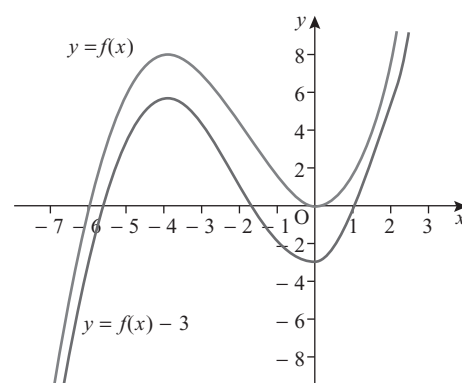
- Q3.** $y = f(x)$ [Shown]
 (i) $y = f(x + 3)$ [Shown]
 (ii) $y = f(x - 2)$ [Shown]



- Q5.** (i) $y = f(x)$ [Shown]
 $y = -f(x)$ [Shown]



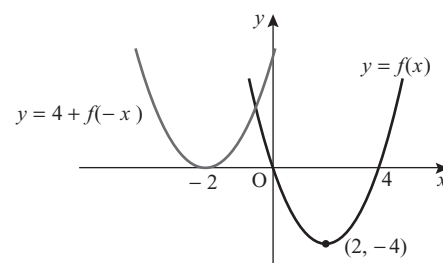
- (ii) $y = f(x)$ [Shown]
 $y = f(x) - 3$ [Shown]



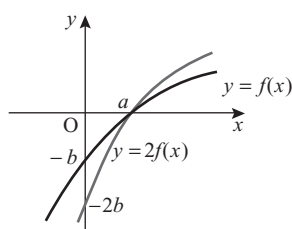
Q6. Graph Ⓒ

Q7. Graph Ⓐ

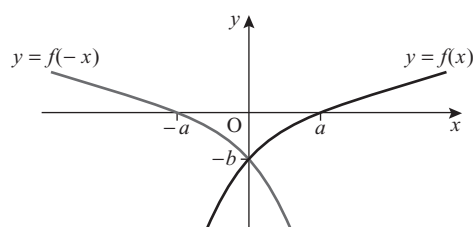
- Q8.** $y = f(x)$ [Shown]
 $y = 4 + f(-x)$ [Shown]



- Q9.** (i) $y = f(x)$ [Shown]
 $y = 2f(x)$ [Shown]



- (ii) $y = f(x)$ [Shown]
 $y = f(-x)$ [Shown]



Q10. (i) $f(x) = (x-2)(x^2+1)$

(a) $x\text{-axis} \Rightarrow f(x) = 0 \Rightarrow (x-2)(x^2+1) = 0$
 $\Rightarrow x = 2 \text{ only} \Rightarrow \text{point } (2, 0)$

(b) $y\text{-axis} \Rightarrow x = 0 \Rightarrow f(0) = (0-2)[(0)^2+1] = (-2)(1) = -2$
 $\Rightarrow \text{point } (0, -2)$

(ii) Graph 

Revision Exercise 1 (Core)

Q1. $f(x) = 2x - 3$ and $g(x) = x^2$
 $\Rightarrow g(f(x)) = g(2x-3) = (2x-3)^2$
 Solve $g(f(x)) = 9$.

$$\begin{aligned} \Rightarrow (2x-3)^2 &= 9 \\ \Rightarrow 4x^2 - 12x + 9 - 9 &= 0 \\ \Rightarrow 4x^2 - 12x &= 0 \\ \Rightarrow x^2 - 3x &= 0 \\ \Rightarrow x(x-3) &= 0 \Rightarrow x = 0 \text{ OR } x = 3 \end{aligned}$$

Q2. (A) and $y = x^2 - 2$ because (i) Curve for $+x^2$ is " \cup -shaped"
 (ii) point on y -axis is $(0, -2)$

(B) and $y = 2 - x^2$ because (i) Curve for $-x^2$ is " \cap -shaped"
 (ii) point on y -axis is $(0, 2)$

(C) and $y = 2x$ because it's a linear graph

Q3. $f(x) = \frac{x}{x+1}$
 $f(1) = \frac{1}{1+1} = \frac{1}{2}$
 $f(2) = \frac{2}{2+1} = \frac{2}{3}$
 $f(3) = \frac{3}{3+1} = \frac{3}{4}$
 $f(4) = \frac{4}{4+1} = \frac{4}{5}$
 $f(5) = \frac{5}{5+1} = \frac{5}{6}$
 $\Rightarrow \text{Range} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \right\}$

Q4. Curve meets x -axis at 0 and -7

$$\Rightarrow y = kx(x+7)$$

$$\text{Point } (4, 4) \Rightarrow 4 = k \cdot 4(4+7)$$

$$\Rightarrow 1 = 11k \Rightarrow k = \frac{1}{11}$$

$$\Rightarrow y = \frac{1}{11}x(x+7) = \frac{x}{11}(x+7)$$

Q5. $g(x) = y = 5 + \frac{x}{2}$

$$\Rightarrow 2y = 10 + x \Rightarrow x = 2y - 10$$

$$\text{hence, } g^{-1}(x) = 2x - 10$$

(i) $g^{-1}(-2) = 2(-2) - 10 = -4 - 10 = -14$

(ii) Solve $g(x) = g^{-1}(x)$.

$$\Rightarrow 5 + \frac{x}{2} = 2x - 10$$

$$\Rightarrow 10 + x = 4x - 20$$

$$\Rightarrow -3x = -30 \Rightarrow x = 10$$

Q6. $y = a^x$

(i) C is on the y -axis $\Rightarrow x = 0 \Rightarrow y = a^0 = 1 \Rightarrow \text{Point C} = (0, 1)$

(ii) B $(2, 16) \Rightarrow 16 = a^2 \Rightarrow a = \sqrt{16} = 4$

Q7. (i) $x = -2, 1, 3$

(ii) $x = 1.4$ or $x = 2.8$

(iii) $x^3 - 2x^2 - 5x = 0 \Rightarrow x^2 - 2x - 5x + 6 = 0 + 6 = 6$

$$\Rightarrow y = 6 \Rightarrow x = -1.5 \text{ or } x = 0$$

No: a horizontal line will cut the graph at more than one point;

Surjective: any horizontal line will cut the graph at least once.

Q8. $y = 2m^x$

(i) Point $(3, 54) \Rightarrow 54 = 2.3^3$

$$\Rightarrow m^3 = 27 \Rightarrow m = (27)^{\frac{1}{3}} = 3$$

(ii) $y = 2.3^x$

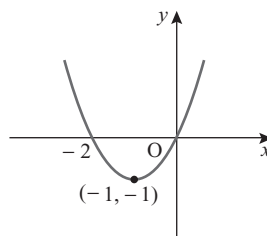
P is on the y -axis $\Rightarrow x = 0 \Rightarrow y = 2.3^0 = 2.1 = 2 \Rightarrow \text{P}(0, 2)$

Q9. (i) $\lim_{x \rightarrow 0} \frac{5x-4}{3+x} = \frac{5(0)-4}{3+0} = \frac{-4}{3}$

(ii) $\lim_{x \rightarrow 1} \frac{x^2-3x+2}{x-1} = \lim_{x \rightarrow 1} \frac{(x-1)(x-2)}{x-1} = \lim_{x \rightarrow 1} (x-2) = 1-2 = -1$

(iii) $\lim_{x \rightarrow 4} \frac{x^3-64}{x^2-16} = \lim_{x \rightarrow 4} \frac{(x-4)(x^2+4x+16)}{(x-4)(x+4)}$
 $= \lim_{x \rightarrow 4} \frac{x^2+4x+16}{x+4} = \frac{(4)^2+4(4)+16}{4+4} = \frac{48}{8} = 6$

Q10. $x^2 + 2x = x^2 + 2x + 1 - 1$
 $= (x+1)^2 - 1$
 \Rightarrow Turning point $= (-1, -1)$



- Q11.** (i) (a) Domain $= \mathbb{R}$; Range is $y \geq -2$
 (b) Domain is $x \leq 2$; Range is \mathbb{R}
 (c) Domain is $-4 \leq x \leq 0$; Range is $0 \leq y \leq 4$

(ii) (a) is the only function; a vertical line will intersect graphs (b) and (c) more than once.

Q12. A and $y = \left(\frac{1}{2}\right)^x$; B and $y = 3^{-x}$ OR $\left(\frac{1}{3}\right)^x$;
 C and $y = 5^x$; D and $y = 2^x$

- Q13.** (i) $x = 2$
 (ii) $x = 3$
 (iii) $x < 4$

Q14. $f(x) = 10x$ and $g(x) = x + 3$

(i) $fg(x) = f(x+3) = 10(x+3) = 10x + 30$
 $y = 10x + 30 \Rightarrow 10x = y - 30$
 $\Rightarrow x = \frac{y-30}{10}$

hence, $(fg)^{-1}(x) = \frac{x-30}{10}$

(ii) $fg(a) = b \Rightarrow fg(a) = 10a + 30 = b$

$(fg)^{-1}(b) = (fg)^{-1}(10a + 30) = \frac{10a + 30 - 30}{10} = \frac{10a}{10} = a$

Q15. $f(x) = x^2 + 3$ and $g(x) = x + 4$

(a) $fg(x) = f(x+4) = (x+4)^2 + 3 = x^2 + 8x + 16 + 3 = x^2 + 8x + 19$

$gf(x) = g(x^2 + 3) = x^2 + 3 + 4 = x^2 + 7$

(b) $fg(x) + gf(x) = 0 \Rightarrow x^2 + 8x + 19 + x^2 + 7 = 0$

$\Rightarrow 2x^2 + 8x + 26 = 0$

$\Rightarrow x^2 + 4x + 13 = 0$

No real roots if $b^2 - 4ac < 0$

$\Rightarrow (4)^2 - 4(1)(13) = 16 - 52 = -36 < 0 \dots \text{True}$

Q16. (i) Range of f is $y \geq 0$

(ii) $y = \frac{1}{2x-3} \Rightarrow 2xy - 3y = 1$

$\Rightarrow 2xy = 3y + 1$

$\Rightarrow x = \frac{3y+1}{2y}$

hence, $g^{-1}(x) = \frac{3x+1}{2x}$

(iii) Domain of g : $x \in R, x \neq \frac{3}{2} \Rightarrow \text{range of } g^{-1}: x \in R, x \neq \frac{3}{2}$

(iv) $fg(x) = f\left[\frac{1}{2x-3}\right] = \frac{1}{(2x-3)^2} = \frac{1}{4x^2 - 12x + 9}$

$fg(x) = 9 \Rightarrow \frac{1}{4x^2 - 12x + 9} = 9$

$\Rightarrow 36x^2 - 108x + 81 = 1$

$\Rightarrow 36x^2 - 108x + 80 = 0$

$\Rightarrow 9x^2 - 27x + 20 = 0$

$\Rightarrow (3x-5)(3x-4) = 0$

$\Rightarrow 3x = 5 \text{ OR } 3x = 4$

$\Rightarrow x = \frac{5}{3} \text{ OR } x = \frac{4}{3}$

Revision Exercise 1 (Advanced)

Q1. $f(x) = x - 1, g(x) = 2x^2 - x - 1$ and $h(x) = \log_3 x$

(i) $hf(x) = h(x-1) = \log_3(x-1)$

$hg(x) = h(2x^2 - x - 1) = \log_3(2x^2 - x - 1)$

(ii) Solve $hg(x) - hf(x) = 2$.

$$\Rightarrow \log_3(2x^2 - x - 1) - \log_3(x - 1) = 2$$

$$\Rightarrow \log_3 \frac{2x^2 - x - 1}{x - 1} = 2$$

$$\Rightarrow \frac{2x^2 - x - 1}{x - 1} = 3^2 = 9$$

$$\Rightarrow 2x^2 - x - 1 = 9x - 9$$

$$\Rightarrow 2x^2 - 10x + 8 = 0$$

$$\Rightarrow x^2 - 5x + 4 = 0$$

$$\Rightarrow (x - 1)(x - 4) = 0 \Rightarrow x = 1 \text{ OR } x = 4$$

Q2. (a) and graph $\textcircled{C} \Rightarrow x\text{-axis } (-4, 0), (8, 0)$

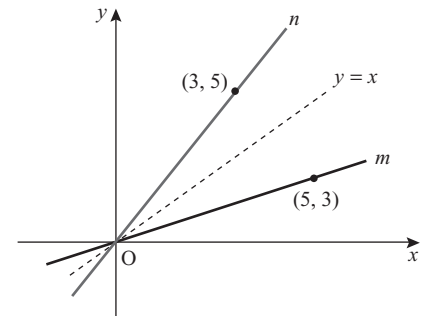
(b) and graph $\textcircled{B} \Rightarrow y\text{-axis } (0, 2)$

(c) and graph $\textcircled{D} \Rightarrow \text{minimum point } (1, -10)$

(d) and graph $\textcircled{A} \Rightarrow x\text{-axis } (-3, 0) \text{ and } (3, 0)$

Q3. (i) Graph of line m : points $(0, 0), (5, 3)$

Graph of line n ; the inverse of m : points $(0, 0), (3, 5)$



(ii) $f(x) = +\sqrt{16 - x^2}$

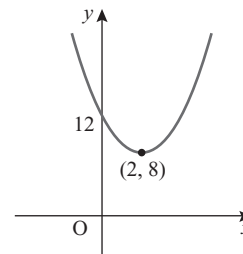
Domain of $f : -4 \leq x \leq 4$

Range of $f : 0 \leq y \leq 4$

Q4. $x^2 - 4x + 12 = x^2 - 4x + 4 + 8$

$$= (x - 2)^2 + 8$$

Turning point = $(2, 8)$



Q5. x cm cut from each corner

$\Rightarrow \text{Length} = (24 - 2x) \text{ cm}, \text{ Width} = (18 - 2x) \text{ cm}, \text{ height} = x \text{ cm}$

$$\text{Volume } V = (24 - 2x)(18 - 2x)(x)$$

$$18 - 2x > 0$$

$$\Rightarrow -2x > -18 \Rightarrow 2x < 18 \Rightarrow x < 9$$

$$\Rightarrow \text{domain: } 0 < x < 9$$

Q6. (i) $y = a.b^x$

Point (0,2) $\Rightarrow 2 = a.b^0 = a.1 = a \Rightarrow a = 2$

$y = 2.b^x$

Point (3, 54) $\Rightarrow 54 = 2.b^3$

$$\Rightarrow b^3 = 27 \Rightarrow b = (27)^{\frac{1}{3}} = 3$$

(ii) (a) $x = 1$

(b) Domain: $x \in \mathbb{R} \setminus \{1\}$

(c) Range = $\mathbb{R} \setminus \{0\}$

(d) Yes; a horizontal line will intersect the graph at most once.

Q7. (i) Area of canvas = $2x^2 + \ell x = 9$

$$\Rightarrow \ell x = 9 - 2x^2$$

$$\Rightarrow \ell = \frac{9 - 2x^2}{x} = \frac{9}{x} - \frac{2x^2}{x} = \left(\frac{9}{x} - 2x \right) \text{cm}$$

$$\text{Volume } V = \left(\frac{9}{x} - 2x \right) (x)(x) = 9x - 2x^3$$

(ii) On x -axis, $y = 0 \Rightarrow 9x - 2x^3 = 0$

$$\Rightarrow x(9 - 2x^2) = 0$$

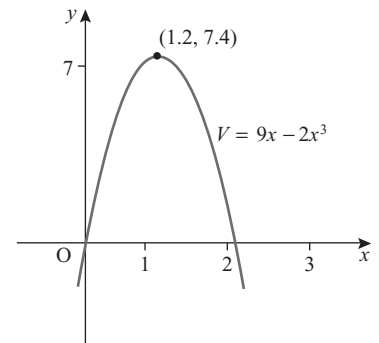
$$\Rightarrow x = 0, x^2 = 4.5$$

$$\Rightarrow x = \sqrt{4.5} = 2.1$$

$$x = 1 \Rightarrow y = 9(1) - 2(1)^3 = 7$$

$$x = 2 \Rightarrow y = 9(2) - 2(2)^3 = 2$$

Points are (0,0), (1,7), (2,2), (2.1,0)



(iii) (a) $x = 1.2$

(b) $y = 7.4 \text{ m}^3 = \text{largest volume.}$

Q8. $f(x) = 3x - 1$ and $g(x) = x^2 + 1$

(a) Range of g ; $y \geq 1$

(b) $gf(x) = g(3x - 1) = (3x - 1)^2 + 1 = 9x^2 - 6x + 2$

$fg(x) = f(x^2 + 1) = 3(x^2 + 1) - 1 = 3x^2 + 2$

Solve $gf(x) = fg(x)$.

$$\Rightarrow 9x^2 - 6x + 2 = 3x^2 + 2$$

$$\Rightarrow 6x^2 - 6x = 0$$

$$\Rightarrow x^2 - x = 0$$

$$\Rightarrow x(x - 1) = 0 \Rightarrow x = 0 \text{ OR } x = 1$$

(c) $|f(x)| = 8$

$$|3x - 1| = 8$$

$$\Rightarrow +(3x - 1) = 8 \text{ OR } -(3x - 1) = 8$$

$$\Rightarrow 3x - 1 = 8 \quad \Rightarrow -3x + 1 = 8$$

$$\Rightarrow 3x = 9 \quad \Rightarrow -3x = 7$$

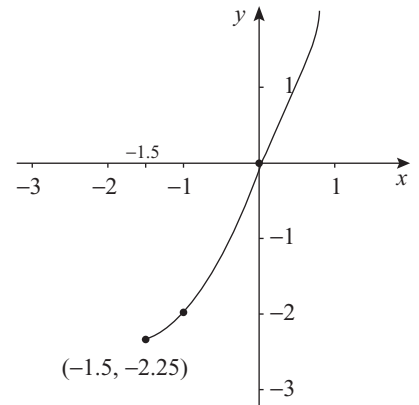
$$\Rightarrow x = 3 \quad \Rightarrow x = -\frac{7}{3}$$

(d) $h(x) = x^2 + 3x$

one-to-one \Rightarrow minimum point occurs

$$\begin{aligned} \text{at } x = -1.5 \Rightarrow h(-1.5) &= (-1.5)^2 + 3(-1.5) \\ &= 2.25 - 4.5 = -2.25 \end{aligned}$$

$$\text{hence, } q = -1.5 = -\frac{3}{2}$$



Q9. (i) $y = k(x - 1)^2(x + t) = k(x - 1)^2(x - 5) \Rightarrow t = -5$

Point $(0, 10) \Rightarrow 10 = k(0 - 1)^2(0 - 5)$

$$\Rightarrow 10 = -5k \Rightarrow k = -2$$

(ii) $\lim_{x \rightarrow 3} \frac{x^2 + 2x - 15}{x^2 - 9} = \lim_{x \rightarrow 3} \frac{(x + 5)(x - 3)}{(x + 3)(x - 3)}$

$$= \lim_{x \rightarrow 3} \frac{x + 5}{x + 3} = \frac{8}{6} = \frac{4}{3}$$

Q10. (i) Suitable domain; $-5 \leq x \leq 5$

Corresponding range; $[0,5]$

(ii) (a) Curve $y = 3^x$ cuts the

y -axis at $x = 0 \Rightarrow y = 3^0 = 1$

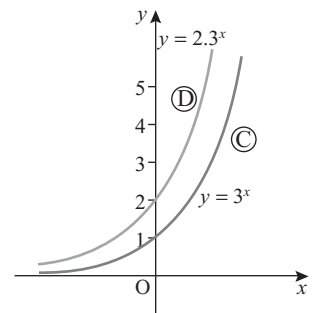
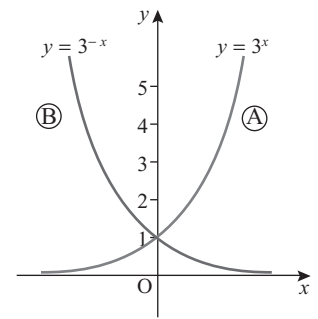
\Rightarrow point = $(0, 1)$

(b) Graph Ⓐ $\Rightarrow y = 3^x$

Graph Ⓑ $\Rightarrow y = 3^{-x}$

Graph Ⓒ $\Rightarrow y = 3^x$

Graph Ⓓ $\Rightarrow y = 2.3^x$



Q11. $P = Ae^{\frac{t}{20}} \Rightarrow$ when $t = 0 \Rightarrow P = A.e^0 = A$

From table $\Rightarrow A = 5$ when $t = 0$

$$P = 5e^{\frac{t}{20}} \Rightarrow t = 5 \Rightarrow P = 5.e^{\frac{5}{20}} = 5e^{0.25} = 6.4$$

$$t = 10 \Rightarrow P = 5.e^{\frac{10}{20}} = 8.2$$

$$t = 15 \Rightarrow P = 5.e^{\frac{15}{20}} = 10.6$$

$$t = 20 \Rightarrow P = 5.e^{\frac{20}{20}} = 13.6$$

t	0	5	10	15	20
P	5	6.4	8.2	10.6	13.6

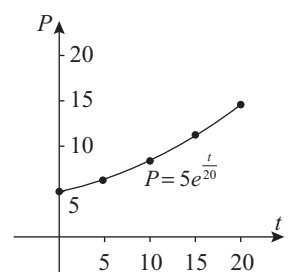
$$10 = 5e^{\frac{t}{20}}$$

$$\Rightarrow 2 = e^{\frac{t}{20}}$$

$$\Rightarrow \ln(2) = \ln e^{\frac{t}{20}}$$

$$\Rightarrow 0.693 = \frac{t}{20}(\ln e) = \frac{t}{20}$$

$$\Rightarrow t = 20(0.693) = 13.86 = 13.9 \text{ days}$$



Q12. (i) $x^2 - 7x + 12 > 0 \Rightarrow (x - 4)(x - 3) > 0 \Rightarrow x \geq 4 \text{ or } x \leq 3$

(ii) (a) A vertical line will intersect the graph once only.

(b) NO

(c) YES

(d) Injective if $\frac{\pi}{2} \leq x \leq 3\frac{\pi}{2}$

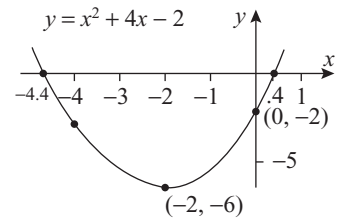
Q13. Graph Ⓒ Check $(1,3)$ in $\log_3 y = x \Rightarrow \log_3 3 = 1 = x \Rightarrow \text{True}$

Revision Exercise 1 (Extended–Response)

Q1. (a) $x^2 + 4x - 2 = x^2 + 4x + 4 - 2 - 4$

$$= (x + 2)^2 - 6 = (x + a)^2 + b$$

hence, $a = 2$, $b = -6$
 \Rightarrow Turning point $= (-2, -6)$



(b) On y -axis, $x = 0 \Rightarrow y = (0)^2 - 4(0) - 2 = -2$
Point $(0, -2)$

On x -axis, $y = 0 \Rightarrow x^2 + 4x - 2 = 0$

$$\Rightarrow x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{24}}{2} = -2 \pm \sqrt{6}$$

$$-2 - \sqrt{6} = -4.4 \text{ and } -2 + \sqrt{6} = 0.4$$

\Rightarrow Points on x -axis: $(-4.4, 0)$ and $(0.4, 0)$

(c) Discriminant $= \sqrt{24}$; Since discriminant > 0 , the curve will intersect the x -axis at two distinct points.

(d) $x^2 + 4x + k = 0$ has no real roots $\Rightarrow b^2 - 4ac < 0$

$$\Rightarrow (4)^2 - 4(1)(k) < 0$$

$$\Rightarrow 16 - 4k < 0$$

$$\Rightarrow -4k < -16$$

$$\Rightarrow 4k > 16 \Rightarrow k > 4$$

Q2. (a) Curve meets x -axis at $A(-10, 0), B(10, 0)$

$$\Rightarrow y = a(x + 10)(x - 10) = a(x^2 - 100)$$

$|OZ| = 9 \Rightarrow$ Point $Z = (0, 9)$

$$\Rightarrow 9 = a((0)^2 - 100)$$

$$\Rightarrow 9 = -100a \Rightarrow a = \frac{-9}{100} = -0.09$$

$$\Rightarrow y = -0.09x^2 + 9 \Rightarrow b = 9$$

(b) Point $C = (-7, 0) \Rightarrow y = -0.09(-7)^2 + 9 = 4.59$

$$\Rightarrow |DE| = 4.59 - 1.8 = 2.79$$

(c) $|OH| = 6.3 \Rightarrow y = 6.3$

$$\Rightarrow -0.09x^2 + 9 = 6.3$$

$$\Rightarrow -0.09x^2 = 6.3 - 9 = -2.7$$

$$\Rightarrow x^2 = \frac{2.7}{0.09} = 30$$

$$\Rightarrow x = \sqrt{30}$$

$$\Rightarrow |FG| = 2\sqrt{30} = 2(5.477) = 10.954 = 10.95 \text{ m}$$

Q3. (a) $y = a \log_2(x - b)$

$$\text{Point } (5, 2) \Rightarrow 2 = a \log_2(5 - b)$$

$$\Rightarrow \log_2(5 - b) = \frac{2}{a}$$

$$\Rightarrow 5 - b = 2^{\frac{2}{a}} = (2^2)^{\frac{1}{a}} = 4^{\frac{1}{a}} \Rightarrow b = 5 - 4^{\frac{1}{a}}$$

$$\text{Point } (7, 4) \Rightarrow 4 = a \log_2(7 - b)$$

$$\Rightarrow \log_2(7 - b) = \frac{4}{a}$$

$$\Rightarrow 7 - b = 2^{\frac{4}{a}} = (2^2)^{\frac{2}{a}} = 4^{\frac{2}{a}} = \left(4^{\frac{1}{a}}\right)^2$$

$$\Rightarrow b = 7 - \left(4^{\frac{1}{a}}\right)^2$$

$$\Rightarrow 5 - 4^{\frac{1}{a}} = 7 - \left(4^{\frac{1}{a}}\right)^2$$

$$\Rightarrow \left(4^{\frac{1}{a}}\right)^2 - 4^{\frac{1}{a}} - 2 = 0$$

$$\text{Let } k = 4^{\frac{1}{a}} \Rightarrow k^2 - k - 2 = 0$$

$$\Rightarrow (k - 2)(k + 1) = 0$$

$$\Rightarrow k = 2 \text{ OR } k = -1 \text{ (Not Valid)}$$

$$\Rightarrow 4^{\frac{1}{a}} = 2$$

$$\Rightarrow (2^2)^{\frac{1}{a}} = 2^{\frac{2}{a}} = 2^1$$

$$\Rightarrow \frac{2}{a} = 1 \Rightarrow a = 2$$

$$\Rightarrow y = 2 \log_2(x - b)$$

$$\text{point } (5, 2) \Rightarrow 2 = 2 \log_2(5 - b)$$

$$\Rightarrow \log_2(5 - b) = \frac{2}{2} = 1$$

$$\Rightarrow 5 - b = 2^1 = 2 \Rightarrow b = 3$$

(b) Statement (iii) is not true

Q4. (a) Graph \textcircled{D}

(b) (i) (a) $N_o = 20,000$

(b) Decrease by 20% $\Rightarrow N = 80\%$ of $20,000 = 16,000$

$$t = 1 \Rightarrow 16,000 = 20,000 e^{k(1)}$$

$$\Rightarrow \frac{16,000}{20,000} = 0.8 = e^k$$

$$\Rightarrow \ln 0.8 = \ln e^k = k \ln e = k \cdot 1 = k$$

$$\Rightarrow k = \ln(0.8) = -0.2231 = -0.223$$

(b) (ii) $N = 20,000 e^{-0.223t}$

$$N = 5,000 \Rightarrow 5,000 = 20,000 e^{-0.223t}$$

$$\Rightarrow e^{-0.223t} = \frac{5,000}{20,000} = \frac{1}{4} = 0.25$$

$$\Rightarrow \ln e^{-0.223t} = \ln(0.25)$$

$$\Rightarrow -0.223t(\ln e) = -1.386$$

$$\Rightarrow 0.223t = 1.386$$

$$\Rightarrow t = \frac{1.386}{0.223} = 6.21 = 6.2 \text{ years}$$

Q5.

$$f(x) = x^3 \quad \text{and} \quad g(x) = \frac{1}{x-3}$$

(a) Range of $f = R$

$$(b) (i) f \circ g(x) = f\left(\frac{1}{x-3}\right) = \left(\frac{1}{x-3}\right)^3 = \frac{1}{(x-3)^3}$$

$$(ii) \text{ Solve } f \circ g(x) = 64 \Rightarrow \frac{1}{(x-3)^3} = 64$$

$$\Rightarrow (x-3)^3 = \frac{1}{64}$$

$$\Rightarrow x-3 = \sqrt[3]{\frac{1}{64}} = \frac{1}{4}$$

$$\Rightarrow x = 3 + \frac{1}{4} = \frac{13}{4}$$

$$(c)(i) g(x) = y = \frac{1}{x-3}$$

$$\Rightarrow xy - 3y = 1$$

$$\Rightarrow xy = 1 + 3y$$

$$\Rightarrow x = \frac{1+3y}{y}$$

$$\text{hence, } g^{-1}(x) = \frac{1+3x}{x}$$

(ii) Range of $g^{-1} = \text{domain of } g = R, x \neq 3$

$$\begin{aligned}
 \text{(iii)} \quad g g^{-1}(x) &= g\left(\frac{1+3x}{x}\right) = \frac{1}{\frac{1+3x}{x} - 3} \\
 &= \frac{1}{\frac{1+3x-3x}{x}} = \frac{1}{\frac{1}{x}} = x
 \end{aligned}$$

(iv) Graph of $g^{-1}(x)$ is not continuous at $x = 0$

Q6. $M = Ae^{-pt}$

(i) $A = €130,000$

(ii) $t = 1 \Rightarrow €122,000 = €130,000 e^{-p(1)}$

$$\Rightarrow e^{-p} = \frac{122,000}{130,000} = \frac{61}{65}$$

$$\Rightarrow \ln e^{-p} = \ln \frac{61}{65}$$

$$\Rightarrow -p(\ln e) = -0.0635$$

$$\Rightarrow p = 0.0635 = 0.064$$

(iii) 1 January 2006 to end of 2011 is 6 years

$$\begin{aligned}
 \Rightarrow t = 6 \Rightarrow M &= 130,000 e^{-0.064(6)} \\
 &= 130,000 e^{-0.384} \\
 &= €88,547.085 = €88,500
 \end{aligned}$$

Q7. (a) Graph \textcircled{B} : $y = \log_5(x - 2)$

Check point $(3,0) \Rightarrow \log_5(3 - 2) = \log_5 1 = 0$, True

Check point $(7,1) \Rightarrow \log_5(7 - 2) = \log_5 5 = 1$, True

(b) (i) $y = 4^x \cap y = 3^{2-x}$

$$\Rightarrow 4^x = 3^{2-x}$$

$$\Rightarrow \log_a 4^x = \log_a 3^{2-x}$$

$$\Rightarrow x \log_a 4 = (2 - x) \log_a 3$$

$$\Rightarrow x \log_a 4 = 2 \log_a 3 - x \log_a 3$$

$$\Rightarrow x \log_a 4 + x \log_a 3 = 2 \log_a 3$$

$$\Rightarrow x(\log_a 4 + \log_a 3) = \log_a 3^2$$

$$\Rightarrow x(\log_a 4.3) = \log_a 9 \Rightarrow x = \frac{\log_a 9}{\log_a 12}$$

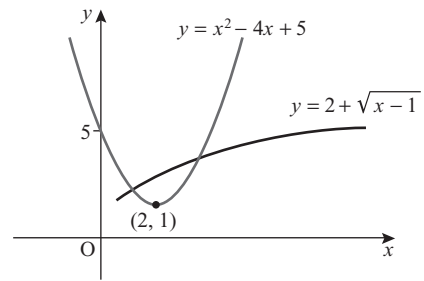
(ii) $x = \frac{\log_a 9}{\log_a 12} = \log_{12} 9 = 0.884228$

$$\Rightarrow y = 4^{0.884228} = 3.40 = 3.4$$

Q8.

$$\begin{aligned}
 \text{(a)} \quad y &= x^2 - 4x + 5 \\
 &= x^2 - 4x + 4 + 1 \\
 &= (x - 2)^2 + 1 \\
 &\Rightarrow \text{Turning point} = (2, 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad (x - 2)^2 + 1 &= y = f(x) \\
 &\Rightarrow (x - 2)^2 = y - 1 \\
 &\Rightarrow x - 2 = \sqrt{y - 1} \\
 &\Rightarrow x = 2 + \sqrt{y - 1} \\
 &\Rightarrow f^{-1}(x) = 2 + \sqrt{x - 1}
 \end{aligned}$$

(c)**Q9.**

$$C = C_0 e^{-kt}$$

$$\text{(i)} \quad C_0 = 5 \text{ kg/ha}$$

$$t = 1 \Rightarrow 2.8 = 5e^{-k(1)}$$

$$\Rightarrow e^{-k} = \frac{2.8}{5} = 0.56$$

$$\Rightarrow \ln e^{-k} = \ln(0.56)$$

$$\Rightarrow -k(\ln e) = -0.57981$$

$$\Rightarrow k = 0.5798$$

$$\text{(ii)} \quad C = 5e^{-0.5798t}$$

$$\Rightarrow 0.2 = 5e^{-0.5798t}$$

$$\Rightarrow e^{-0.5798t} = \frac{0.2}{5} = 0.04$$

$$\Rightarrow \ln e^{-0.5798t} = \ln(0.04)$$

$$\Rightarrow -0.5798t(\ln e) = -3.218876$$

$$\Rightarrow 0.5798t = 3.218876$$

$$\Rightarrow t = \frac{3.218876}{0.5798} = 5.551 = 5.6 \text{ years}$$

Chapter 2: Differential Calculus

Exercise 2.1

- Q1.** (i) $A(2, 4) B(1, 0) \Rightarrow \text{average rate of change} = \frac{0-4}{1-2} = \frac{-4}{-1} = 4$
- (ii) $B(1, 0) C(-1, -2) \Rightarrow \text{average rate of change} = \frac{-2-0}{-1-1} = \frac{-2}{-2} = 1$
- (iii) $C(-1, -2) D(-3, 4) \Rightarrow \text{average rate of change} = \frac{4+2}{-3+1} = \frac{6}{-2} = -3$

Q2. Average rate of change $= \frac{20-4}{5+2} = \frac{16}{7}$

- Q3.** (i) Average rate of change $= \frac{5-30}{2+5} = \frac{-25}{7}$
- (ii) Average rate of change $= \frac{15-3}{3-0} = \frac{12}{3} = 4$

Q4. $t = 0 \Rightarrow d(0) = \frac{-300}{0+6} + 50 = -50 + 50 = 0 \Rightarrow (0, 0)$

$t = 10 \Rightarrow d(10) = \frac{-300}{10+6} + 50 = -18.75 + 50 = 31.25 \Rightarrow (10, 31.25)$

$\Rightarrow \text{average rate of change} = \frac{31.25-0}{10-0} = \frac{31.25}{10} = \frac{25}{8}$

- Q5.** (i) 8 years
- (ii) $(5, 80), (10, 140)$
- $\Rightarrow \text{average rate of change} = \frac{140-80}{10-5} = \frac{60}{5} = 12 \text{ (cm/year)}$

- Q6.** (i) $S(x) = 6x^2$
- (ii) $x = 2 \Rightarrow S(2) = 6(2)^2 = 24 \Rightarrow (2, 24)$
- $x = 5 \Rightarrow S(5) = 6(5)^2 = 150 \Rightarrow (5, 150)$
- $\Rightarrow \text{average rate of change} = \frac{150-24}{5-2} = \frac{126}{3} = 42 \text{ cm}^2/\text{sec}$

- Q7.** (i) $P(4, 8), Q(3, 3) \Rightarrow \text{slope PQ} = \frac{3-8}{3-4} = \frac{-5}{-1} = 5$
- (ii) $P(3.5, 5.25), Q(3, 3) \Rightarrow \text{slope PQ} = \frac{3-5.25}{3-3.5} = \frac{-2.25}{-0.5} = 4.5$
- (iii) $P(3.1, 3.41), Q(3, 3) \Rightarrow \text{slope PQ} = \frac{3-3.41}{3-3.1} = \frac{-0.41}{-0.1} = 4.1$
- (iv) Slope = 4

Exercise 2.2

Q1. (i) $f(x) = 5x$

$$\Rightarrow f(x+h) = 5(x+h) = 5x + 5h$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= 5x + 5h - 5x \\ &= 5h\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{5h}{h} = 5$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} 5 = 5$$

$$\therefore f'(x) = 5$$

(ii) $f(x) = 3x - 4$

$$\Rightarrow f(x+h) = 3(x+h) - 4 = 3x + 3h - 4$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= 3x + 3h - 4 - (3x - 4) \\ &= 3x + 3h - 4 - 3x + 4 \\ &= 3h\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{3h}{h} = 3$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} 3 = 3$$

$$\therefore f'(x) = 3$$

(iii) $f(x) = 6 - 4x$

$$\Rightarrow f(x+h) = 6 - 4(x+h) = 6 - 4x - 4h$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= 6 - 4x - 4h - (6 - 4x) \\ &= 6 - 4x - 4h - 6 + 4x \\ &= -4h\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{-4h}{h} = -4$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} -4 = -4$$

$$\therefore f'(x) = -4$$

Q2. (i) $f(x) = x^2$

$$\Rightarrow f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= x^2 + 2xh + h^2 - x^2 \\ &= 2xh + h^2\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x$$

$$\therefore f'(x) = 2x$$

$$(ii) \quad f(x) = 2x^2 + 9x$$

$$\Rightarrow f(x+h) = 2(x+h)^2 + 9(x+h) = 2x^2 + 4xh + h^2 + 9x + 9h$$

$$\begin{aligned} \Rightarrow f(x+h) - f(x) &= 2x^2 + 4xh + 2h^2 + 9x + 9h - (2x^2 + 9x) \\ &= 2x^2 + 4xh + 2h^2 + 9x + 9h - 2x^2 - 9x \\ &= 4xh + 2h^2 + 9h \end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{4xh + 2h^2 + 9h}{h} = 4x + 2h + 9$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (4x + 2h + 9) = 4x + 9$$

$$\therefore f'(x) = 4x + 9$$

$$(iii) \quad f(x) = 3x^2 - 4x - 6$$

$$\Rightarrow f(x+h) = 3(x+h)^2 - 4(x+h) - 6 = 3x^2 + 6xh + 3h^2 - 4x - 4h - 6$$

$$\begin{aligned} \Rightarrow f(x+h) - f(x) &= 3x^2 + 6xh + 3h^2 - 4x - 4h - 6 - (3x^2 - 4x - 6) \\ &= 3x^2 + 6xh + 3h^2 - 4x - 4h - 6 - 3x^2 + 4x + 6 \\ &= 6xh + 3h^2 - 4h \end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{6xh + 3h^2 - 4h}{h} = 6x + 3h - 4$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (6x + 3h - 4) = 6x - 4$$

$$\therefore f'(x) = 6x - 4$$

Q3. (i) $f(x) = x^2 - 2x + 5$

$$\Rightarrow f(x+h) = (x+h)^2 - 2(x+h) + 5 = x^2 + 2xh + h^2 - 2x - 2h + 5$$

$$\begin{aligned} \Rightarrow f(x+h) - f(x) &= x^2 + 2xh + h^2 - 2x - 2h + 5 - (x^2 - 2x + 5) \\ &= x^2 + 2xh + h^2 - 2x - 2h + 5 - x^2 + 2x - 5 \\ &= 2xh + h^2 - 2h \end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 - 2h}{h} = 2x + h - 2$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (2x + h - 2) = 2x - 2$$

$$\therefore f'(x) = 2x - 2$$

(ii) Point (2,5) \Rightarrow slope $= 2(2) - 2 = 4 - 2 = 2$

(iii) Equation of Tangent: $y - 5 = 2(x - 2)$

$$\Rightarrow y - 5 = 2x - 4$$

$$\Rightarrow 2x - y + 1 = 0$$

Q4.

$$f(x) = kx^2$$

$$\Rightarrow f(x+h) = k(x+h)^2 = kx^2 + 2kxh + kh^2$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= kx^2 + 2kxh + kh^2 - kx^2 \\ &= 2kxh + kh^2\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2kxh + kh^2}{h} = 2kx + kh$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (2kx + kh) = 2kx$$

$$\therefore f'(x) = 2kx$$

Q5. (i) $f(x) = -x^2$

$$\Rightarrow f(x+h) = -(x+h)^2 = -(x^2 + 2xh + h^2) = -x^2 - 2xh - h^2$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= -x^2 - 2xh - h^2 - (-x^2) \\ &= -x^2 - 2xh - h^2 + x^2 = -2xh - h^2\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{-2xh - h^2}{h} = -2x - h$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (-2x - h) = -2x$$

$$\therefore f'(x) = -2x$$

(ii) $f(x) = 4x - x^2$

$$\Rightarrow f(x+h) = 4(x+h) - (x+h)^2 = 4x + 4h - x^2 - 2xh - h^2$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= 4x + 4h - x^2 - 2xh - h^2 - (4x - x^2) \\ &= 4x + 4h - x^2 - 2xh - h^2 - 4x + x^2 \\ &= 4h - 2xh - h^2\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{4h - 2xh - h^2}{h} = 4 - 2x - h$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (4 - 2x - h) = 4 - 2x$$

$$\therefore f'(x) = 4 - 2x$$

(iii) $f(x) = 2 - x - 3x^2$

$$\Rightarrow f(x+h) = 2 - (x+h) - 3(x+h)^2 = 2 - x - h - 3x^2 - 6xh - 3h^2$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= 2 - x - h - 3x^2 - 6xh - 3h^2 - (2 - x - 3x^2) \\ &= 2 - x - h - 3x^2 - 6xh - 3h^2 - 2 + x + 3x^2 \\ &= -h - 6xh - 3h^2\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{-h - 6xh - 3h^2}{h} = -1 - 6x - 3h$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (-1 - 6x - 3h) = -1 - 6x$$

$$\therefore f'(x) = -1 - 6x$$

Q6.

$$f(x) = 2x^2 - 3x - 2$$

$$\Rightarrow f(x+h) = 2(x+h)^2 - 3(x+h) - 2 = 2x^2 + 4xh + 2h^2 - 3x - 3h - 2$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= 2x^2 + 4xh + 2h^2 - 3x - 3h - 2 - (2x^2 - 3x - 2) \\ &= 2x^2 + 4xh + 2h^2 - 3x - 3h - 2 - 2x^2 + 3x + 2 \\ &= 4xh + 2h^2 - 3h\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{4xh + 2h^2 - 3h}{h} = 4x + 2h - 3$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (4x + 2h - 3) = 4x - 3$$

$$\therefore f'(x) = 4x - 3$$

(i) Point (3, 7) \Rightarrow slope $= 4(3) - 3 = 12 - 3 = 9$

(ii) Equation of Tangent: $y - 7 = 9(x - 3)$

$$\Rightarrow y - 7 = 9x - 27$$

$$\Rightarrow 9x - y - 20 = 0$$

Q7.

$$A = f(r) = \pi r^2$$

$$\Rightarrow f(r+h) = \pi(r+h)^2 = \pi(r^2 + 2rh + h^2) = \pi r^2 + 2\pi rh + \pi h^2$$

$$\begin{aligned}\Rightarrow f(r+h) - f(r) &= \pi r^2 + 2\pi rh + \pi h^2 - \pi r^2 \\ &= 2\pi rh + \pi h^2\end{aligned}$$

$$\Rightarrow \frac{f(r+h) - f(r)}{h} = \frac{2\pi rh + \pi h^2}{h} = 2\pi r + \pi h$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(r+h) - f(r)}{h} = \lim_{h \rightarrow 0} (2\pi r + \pi h) = 2\pi r$$

$$\therefore f'(r) = \frac{dA}{dr} = 2\pi r$$

Q8.

$$f(x) = x^2 - 3x + 1$$

$$\Rightarrow f(x+h) = (x+h)^2 - 3(x+h) + 1 = x^2 + 2xh + h^2 - 3x - 3h + 1$$

$$\begin{aligned}\Rightarrow f(x+h) - f(x) &= x^2 + 2xh + h^2 - 3x - 3h + 1 - (x^2 - 3x + 1) \\ &= x^2 + 2xh + h^2 - 3x - 3h + 1 - x^2 + 3x - 1 \\ &= 2xh + h^2 - 3h\end{aligned}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 - 3h}{h} = 2x + h - 3$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (2x + h - 3) = 2x - 3$$

$$\therefore f'(x) = 2x - 3$$

Slope of tangent is zero $\Rightarrow 2x - 3 = 0$

$$\Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2} = 1\frac{1}{2}$$

$$\Rightarrow f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 1$$

$$= \frac{9}{4} - \frac{9}{2} + 1 = -1\frac{1}{4} \Rightarrow \text{Point} = \left(1\frac{1}{2}, -1\frac{1}{4}\right)$$

Exercise 2.3

Q1. (i) $y = 5x^5 \Rightarrow \frac{dy}{dx} = 25x^4$

(ii) $y = 5x^2 - 4x \Rightarrow \frac{dy}{dx} = 10x - 4$

(iii) $y = 6x^2 + 5x - 4 \Rightarrow \frac{dy}{dx} = 12x + 5$

(iv) $y = x^3 - 8x + 2 \Rightarrow \frac{dy}{dx} = 3x^2 - 8$

(v) $y = x^2 + 2x + \frac{1}{x} = x^2 + 2x + x^{-1}$
 $\frac{dy}{dx} = 2x + 2 - 1x^{-2} = 2x + 2 - \frac{1}{x^2}$

(vi) $y = 2x^3 + x^2 + \frac{1}{x^2} = 2x^3 + x^2 + x^{-2}$
 $\frac{dy}{dx} = 6x^2 + 2x - 2x^{-3} = 6x^2 + 2x - \frac{2}{x^3}$

Q2. (i) $f(x) = 7x^2 - \frac{3}{x} = 7x^2 - 3x^{-1}$
 $\Rightarrow f'(x) = 14x + 3x^{-2} = 14x + \frac{3}{x^2}$

(ii) $f(x) = 3\sqrt{x} = 3x^{\frac{1}{2}}$
 $\Rightarrow f'(x) = 3 \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{3}{2\sqrt{x}}$

(iii) $f(x) = 2\sqrt{x} + \frac{2}{x^2} = 2x^{\frac{1}{2}} + 2x^{-2}$
 $\Rightarrow f'(x) = 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} - 4x^{-3} = \frac{1}{\sqrt{x}} - \frac{4}{x^3}$

(iv) $f(x) = x^2 - 5\sqrt{x} = x^2 - 5x^{\frac{1}{2}}$
 $\Rightarrow f'(x) = 2x - \frac{5}{2} x^{-\frac{1}{2}} = 2x - \frac{5}{2\sqrt{x}}$

(v) $f(x) = \frac{3}{\sqrt{x}} = 3x^{-\frac{1}{2}}$
 $\Rightarrow f'(x) = -\frac{3}{2} x^{-\frac{3}{2}} = \frac{-3}{2x^{\frac{3}{2}}} = \frac{-3}{2\sqrt{x^3}}$

(vi) $f(x) = 3x^{-2} + \frac{1}{2\sqrt{x}} = 3x^{-2} + \frac{1}{2} x^{-\frac{1}{2}}$
 $\Rightarrow f'(x) = -6x^{-3} - \frac{1}{4} x^{-\frac{3}{2}} = \frac{-6}{x^3} - \frac{1}{4\sqrt{x^3}}$

Q3. $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x$
 $\frac{dy}{dx} = \frac{1}{3} \cdot 3x^2 + \frac{1}{2} \cdot 2x - 6 = x^2 + x - 6$

Q4. (i) $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$
 $\Rightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}$

(ii) $f(x) = 3\sqrt{x} - \frac{1}{x^2} = 3x^{\frac{1}{2}} - x^{-2}$
 $\Rightarrow f'(x) = \frac{3}{2}x^{-\frac{1}{2}} + 2x^{-3} = \frac{3}{2\sqrt{x}} + \frac{2}{x^3}$

(iii) $f(x) = \frac{4}{x} + \frac{3}{\sqrt{x}} = 4x^{-1} + 3x^{-\frac{1}{2}}$
 $\Rightarrow f'(x) = -4x^{-2} - \frac{3}{2}x^{-\frac{3}{2}} = -\frac{4}{x^2} - \frac{3}{2\sqrt{x^3}}$

(iv) $f(x) = 6 - \frac{3}{x} = 6 - 3x^{-1}$
 $\Rightarrow f'(x) = 0 + 3x^{-2} = \frac{3}{x^2}$

(v) $f(x) = 2\sqrt{x} + \sqrt[3]{x} = 2x^{\frac{1}{2}} + x^{\frac{1}{3}}$
 $\Rightarrow f'(x) = 1x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}$

(vi) $f(x) = x^2 + 3 - \frac{4}{x^{-2}} = x^2 + 3 - 4x^2 = 3 - 3x^2$
 $\Rightarrow f'(x) = 0 - 6x = -6x$

Q5. $y = \sqrt{x}(1 + \sqrt{x}) = \sqrt{x} + x = x^{\frac{1}{2}} + x$
 $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + 1 = \frac{1}{2\sqrt{x}} + 1$
 When $x = 4 \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{4}} + 1 = \frac{1}{4} + 1 = 1\frac{1}{4}$

Q6. $f(x) = x^3 + 2\sqrt{x} = x^3 + 2x^{\frac{1}{2}}$
 $\Rightarrow f'(x) = 3x^2 + 1 \cdot x^{-\frac{1}{2}} = 3x^2 + \frac{1}{\sqrt{x}}$
 $\Rightarrow f'(4) = 3(4)^2 + \frac{1}{\sqrt{4}} = 48 + \frac{1}{2} = 48\frac{1}{2}$

Q7. $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$
 $\Rightarrow f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$
 $\Rightarrow f'(4) = -\frac{1}{2\sqrt{(4)^3}} = -\frac{1}{16}$

Q8. $y = x^{\frac{5}{2}} \Rightarrow \frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} = \frac{5}{2}\sqrt{x^3}$
When $x = 2 \Rightarrow \frac{dy}{dx} = \frac{5}{2}\sqrt{(2)^3}$
 $= \frac{5}{2}\sqrt{8} = \frac{5}{2} \cdot 2\sqrt{2} = 5\sqrt{2}$
 $\Rightarrow p = 5$

Q9. $f(x) = x^2 + kx \Rightarrow f'(x) = 2x + k$
 $\Rightarrow f'(-1) = 2(-1) + k = 3$
 $\Rightarrow -2 + k = 3 \Rightarrow k = 5$

Q10. $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$
 $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{\frac{3}{2}}}$
 $= \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$
 $= \frac{x-1}{2x\sqrt{x}}$

Q11. $y = x^2 - 2x - 3$
 $\Rightarrow \frac{dy}{dx} = 2x - 2$
Point $(2, 3) \Rightarrow \text{slope} = \frac{dy}{dx} = 2(2) - 2 = 2$

Q12. $y = 2x^2 - 3x + 4$
 $\Rightarrow \frac{dy}{dx} = 4x - 3$
Point $(1, 3) \Rightarrow \text{slope} = \frac{dy}{dx} = 4(1) - 3 = 1$
Equation of tangent: $y - 3 = 1(x - 1)$
 $\Rightarrow y - 3 = x - 1$
 $\Rightarrow x - y + 2 = 0$

Q13.

$$y = 6 + x - x^2$$

$$\Rightarrow \frac{dy}{dx} = 1 - 2x$$

$$\text{Point } (2, 4) \Rightarrow \text{slope} = \frac{dy}{dx} = 1 - 2(2) = -3$$

$$\text{Equation of tangent: } y - 4 = -3(x - 2)$$

$$\Rightarrow y - 4 = -3x + 6$$

$$\Rightarrow 3x + y - 10 = 0$$

Q14.

$$y = 8 + 2x - x^2$$

$$\Rightarrow \frac{dy}{dx} = 2 - 2x$$

$$\text{Slope} = 6 \Rightarrow 2 - 2x = 6$$

$$\Rightarrow -2x = 6 - 2 = 4$$

$$\Rightarrow 2x = -4 \Rightarrow x = -2$$

Q15.

$$y = x^2 - x \Rightarrow \frac{dy}{dx} = 2x - 1$$

$$\text{Slope} = 1 \Rightarrow 2x - 1 = 1$$

$$\Rightarrow 2x = 2 \Rightarrow x = 1$$

$$\Rightarrow y = (1)^2 - 1 = 0 \Rightarrow \text{Point } (1, 0)$$

Q16.

$$y = 2x^2 - x - 4 \Rightarrow \frac{dy}{dx} = 4x - 1$$

$$\text{Slope} = 3 \Rightarrow 4x - 1 = 3$$

$$\Rightarrow 4x = 4 \Rightarrow x = 1$$

$$\Rightarrow y = 2(1)^2 - 1 - 4 = 2 - 1 - 4 = -3 \Rightarrow \text{Point } (1, -3)$$

Q17.

$$y = x^2 + ax \Rightarrow \frac{dy}{dx} = 2x + a$$

$$\text{When } x = -1 \Rightarrow \text{slope} = \frac{dy}{dx} = 2(-1) + a = 3$$

$$\Rightarrow -2 + a = 3 \Rightarrow a = 5$$

Q18.

$$y = x^2 - 3x + 4 \Rightarrow \frac{dy}{dx} = 2x - 3$$

$$\text{When } x = 1\frac{1}{2} \Rightarrow \frac{dy}{dx} = 2\left(1\frac{1}{2}\right) - 3 = 3 - 3 = 0$$

Hence, tangent is parallel to the x -axis.

Q19. $y = 2x^2 - 8x + 3 \Rightarrow \frac{dy}{dx} = 4x - 8$

Line: $4x - y + 2 = 0 \Rightarrow \text{slope} = -\frac{a}{b} = -\frac{4}{-1} = 4$

Hence, $4x - 8 = 4$

$\Rightarrow 4x = 12 \Rightarrow x = 3$

$\Rightarrow y = 2(3)^2 - 8(3) + 3 = 18 - 24 + 3 = -3 \Rightarrow \text{Point } (3, -3)$

Q20. $y = 2x^2 + 3x \Rightarrow \frac{dy}{dx} = 4x + 3$

Tangent is parallel to the x -axis $\Rightarrow \text{slope} = 0$

$\Rightarrow 4x + 3 = 0$

$\Rightarrow 4x = -3 \Rightarrow x = -\frac{3}{4}$

$\Rightarrow y = 2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) = \frac{9}{8} - \frac{9}{4} = \frac{-9}{8} \Rightarrow P = \left(-\frac{3}{4}, \frac{-9}{8}\right)$

Q21. $y = a\sqrt{x} + b = ax^{\frac{1}{2}} + b$

$\Rightarrow \frac{dy}{dx} = a \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{a}{2\sqrt{x}}$

$x = 4 \Rightarrow \frac{dy}{dx} = \frac{a}{2\sqrt{4}} = \frac{a}{4} = 3 \Rightarrow a = 12$

Point $(4, 6) \Rightarrow 6 = a\sqrt{4} + b$

$\Rightarrow 6 = 12 \cdot 2 + b$

$\Rightarrow -b = -6 + 24 \Rightarrow b = -18$

Q22. $y = \frac{3}{x} = 3x^{-1} \Rightarrow \frac{dy}{dx} = -3x^{-2} = \frac{-3}{x^2}$

Point $\left(2, \frac{3}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{-3}{(2)^2} = \frac{-3}{4}$

Equation of Tangent: $y - \frac{3}{2} = -\frac{3}{4}(x - 2)$

$\Rightarrow 4y - 6 = -3x + 6$

$\Rightarrow 3x + 4y - 12 = 0$

Point A on x -axis $\Rightarrow y = 0 \Rightarrow 3x = 12$

$\Rightarrow x = 4 \Rightarrow A(4, 0)$

Point B on y -axis $\Rightarrow x = 0 \Rightarrow 4y = 12$

$\Rightarrow y = 3 \Rightarrow B(0, 3)$

Area Triangle AOB $= \frac{1}{2} |x_1 y_2 - x_2 y_1|$

$= \frac{1}{2} |(4)(3) - 0 \cdot 0|$

$= \frac{1}{2} |12| = 6 \text{ sq. units}$

Exercise 2.4

Q1. (i) $y = (3x + 4)(x - 2)$

Product Rule $\Rightarrow u = 3x + 4$ and $v = x - 2$

$$\Rightarrow \frac{du}{dx} = 3 \quad \Rightarrow \frac{dv}{dx} = 1$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (3x + 4).(1) + (x - 2).(3) \\ &= 3x + 4 + 3x - 6 = 6x - 2 \end{aligned}$$

(ii) $y = (3x - 4)(4x + 5)$

Product Rule $\Rightarrow u = 3x - 4$ and $v = 4x + 5$

$$\Rightarrow \frac{du}{dx} = 3 \quad \Rightarrow \frac{dv}{dx} = 4$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (3x - 4).(4) + (4x + 5).(3) \\ &= 12x - 16 + 12x + 15 = 24x - 1 \end{aligned}$$

(iii) $y = (x^2 + 2)(x - 1)$

Product Rule $\Rightarrow u = x^2 + 2$ and $v = x - 1$

$$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dv}{dx} = 1$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (x^2 + 2).(1) + (x - 1).(2x) \\ &= x^2 + 2 + 2x^2 - 2x = 3x^2 - 2x + 2 \end{aligned}$$

(iv) $y = (2x - 1)(x^2 - 2)$

Product Rule $\Rightarrow u = 2x - 1$ and $v = x^2 - 2$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dv}{dx} = 2x$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (2x - 1).(2x) + (x^2 - 2).(2) \\ &= 4x^2 - 2x + 2x^2 - 4 \\ &= 6x^2 - 2x - 4 \end{aligned}$$

(v) $y = (1 - x)(2 - x^2)$

Product Rule $\Rightarrow u = 1 - x$ and $v = 2 - x^2$

$$\Rightarrow \frac{du}{dx} = -1 \quad \Rightarrow \frac{dv}{dx} = -2x$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (1 - x)(-2x) + (2 - x^2).(-1) \\ &= -2x + 2x^2 - 2 + x^2 \\ &= 3x^2 - 2x - 2 \end{aligned}$$

$$(vi) \quad y = (x^3 - 1)(2x + 1)$$

$$\text{Product Rule} \Rightarrow u = x^3 - 1 \quad \text{and} \quad v = 2x + 1$$

$$\Rightarrow \frac{du}{dx} = 3x^2 \quad \Rightarrow \frac{dv}{dx} = 2$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (x^3 - 1).(2) + (2x + 1).(3x^2) \\ &= 2x^3 - 2 + 6x^3 + 3x^2 \\ &= 8x^3 + 3x^2 - 2 \end{aligned}$$

$$\mathbf{Q2.} \quad (i) \quad f(x) = \frac{3x}{2x+6}$$

$$\text{Quotient Rule} \Rightarrow u = 3x \quad \text{and} \quad v = 2x + 6$$

$$\Rightarrow \frac{du}{dx} = 3 \quad \Rightarrow \frac{dv}{dx} = 2$$

$$\begin{aligned} f'(x) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(2x+6)(3) - (3x).(2)}{(2x+6)^2} \\ &= \frac{6x+18-6x}{(2x+6)^2} = \frac{18}{(2x+6)^2} \end{aligned}$$

$$(ii) \quad f(x) = \frac{2x+3}{x-1}$$

$$\text{Quotient Rule} \Rightarrow u = 2x + 3 \quad \text{and} \quad v = x - 1$$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dv}{dx} = 1$$

$$\begin{aligned} f'(x) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x-1)(2) - (2x+3).(1)}{(x-1)^2} \\ &= \frac{2x-2-2x-3}{(x-1)^2} = \frac{-5}{(x-1)^2} \end{aligned}$$

$$(iii) \quad f(x) = \frac{x^2}{2x+3}$$

$$\text{Quotient Rule} \Rightarrow u = x^2 \quad \text{and} \quad v = 2x + 3$$

$$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dv}{dx} = 2$$

$$\begin{aligned} f'(x) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(2x+3).(2x) - (x^2).(2)}{(2x+3)^2} \\ &= \frac{4x^2 + 6x - 2x^2}{(2x+3)^2} = \frac{2x^2 + 6x}{(2x+3)^2} \end{aligned}$$

$$(iv) \quad f(x) = \frac{2x^2 - 1}{2x - 3}$$

$$\text{Quotient Rule} \Rightarrow u = 2x^2 - 1 \quad \text{and} \quad v = 2x - 3$$

$$\Rightarrow \frac{du}{dx} = 4x \quad \Rightarrow \frac{dv}{dx} = 2$$

$$\begin{aligned} f'(x) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(2x-3)(4x) - (2x^2-1)(2)}{(2x-3)^2} \\ &= \frac{8x^2 - 12x - 4x^2 + 2}{(2x-3)^2} = \frac{4x^2 - 12x + 2}{(2x-3)^2} \end{aligned}$$

$$(v) \quad f(x) = \frac{2x^3}{1-2x}$$

$$\text{Quotient Rule} \Rightarrow u = 2x^3 \quad \text{and} \quad v = 1 - 2x$$

$$\Rightarrow \frac{du}{dx} = 6x^2 \quad \Rightarrow \frac{dv}{dx} = -2$$

$$\begin{aligned} f'(x) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(1-2x)(6x^2) - (2x^3)(-2)}{(1-2x)^2} \\ &= \frac{6x^2 - 12x^3 + 4x^3}{(1-2x)^2} \\ &= \frac{-8x^3 + 6x^2}{(1-2x)^2} \end{aligned}$$

$$(vi) \quad f(x) = \frac{3x+2}{x^2-3}$$

$$\text{Quotient Rule} \Rightarrow u = 3x + 2 \quad \text{and} \quad v = x^2 - 3$$

$$\Rightarrow \frac{du}{dx} = 3 \quad \Rightarrow \frac{dv}{dx} = 2x$$

$$\begin{aligned} f'(x) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x^2-3)(3) - (3x+2)(2x)}{(x^2-3)^2} \\ &= \frac{3x^2 - 9 - 6x^2 - 4x}{(x^2-3)^2} \\ &= \frac{-3x^2 - 4x - 9}{(x^2-3)^2} \end{aligned}$$

Q3.

$$y = \frac{x^2 + 1}{3x - 1}$$

$$\text{Quotient Rule} \Rightarrow u = x^2 + 1 \quad \text{and} \quad v = 3x - 1$$

$$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dv}{dx} = 3$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(3x - 1) \cdot (2x) - (x^2 + 1) \cdot (3)}{(3x - 1)^2} \\ &= \frac{6x^2 - 2x - 3x^2 - 3}{(3x - 1)^2} \\ &= \frac{3x^2 - 2x - 3}{(3x - 1)^2} \end{aligned}$$

$$\text{At } x = 0 \Rightarrow \frac{dy}{dx} = \frac{3(0)^2 - 2(0) - 3}{[3(0) - 1]^2} = \frac{-3}{(-1)^2} = \frac{-3}{1} = -3$$

Q4.

$$y = \sqrt{x} \cdot (2x - 1)$$

$$\text{Product Rule} \Rightarrow u = \sqrt{x} = x^{\frac{1}{2}} \quad \text{and} \quad v = 2x - 1$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \quad \Rightarrow \frac{dv}{dx} = 2$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (\sqrt{x}) \cdot 2 + (2x - 1) \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{(2\sqrt{x}) \cdot (2\sqrt{x}) + 2x - 1}{2\sqrt{x}} \\ &= \frac{4x + 2x - 1}{2\sqrt{x}} = \frac{6x - 1}{2\sqrt{x}} \end{aligned}$$

Q5.

$$y = (\sqrt{x} + 4)(\sqrt{x} - 4)$$

$$\text{Product Rule} \Rightarrow u = \sqrt{x} + 4 = x^{\frac{1}{2}} + 4 \quad \text{and} \quad v = \sqrt{x} - 4 = x^{\frac{1}{2}} - 4$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \quad \Rightarrow \frac{dv}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (\sqrt{x} + 4) \cdot \left(\frac{1}{2\sqrt{x}} \right) + (\sqrt{x} - 4) \cdot \left(\frac{1}{2\sqrt{x}} \right) \\ &= \frac{1}{2} + \frac{4}{2\sqrt{x}} + \frac{1}{2} - \frac{4}{2\sqrt{x}} = 1 \end{aligned}$$

Q6. $y = \frac{x}{1-x^2}$

Quotient Rule $\Rightarrow u = x$ and $v = 1 - x^2$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(1-x^2) \cdot (1) - (x)(-2x)}{(1-x^2)^2}$$

$$= \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2} > 0$$

Q7. (i) $y = (x+4)^2$

Chain Rule \Rightarrow let $u = x+4$ and $y = u^2$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 1 = 2u = 2(x+4)$$

(ii) $y = (2x-1)^3$

Chain Rule \Rightarrow let $u = 2x-1$ and $y = u^3$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dy}{du} = 3u^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot 2 = 6u^2 = 6(2x-1)^2$$

(iii) $y = (3x+5)^3$

Chain Rule \Rightarrow let $u = 3x+5$ and $y = u^3$

$$\Rightarrow \frac{du}{dx} = 3 \quad \Rightarrow \frac{dy}{du} = 3u^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot 3 = 9u^2 = 9(3x+5)^2$$

(iv) $y = (x^2-1)^2$

Chain Rule \Rightarrow let $u = x^2-1$ and $y = u^2$

$$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 2x = 2(x^2-1) \cdot 2x = 4x \cdot (x^2-1)$$

(v) $y = (2x^2+3)^4$

Chain Rule \Rightarrow let $u = 2x^2+3$ and $y = u^4$

$$\Rightarrow \frac{du}{dx} = 4x \quad \Rightarrow \frac{dy}{du} = 4u^3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 \cdot 4x = 4(2x^2+3)^3 \cdot 4x = 16x(2x^2+3)^3$$

$$(vi) \quad y = (1-3x)^5$$

Chain Rule \Rightarrow let $u = 1-3x$ and $y = u^5$

$$\Rightarrow \frac{du}{dx} = -3 \quad \Rightarrow \frac{dy}{du} = 5u^4$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 \cdot -3 = -15u^4 = -15(1-3x)^4$$

$$\mathbf{Q8.} \quad (i) \quad f(x) = \sqrt{4x+1} = (4x+1)^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}(4x+1)^{-\frac{1}{2}} \cdot 4 = \frac{2}{\sqrt{4x+1}}$$

$$(ii) \quad f(x) = \sqrt{x^2-4} = (x^2-4)^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}(x^2-4)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2-4}}$$

$$(iii) \quad f(x) = \sqrt{x^3-2x} = (x^3-2x)^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}(x^3-2x)^{-\frac{1}{2}} \cdot (3x^2-2) \\ = \frac{3x^2-2}{2\sqrt{x^3-2x}}$$

$$\mathbf{Q9.} \quad (i) \quad y = 2x(2x+5)^3$$

Product Rule \Rightarrow let $u = 2x$ and $v = (2x+5)^3$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dv}{dx} = 3(2x+5)^2 \cdot 2 \\ = 6(2x+5)^2$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = 2x \cdot 6(2x+5)^2 + (2x+5)^3 \cdot 2 \\ = 2(2x+5)^3 + 12x(2x+5)^2$$

$$(ii) \quad y = (x^2-1)(3x+2)^2$$

Product Rule \Rightarrow let $u = x^2-1$ and $v = (3x+2)^2$

$$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dv}{dx} = 2(3x+2)^1 \cdot 3 = 6(3x+2)$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = (x^2-1) \cdot 6(3x+2) + (3x+2)^2 \cdot 2x \\ = 6(3x+2)(x^2-1) + 2x(3x+2)^2$$

$$(iii) \quad y = (x+4)^2 \cdot (x-2)$$

Product Rule \Rightarrow let $u = (x+4)^2$ and $v = x-2$

$$\Rightarrow \frac{du}{dx} = 2(x+4)^1 \cdot 1 \quad \Rightarrow \frac{dv}{dx} = 1 \\ = 2(x+4)$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = (x+4)^2 \cdot 1 + (x-2) \cdot 2(x+4) \\ = (x+4)^2 + 2(x-2)(x+4)$$

Q10. $y = (x^2 - 3)^3 \Rightarrow \frac{dy}{dx} = 3(x^2 - 3)^2 \cdot 2x = 6x(x^2 - 3)^2$
 When $x = 1 \Rightarrow \frac{dy}{dx} = 6(1)[(1)^2 - 3]^2 = 6(-2)^2 = 24$

Q11. $y = \frac{(2x-1)^2}{3x+4}$
 Quotient Rule $\Rightarrow u = (2x-1)^2$ and $v = 3x+4$
 $\frac{du}{dx} = 2(2x-1)^1 \cdot 2 \Rightarrow \frac{dv}{dx} = 3$
 $= 4(2x-1)$
 $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(3x+4) \cdot 4(2x-1) - (2x-1)^2 \cdot 3}{(3x+4)^2}$
 At $x = 0 \Rightarrow \frac{dy}{dx} = \frac{(3(0)+4) \cdot 4(2(0)-1) - (2(0)-1)^2 \cdot 3}{(3(0)+4)^2}$
 $= \frac{(4)(4)(-1) - (1)(3)}{(4)^2} = \frac{-16-3}{16} = \frac{-19}{16}$

Q12. $y = (2x^2 - 3)^7 \Rightarrow \frac{dy}{dx} = 7(2x^2 - 3)^6 \cdot 4x$
 At $x = -1 \Rightarrow \frac{dy}{dx} = 7(2(-1)^2 - 3)^6 \cdot 4(-1)$
 $= 7(2 - 3)^6 \cdot (-4)$
 $= 7(-1)^6 \cdot (-4) = 7 \cdot (1) \cdot (-4) = -28$

Q13. $f(x) = x\sqrt{x+1}$
 Product Rule $\Rightarrow u = x$ and $v = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$
 $\Rightarrow \frac{du}{dx} = 1 \Rightarrow \frac{dv}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}}$
 $f'(x) = u \frac{dv}{dx} + v \frac{du}{dx} = x \cdot \frac{1}{2\sqrt{x+1}} + \sqrt{x+1} \cdot 1$
 $= \frac{x + 2\sqrt{x+1} \cdot \sqrt{x+1}}{2\sqrt{x+1}}$
 $= \frac{x + 2(x+1)}{2\sqrt{x+1}}$
 $= \frac{x + 2x + 2}{2\sqrt{x+1}} = \frac{3x + 2}{2\sqrt{x+1}}$

Q14.

$$4x^2 + 2xy = 5$$

$$\Rightarrow 2xy = 5 - 4x^2$$

$$\Rightarrow y = \frac{5 - 4x^2}{2x}$$

$$\text{Quotient Rule} \Rightarrow u = 5 - 4x^2 \quad \text{and} \quad v = 2x$$

$$\Rightarrow \frac{du}{dx} = -8x \quad \Rightarrow \frac{dv}{dx} = 2$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(2x)(-8x) - (5 - 4x^2)(2)}{(2x)^2} \\ &= \frac{-16x^2 - 10 + 8x^2}{4x^2} \\ &= \frac{-8x^2 - 10}{4x^2} = \frac{-4x^2 - 5}{2x^2} \end{aligned}$$

Q15.

$$y = \frac{x}{\sqrt{x} + 1}$$

$$\text{Quotient Rule} \Rightarrow u = x \quad \text{and} \quad v = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(\sqrt{x} + 1)(1) - x \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2} \\ &= \frac{2\sqrt{x}(\sqrt{x} + 1) - x}{2\sqrt{x}(\sqrt{x} + 1)^2} \\ &= \frac{2x + 2\sqrt{x} - x}{2\sqrt{x}(\sqrt{x} + 1)^2} \\ &= \frac{x + 2\sqrt{x}}{2\sqrt{x}(\sqrt{x} + 1)^2} = \frac{\sqrt{x}(\sqrt{x} + 2)}{2\sqrt{x}(\sqrt{x} + 1)^2} \\ &= \frac{\sqrt{x} + 2}{2(\sqrt{x} + 1)^2} \end{aligned}$$

$$\text{At } x = 1 \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1} + 2}{2(\sqrt{1} + 1)^2} = \frac{1 + 2}{2(2)^2} = \frac{3}{8}$$

Q16. $y = \frac{3x+1}{1-2x}$
 Quotient Rule $\Rightarrow u = 3x+1$ and $v = 1-2x$

$$\Rightarrow \frac{du}{dx} = 3 \quad \Rightarrow \frac{dv}{dx} = -2$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(1-2x) \cdot 3 - (3x+1) \cdot (-2)}{(1-2x)^2}$$

$$= \frac{3-6x+6x+2}{(1-2x)^2} = \frac{5}{(1-2x)^2}$$

Q17. $f(x) = \sqrt{3x^2-2} = (3x^2-2)^{\frac{1}{2}}$

$$\Rightarrow f'(x) = \frac{1}{2}(3x^2-2)^{-\frac{1}{2}} \cdot 6x = \frac{3x}{\sqrt{3x^2-2}}$$

 When $x = 1 \Rightarrow f'(1) = \frac{3(1)}{\sqrt{3(1)^2-2}} = \frac{3}{\sqrt{1}} = \frac{3}{1} = 3$

Q18. $y = (x-1)^{\frac{3}{2}} - 3(x-1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{3}{2}(x-1)^{-\frac{1}{2}} - 3 \cdot \frac{1}{2}(x-1)^{-\frac{1}{2}}$$

$$= \frac{3\sqrt{x-1}}{2} - \frac{3}{2\sqrt{x-1}} = \frac{3(\sqrt{x-1})(\sqrt{x-1}) - 3}{2\sqrt{x-1}}$$

$$= \frac{3(x-1) - 3}{2\sqrt{x-1}}$$

$$= \frac{3x - 3 - 3}{2\sqrt{x-1}}$$

$$= \frac{3x - 6}{2\sqrt{x-1}}$$

$$= \frac{3(x-2)}{2\sqrt{x-1}}$$

Q19. $y = ax^3 + 2bx^2 + 3cx$

$$\frac{dy}{dx} = 3ax^2 + 4bx + 3c = 6x^2 + 6x - 6$$

$$\Rightarrow 3a = 6 \quad \text{and} \quad 4b = 6 \quad \text{and} \quad 3c = -6$$

$$\Rightarrow a = 2 \quad \Rightarrow b = \frac{6}{4} = 1\frac{1}{2} \quad \Rightarrow c = -2$$

Q20.

$$y = \frac{4x}{x+3}$$

Quotient Rule $\Rightarrow u = 4x$ and $v = x + 3$

$$\Rightarrow \frac{du}{dx} = 4 \quad \Rightarrow \frac{dv}{dx} = 1$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x+3) \cdot (4) - (4x) \cdot 1}{(x+3)^2} \\ &= \frac{4x+12-4x}{(x+3)^2} = \frac{12}{(x+3)^2} \end{aligned}$$

$$f(x) = \sqrt{\frac{4x}{x+3}} = \left(\frac{4x}{x+3} \right)^{\frac{1}{2}}$$

$$\Rightarrow f'(x) = \frac{1}{2} \left(\frac{4x}{x+3} \right)^{-\frac{1}{2}} \cdot \frac{12}{(x+3)^2}$$

$$= \frac{6}{\sqrt{\frac{4x}{x+3}} \cdot (x+3)^2}$$

$$f'(1) = \frac{6}{\sqrt{\frac{4(1)}{1+3}} \cdot (1+3)^2}$$

$$= \frac{6}{\sqrt{\frac{4}{4}} \cdot 16}$$

$$= \frac{6}{1 \cdot 16} = \frac{3}{8}$$

Q21.

$$y = (8 - 2x^2)^{\frac{2}{3}}$$

$$\frac{dy}{dx} = \frac{2}{3} (8 - 2x^2)^{-\frac{1}{3}} \cdot (-4x)$$

$$= \frac{-8x}{3} (8 - 2x^2)^{-\frac{1}{3}} \Rightarrow \text{Answer is A}$$

Q22.

$$f(x) = 3x + 1 \text{ and } g(x) = x^2 - 2$$

(a) (i) $p(x) = f(g(x))$

$$= f(x^2 - 2)$$

$$= 3(x^2 - 2) + 1$$

$$= 3x^2 - 6 + 1 = 3x^2 - 5$$

(ii) $q(x) = g(f(x))$

$$= g(3x + 1)$$

$$= (3x + 1)^2 - 2$$

$$= 9x^2 + 6x + 1 - 2 = 9x^2 + 6x - 1$$

$$\begin{aligned}
 \text{(b)} \quad p(x) &= 3x^2 - 5 \Rightarrow p'(x) = 6x \\
 q(x) &= 9x^2 + 6x - 1 \Rightarrow q'(x) = 18x + 6 \\
 \text{Solve } p'(x) &= q'(x) \\
 &\Rightarrow 6x = 18x + 6 \\
 &\Rightarrow -12x = 6 \\
 &\Rightarrow -2x = 1 \\
 &\Rightarrow x = -\frac{1}{2}
 \end{aligned}$$

Exercise 2.5

$$\begin{aligned}
 \text{Q1.} \quad y &= x^3 + 2x^2 \\
 \Rightarrow \frac{dy}{dx} &= 3x^2 + 4x \\
 \Rightarrow \frac{d^2y}{dx^2} &= 6x + 4
 \end{aligned}$$

$$\begin{aligned}
 \text{Q2.} \quad y &= x^4 - 3x^2 + 6 \\
 \Rightarrow \frac{dy}{dx} &= 4x^3 - 6x \\
 \Rightarrow \frac{d^2y}{dx^2} &= 12x^2 - 6
 \end{aligned}$$

$$\begin{aligned}
 \text{Q3.} \quad y &= \frac{1}{x} = x^{-1} \\
 \Rightarrow \frac{dy}{dx} &= -1x^{-2} \\
 \Rightarrow \frac{d^2y}{dx^2} &= 2x^{-3} = \frac{2}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Q4.} \quad y &= \frac{1}{x^2} + 3x^2 = x^{-2} + 3x^2 \\
 \Rightarrow \frac{dy}{dx} &= -2x^{-3} + 6x \\
 \Rightarrow \frac{d^2y}{dx^2} &= 6x^{-4} + 6 = \frac{6}{x^4} + 6
 \end{aligned}$$

$$\begin{aligned}
 \text{Q5.} \quad y &= 3x + \frac{1}{x} + 4 = 3x + x^{-1} + 4 \\
 \Rightarrow \frac{dy}{dx} &= 3 - x^{-2} \\
 \Rightarrow \frac{d^2y}{dx^2} &= 2x^{-3} = \frac{2}{x^3}
 \end{aligned}$$

Q6. $y = \sqrt{x} = x^{\frac{1}{2}}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$
 $\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{4} x^{-\frac{3}{2}} = -\frac{1}{4\sqrt{x^3}}$

Q7. $y = \sqrt{2x+3} = (2x+3)^{\frac{1}{2}}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{2} (2x+3)^{-\frac{1}{2}} \cdot 2 = (2x+3)^{-\frac{1}{2}}$
 $\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{2} (2x+3)^{-\frac{3}{2}} \cdot 2 = \frac{-1}{\sqrt{(2x+3)^3}}$

Q8. $y = (3x-2)^3$
 $\Rightarrow \frac{dy}{dx} = 3(3x-2)^2 \cdot 3 = 9(3x-2)^2$
 $\Rightarrow \frac{d^2y}{dx^2} = 18(3x-2)^1 \cdot 3 = 54(3x-2)$

Q9. $y = \frac{1}{x+4} = (x+4)^{-1}$
 $\Rightarrow \frac{dy}{dx} = -1(x+4)^{-2}$
 $\Rightarrow \frac{d^2y}{dx^2} = 2(x+4)^{-3} = \frac{2}{(x+4)^3}$

Q10. $y = x^4 - x^3 + 4x - 1$
 $\Rightarrow \frac{dy}{dx} = 4x^3 - 3x^2 + 4$
 $\Rightarrow \frac{d^2y}{dx^2} = 12x^2 - 6x$

Solve $\Rightarrow \frac{d^2y}{dx^2} = 0 \Rightarrow 12x^2 - 6x = 0$
 $\Rightarrow 6x(2x-1) = 0$
 $\Rightarrow x = 0, \frac{1}{2}$

Q11. $y = 3x + \frac{4}{x} = 3x + 4x^{-1}$

$$\Rightarrow \frac{dy}{dx} = 3 - 4x^{-2} = 3 - \frac{4}{x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 8x^{-3} = \frac{8}{x^3}$$

Hence, $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y$

$$= x^2 \cdot \left(\frac{8}{x^3} \right) + x \cdot \left(3 - \frac{4}{x^2} \right) - \left(3x + \frac{4}{x} \right)$$

$$= \frac{8}{x} + 3x - \frac{4}{x} - 3x - \frac{4}{x}$$

$$= \frac{8}{x} - \frac{8}{x} = 0$$

Q12. $f(x) = \frac{2}{x} + 4\sqrt{x} = 2x^{-1} + 4x^{\frac{1}{2}}$

$$\Rightarrow f'(x) = -2x^{-2} + 2x^{-\frac{1}{2}}$$

$$\Rightarrow f''(x) = 4x^{-3} - x^{-\frac{3}{2}} = \frac{4}{x^3} - \frac{1}{\sqrt{x^3}}$$

Hence, $f''(4) = \frac{4}{(4)^3} - \frac{1}{\sqrt{(4)^3}}$

$$= \frac{4}{64} - \frac{1}{\sqrt{64}} = \frac{1}{16} - \frac{1}{8} = -\frac{1}{16}$$

Q13. $y = x^4 \Rightarrow \frac{dy}{dx} = 4x^3 \Rightarrow \frac{d^2y}{dx^2} = 12x^2$

Hence, $\frac{4x^4}{3} \left(\frac{d^2y}{dx^2} \right) - \left(\frac{dy}{dx} \right)^2$

$$= \frac{4x^4}{3} (12x^2) - (4x^3)^2$$

$$= \frac{48}{3} x^6 - 16x^6$$

$$= 16x^6 - 16x^6 = 0$$

Q14. $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{5}{2}} = \frac{3}{4\sqrt{x^5}}$$

Hence, $2x\left(\frac{d^2y}{dx^2}\right) + 3\frac{dy}{dx}$

$$= 2x\left(\frac{3}{4}x^{-\frac{5}{2}}\right) + 3\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$$

$$= \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{2}x^{-\frac{3}{2}} = 0$$

Exercise 2.6

Q1. (i) $y = \sin 2x \Rightarrow \frac{dy}{dx} = \cos 2x \cdot 2 = 2 \cos 2x$

(ii) $y = \cos 6x \Rightarrow \frac{dy}{dx} = -\sin 6x \cdot 6 = -6 \sin 6x$

(iii) $y = \tan 4x \Rightarrow \frac{dy}{dx} = \sec^2 4x \cdot 4 = 4 \sec^2 4x$

(iv) $y = \sin(2x + 3) \Rightarrow \frac{dy}{dx} = \cos(2x + 3) \cdot 2 = 2 \cos(2x + 3)$

(v) $y = \cos(3x - 1) \Rightarrow \frac{dy}{dx} = -\sin(3x - 1) \cdot 3 = -3 \sin(3x - 1)$

(vi) $y = \tan(x^2) \Rightarrow \frac{dy}{dx} = \sec^2(x^2) \cdot 2x = 2x \sec^2(x^2)$

(vii) $y = \sin\left(\frac{1}{2}x\right) \Rightarrow \frac{dy}{dx} = \cos\left(\frac{1}{2}x\right) \cdot \frac{1}{2} = \frac{1}{2} \cos \frac{1}{2}x$

(viii) $y = \cos(x^2 - 1) \Rightarrow \frac{dy}{dx} = -\sin(x^2 - 1) \cdot 2x = -2x \sin(x^2 - 1)$

(ix) $y = \sin 2x + \cos 4x \Rightarrow \frac{dy}{dx} = \cos 2x \cdot 2 - \sin 4x \cdot 4$
 $= 2 \cos 2x - 4 \sin 4x$

Q2. (i) $y = \sin^2 x \Rightarrow \frac{dy}{dx} = 2 \sin x \cos x$

(ii) $y = \cos^3 x \Rightarrow \frac{dy}{dx} = 3 \cos^2 x \cdot -\sin x = -3 \cos^2 x \sin x$

(iii) $y = \tan^4 x \Rightarrow \frac{dy}{dx} = 4 \tan^3 x \sec^2 x$

$$\begin{aligned} \text{(iv)} \quad y = \sin^3(4x) &\Rightarrow \frac{dy}{dx} = 3 \sin^2(4x) \cdot \cos(4x) \cdot 4 \\ &= 12 \sin^2(4x) \cos(4x) \end{aligned}$$

$$\begin{aligned} \text{(v)} \quad y = \cos^2(2x+1) &\Rightarrow \frac{dy}{dx} = 2 \cos(2x+1) \cdot -\sin(2x+1) \cdot 2 \\ &= -4 \cos(2x+1) \sin(2x+1) \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad y = \tan^3(4x+3) &\Rightarrow \frac{dy}{dx} = 3 \tan^2(4x+3) \cdot \sec^2(4x+3) \cdot 4 \\ &= 12 \tan^2(4x+3) \sec^2(4x+3) \end{aligned}$$

$$\begin{aligned} \text{Q3. (i)} \quad y = 2 \sin 3\theta + \cos 2\theta &\Rightarrow \frac{dy}{d\theta} = 2 \cos 3\theta \cdot 3 - \sin 2\theta \cdot 2 \\ &= 6 \cos 3\theta - 2 \sin 2\theta \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad y &= \tan^2 \theta + \tan 2\theta \\ \Rightarrow \frac{dy}{d\theta} &= 2 \tan \theta \cdot \sec^2 \theta + \sec^2 2\theta \cdot 2 \\ &= 2 \tan \theta \sec^2 \theta + 2 \sec^2 2\theta \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad y &= \cos 4\theta - \cos \frac{\theta}{4} \\ \frac{dy}{d\theta} &= -\sin 4\theta \cdot 4 + \sin \frac{\theta}{4} \cdot \frac{1}{4} \\ &= -4 \sin 4\theta + \frac{1}{4} \sin \frac{\theta}{4} \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad y &= \tan^3 \theta + 5 \\ \Rightarrow \frac{dy}{d\theta} &= 3 \tan^2 \theta \cdot \sec^2 \theta \end{aligned}$$

$$\begin{aligned} \text{Q4. (i)} \quad y &= x \sin 2x \\ \text{Product Rule} &\Rightarrow \text{let } u = x \quad \text{and} \quad v = \sin 2x \\ &\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = \cos 2x \cdot 2 \\ &= 2 \cos 2x \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = x \cdot 2 \cos 2x + \sin 2x \cdot 1 \\ &= 2x \cos 2x + \sin 2x \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad y &= x^2 \cos x \\ \text{Product Rule} &\Rightarrow \text{let } u = x^2 \quad \text{and} \quad v = \cos x \\ &\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dv}{dx} = -\sin x \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = x^2 \cdot -\sin x + \cos x \cdot 2x \\ &= 2x \cos x - x^2 \sin x \end{aligned}$$

$$(iii) \quad y = (x+3) \sin x$$

$$\text{Product Rule} \Rightarrow \text{let } u = x+3 \quad \text{and} \quad v = \sin x$$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \quad \frac{dv}{dx} = \cos x$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (x+3) \cdot \cos x + \sin x \cdot 1 \\ &= (x+3) \cos x + \sin x \end{aligned}$$

Q5.

$$y = \sin x \cos x$$

$$\text{Product Rule} \Rightarrow \text{let } u = \sin x \quad \text{and} \quad v = \cos x$$

$$\Rightarrow \frac{du}{dx} = \cos x \quad \Rightarrow \quad \frac{dv}{dx} = -\sin x$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = \sin x \cdot -\sin x + \cos x \cdot \cos x \\ &= \cos^2 x - \sin^2 x \\ &= \cos 2x \end{aligned}$$

Q6.

$$f(x) = \cos x \cdot \tan x$$

$$= \cos x \cdot \frac{\sin x}{\cos x}$$

$$= \sin x$$

$$\Rightarrow f'(x) = \cos x$$

Q7. (i)

$$y = \sin 2x \quad \Rightarrow \quad \frac{dy}{dx} = \cos 2x \cdot 2 = 2 \cos 2x$$

$$\text{At } x = \pi \quad \Rightarrow \quad \frac{dy}{dx} = 2 \cos 2\pi = 2 \cdot 1 = 2$$

(ii) $y = x \cos x$

$$\text{Product Rule} \Rightarrow u = x \quad \text{and} \quad v = \cos x$$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \quad \frac{dv}{dx} = -\sin x$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = x \cdot -\sin x + \cos x \cdot 1 \\ &= -x \sin x + \cos x \end{aligned}$$

$$\begin{aligned} \text{At } x = \pi \quad \Rightarrow \quad \frac{dy}{dx} &= -\pi \sin \pi + \cos \pi \\ &= -\pi(0) - 1 = -1 \end{aligned}$$

$$(iii) \quad y = \sin^2 x \quad \Rightarrow \quad \frac{dy}{dx} = 2 \sin x \cos x$$

$$\text{At } x = \pi \quad \Rightarrow \quad \frac{dy}{dx} = 2 \sin \pi \cdot \cos \pi = 2(0)(-1) = 0$$

Q8. $y = \tan x = \frac{\sin x}{\cos x}$
 Quotient Rule $\Rightarrow u = \sin x$ and $v = \cos x$
 $\Rightarrow \frac{du}{dx} = \cos x \quad \Rightarrow \frac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{\cos x (\cos x) - \sin x (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Q9. $f(x) = (\sin x + 1)^2$
 $\Rightarrow f'(x) = 2(\sin x + 1) \cdot \cos x$
 $\Rightarrow f'\left(\frac{\pi}{6}\right) = 2\left(\sin \frac{\pi}{6} + 1\right) \cdot \cos \frac{\pi}{6}$

$$= 2\left(\frac{1}{2} + 1\right) \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

Q10. $y = \sin x + 3 \cos x$
 $\Rightarrow \frac{dy}{dx} = \cos x + 3(-\sin x) = \cos x - 3 \sin x$
 Hence, $\cos x \frac{dy}{dx} + y \sin x$

$$= \cos x (\cos x - 3 \sin x) + (\sin x + 3 \cos x) \sin x$$

$$= \cos^2 x - 3 \sin x \cos x + \sin^2 x + 3 \sin x \cos x$$

$$= 1$$

Q11. $y = \sin 2x - 2x$
 $\Rightarrow \frac{dy}{dx} = \cos 2x \cdot 2 - 2 = 2 \cos 2x - 2$

$$= 2(1 - 2 \sin^2 x) - 2$$

$$= 2 - 4 \sin^2 x - 2 = -4 \sin^2 x$$

Q12. $y = \cos\left(\frac{1}{4}\pi x\right)$
 $\Rightarrow \frac{dy}{dx} = -\sin\left(\frac{1}{4}\pi x\right) \cdot \frac{1}{4}\pi$
 At $x = 4 \quad \Rightarrow \frac{dy}{dx} = -\sin\left(\frac{1}{4} \cdot \pi \cdot 4\right) \cdot \frac{1}{4}\pi$

$$= -\sin \pi \cdot \frac{1}{4}\pi = 0 \cdot \frac{1}{4}\pi = 0$$

Q13. $f(x) = \cos^3(2x) \Rightarrow f'(x) = 3\cos^2(2x) \cdot -\sin(2x) \cdot 2$
 $= -6\cos^2(2x) \cdot \sin(2x)$
 At $x = \frac{\pi}{6} \Rightarrow f'\left(\frac{\pi}{6}\right) = -6\cos^2\left(2 \cdot \frac{\pi}{6}\right) \sin\left(2 \cdot \frac{\pi}{6}\right)$
 $= -6\cos^2 \frac{\pi}{3} \sin \frac{\pi}{3} = -6\left(\frac{1}{2}\right)^2 \cdot \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{4}$

Q14. (i) $f(x) = \cos 2x \Rightarrow f'(x) = -\sin 2x \cdot 2 = -2\sin 2x$
 (ii) $g(x) = 2\sin^2 x \Rightarrow g'(x) = 4\sin x \cdot \cos x$
 Hence, $f'(x) + g'(x) = -2\sin 2x + 4\sin x \cos x$
 $= -2 \cdot 2\sin x \cos x + 4\sin x \cos x$
 $= -4\sin x \cos x + 4\sin x \cos x = 0$

Q15. $y = \sin 3x \Rightarrow \frac{dy}{dx} = \cos 3x \cdot 3 = 3\cos 3x$
 $\Rightarrow \frac{d^2y}{dx^2} = 3(-\sin 3x \cdot 3)$
 $= -9\sin 3x = -9y$

Q16. $y = \tan x + \frac{1}{3}\tan^3 x$
 $\Rightarrow \frac{dy}{dx} = \sec^2 x + \frac{1}{3} \cdot 3\tan^2 x \cdot \sec^2 x$
 $= \sec^2 x + \tan^2 x \cdot \sec^2 x$
 $= \sec^2 x (1 + \tan^2 x) = \sec^2 x \cdot \sec^2 x = \sec^4 x$

Q17. $y = 3\sin x + k\sin 3x$
 $\Rightarrow \frac{dy}{dx} = 3\cos x + k\cos 3x \cdot 3$
 $= 3\cos x + 3k\cos 3x$
 When $x = \frac{\pi}{3} \Rightarrow \frac{dy}{dx} = 3\cos \frac{\pi}{3} + 3k\cos 3 \cdot \left(\frac{\pi}{3}\right) = 0$
 $\Rightarrow 3\cos \frac{\pi}{3} + 3k\cos \pi = 0$
 $\Rightarrow 3 \cdot \frac{1}{2} + 3k(-1) = 0$
 $\Rightarrow \frac{3}{2} - 3k = 0$
 $\Rightarrow 3 - 6k = 0$
 $\Rightarrow -6k = -3$
 $\Rightarrow k = \frac{1}{2}$

Q18.

$$y = \frac{\sin x}{2 + \cos x}$$

Quotient Rule $\Rightarrow u = \sin x$ and $v = 2 + \cos x$

$$\Rightarrow \frac{du}{dx} = \cos x \quad \Rightarrow \frac{dv}{dx} = -\sin x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(2 + \cos x) \cdot \cos x - \sin x(-\sin x)}{(2 + \cos x)^2} \\ &= \frac{2 \cos x + \cos^2 x + \sin^2 x}{(2 + \cos x)^2} \\ &= \frac{1 + 2 \cos x}{(2 + \cos x)^2} = \frac{a + b \cos x}{(2 + \cos x)^2} \\ &\Rightarrow a = 1, b = 2\end{aligned}$$

Exercise 2.7

Q1. (i) $y = \sin^{-1} 6x$

Chain Rule $\Rightarrow u = 6x$ and $y = \sin^{-1} u$

$$\Rightarrow \frac{du}{dx} = 6 \quad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot 6 = \frac{6}{\sqrt{1-36x^2}}$$

(ii) $y = \tan^{-1} 3x$

Chain Rule $\Rightarrow u = 3x$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = 3 \quad \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot 3 = \frac{3}{1+9x^2}$$

(iii) $y = \sin^{-1}(2x+1)$

Chain Rule $\Rightarrow u = 2x+1$ and $y = \sin^{-1} u$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot 2 = \frac{2}{\sqrt{1-(2x+1)^2}} \\ &= \frac{2}{\sqrt{1-(4x^2+4x+1)}} \\ &= \frac{2}{\sqrt{1-4x^2-4x-1}} = \frac{2}{\sqrt{-4x^2-4x}}\end{aligned}$$

(iv) $y = \tan^{-1}(x^2)$

Chain Rule $\Rightarrow u = x^2$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot 2x = \frac{2}{1+(x^2)^2} = \frac{2x}{1+x^4}$$

Q2. $y = \sin^{-1}(3x-1)$

Chain Rule $\Rightarrow u = 3x-1$ and $y = \sin^{-1} u$

$$\Rightarrow \frac{du}{dx} = 3 \quad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot 3 = \frac{3}{\sqrt{1-(3x-1)^2}} \\ &= \frac{3}{\sqrt{1-(9x^2-6x+1)}} \\ &= \frac{3}{\sqrt{1-9x^2+6x-1}} \\ &= \frac{3}{\sqrt{6x-9x^2}} \end{aligned}$$

Q3. (i) $y = \sin^{-1} 2x$

Chain Rule $\Rightarrow u = 2x$ and $y = \sin^{-1} u$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

$$\text{At } x=0 \quad \Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-4(0)^2}} = \frac{2}{\sqrt{1-0}} = \frac{2}{1} = 2$$

(ii) $y = \tan^{-1} 4x$

Chain Rule $\Rightarrow u = 4x$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = 4 \quad \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot 4 = \frac{4}{1+16x^2}$$

$$\text{At } x = \frac{1}{4} \quad \Rightarrow \frac{dy}{dx} = \frac{4}{1+16\left(\frac{1}{4}\right)^2} = \frac{4}{1+16 \cdot \frac{1}{16}} = \frac{4}{2} = 2$$

Q4. (i) $f(x) = \sin^{-1} \frac{3}{x}$

Chain Rule $\Rightarrow u = \frac{3}{x} = -3x^{-2}$ and $y = \sin^{-1} u$

$$\Rightarrow \frac{du}{dx} = -3x^{-2} = \frac{-3}{x^2} \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\begin{aligned} f'(x) &= \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{-3}{x^2} = \frac{-3}{x^2 \sqrt{1-\frac{9}{x^2}}} \\ &= \frac{-3}{x^2 \sqrt{\frac{x^2-9}{x^2}}} \\ &= \frac{-3}{x^2 \frac{\sqrt{x^2-9}}{\sqrt{x^2}}} \\ &= \frac{-3}{x^2 \frac{\sqrt{x^2-9}}{x}} \\ &= \frac{-3}{x \sqrt{x^2-9}} \end{aligned}$$

(ii) $f(x) = \tan^{-1} \frac{x}{4}$

Chain Rule $\Rightarrow u = \frac{x}{4}$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = \frac{1}{4} \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\begin{aligned} f'(x) &= \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot \frac{1}{4} \\ &= \frac{1}{1+\left(\frac{x}{4}\right)^2} \cdot \frac{1}{4} \\ &= \frac{1}{1+\frac{x^2}{16}} \cdot \frac{1}{4} \\ &= \frac{1}{\frac{16+x^2}{16}} \cdot \frac{1}{4} = \frac{1}{\frac{x^2+16}{4}} = \frac{4}{x^2+16} \end{aligned}$$

Q5. (i) $y = x \sin^{-1} x$

Product Rule $\Rightarrow u = x$ and $v = \sin^{-1} x$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x \cdot 1 \\ &= \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x \end{aligned}$$

(ii) $y = 2x \cdot \tan^{-1} x$

Product Rule $\Rightarrow u = 2x$ and $v = \tan^{-1} x$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dv}{dx} = \frac{1}{1+x^2}$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = 2x \cdot \frac{1}{1+x^2} + \tan^{-1} x \cdot 2 \\ &= 2 \tan^{-1} x + \frac{2x}{1+x^2} \end{aligned}$$

Q6. $y = (\sin^{-1} x)^2$

Chain Rule $\Rightarrow u = \sin^{-1} x$ and $y = u^2$

$$\Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \frac{dy}{du} = 2u$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot \frac{1}{\sqrt{1-x^2}} = 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} \\ &= \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} \end{aligned}$$

Q7. $y = f(x) = \sin^{-1}(\cos x)$

Chain Rule $\Rightarrow u = \cos x$ and $y = \sin^{-1} u$

$$\Rightarrow \frac{du}{dx} = -\sin x \quad \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\begin{aligned} f'(x) &= \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot -\sin x \\ &= \frac{1}{\sqrt{1-\cos^2 x}} \cdot -\sin x \\ &= \frac{1}{\sqrt{\sin^2 x}} \cdot -\sin x \\ &= \frac{-\sin x}{\sin x} = -1 \quad \Rightarrow k = -1 \end{aligned}$$

Q8. $y = f(x) = \tan^{-1}(\cos x)$

Chain Rule $\Rightarrow u = \cos x$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = -\sin x \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot -\sin x$$

$$= \frac{-\sin x}{1+\cos^2 x}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{-\sin \frac{\pi}{6}}{1+\cos^2 \frac{\pi}{6}} = \frac{-\frac{1}{2}}{1+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{-\frac{1}{2}}{1+\frac{3}{4}}$$

$$= \frac{-\frac{1}{2}}{\frac{7}{4}} = -\frac{2}{7}$$

Q9. $y = \tan^{-1} \frac{1}{x}$

Chain Rule $\Rightarrow u = \frac{1}{x} = x^{-1}$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = -x^{-2} = \frac{-1}{x^2} \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot \frac{-1}{x^2}$$

$$= \frac{1}{1+\frac{1}{x^2}} \cdot \frac{-1}{x^2} = \frac{-1}{x^2+1}$$

At $x = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{(1)^2+1} = -\frac{1}{2}$

Q10. $y = \tan^{-1}(3x^2)$

Chain Rule $\Rightarrow u = 3x^2$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = 6x \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot 6x = \frac{1}{1+(3x^2)^2} \cdot 6x$$

$$= \frac{6x}{1+9x^4}$$

At $x = \frac{1}{3} \Rightarrow \frac{dy}{dx} = \frac{6\left(\frac{1}{3}\right)}{1+9\left(\frac{1}{3}\right)^4} = \frac{2}{1+9 \cdot \frac{1}{81}} = \frac{2}{1+\frac{1}{9}} = \frac{9}{5}$

Q11. $y = \tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1}$
 $\Rightarrow \frac{d^2y}{dx^2} = -1(1+x^2)^{-2} \cdot 2x = \frac{-2x}{(1+x^2)^2}$

Hence, $\frac{d^2y}{dx^2}(1+x^2) + 2x \frac{dy}{dx}$
 $= \frac{-2x}{(1+x^2)^2} \cdot (1+x^2) + 2x \cdot \frac{1}{1+x^2}$
 $= \frac{-2x}{1+x^2} + \frac{2x}{1+x^2} = 0$

Exercise 2.8

Q1. (i) $y = e^{4x} \Rightarrow \frac{dy}{dx} = e^{4x} \cdot 4 = 4e^{4x}$

(ii) $y = e^{-3x} \Rightarrow \frac{dy}{dx} = e^{-3x} \cdot -3 = -3e^{-3x}$

(iii) $y = e^{x^2} \Rightarrow \frac{dy}{dx} = e^{x^2} \cdot 2x = 2xe^{x^2}$

(iv) $y = e^{2x+4} \Rightarrow \frac{dy}{dx} = e^{2x+4} \cdot 2 = 2e^{2x+4}$

(v) $y = e^{x^2+3x} \Rightarrow \frac{dy}{dx} = e^{x^2+3x} \cdot (2x+3) = (2x+3)e^{x^2+3x}$

(vi) $y = e^{\sin x} \Rightarrow \frac{dy}{dx} = e^{\sin x} \cdot \cos x = \cos x e^{\sin x}$

Q2. (i) $y = e^{\frac{x}{2}} \Rightarrow \frac{dy}{dx} = e^{\frac{x}{2}} \cdot \frac{1}{2} = \frac{1}{2} e^{\frac{x}{2}}$

(ii) $y = e^{\sin^2 x} \Rightarrow \frac{dy}{dx} = (e^{\sin^2 x}) 2 \sin x \cdot \cos x$
 $= 2 \sin x \cos x (e^{\sin^2 x})$

(iii) $y = xe^{2x}$

Product Rule $\Rightarrow u = x$ and $v = e^{2x}$

$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = 2e^{2x}$

$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = x \cdot 2e^{2x} + e^{2x} \cdot 1$
 $= e^{2x} (1 + 2x)$

Q3. (i) $y = e^{2x} \sin x$

Product Rule $\Rightarrow u = e^{2x}$ and $v = \sin x$

$$\Rightarrow \frac{du}{dx} = e^{2x} \cdot 2 = 2e^{2x} \quad \Rightarrow \frac{dv}{dx} = \cos x$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = e^{2x} \cdot \cos x + \sin x \cdot 2e^{2x} \\ &= e^{2x} (2 \sin x + \cos x) \end{aligned}$$

(ii) $y = (e^x - 1)^2$

Chain Rule $\Rightarrow u = e^x - 1$ and $y = u^2$

$$\Rightarrow \frac{du}{dx} = e^x \quad \Rightarrow \frac{dy}{du} = 2u$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot e^x \\ &= 2(e^x - 1)e^x = 2e^x(e^x - 1) \end{aligned}$$

(iii) $y = \frac{e^{2x+1}}{e^x} = e^{2x+1-x} = e^{x+1}$

$$\Rightarrow \frac{dy}{dx} = e^{x+1}$$

Q4. (i) $y = e^{2x}(1 + e^x) = e^{2x} + e^{3x}$

$$\Rightarrow \frac{dy}{dx} = 2e^{2x} + 3e^{3x}$$

(ii) $t = \frac{e^{2x}}{x}$

Quotient Rule $\Rightarrow u = e^{2x}$ and $v = x$

$$\Rightarrow \frac{du}{dx} = 2e^{2x} \quad \Rightarrow \frac{dv}{dx} = 1$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{x \cdot 2e^{2x} - e^{2x} \cdot 1}{x^2} \\ &= \frac{e^{2x}(2x - 1)}{x^2} \end{aligned}$$

(iii) $y = x^2 e^{\cos x}$

Product Rule $\Rightarrow u = x^2$ and $v = e^{\cos x}$

$$\begin{aligned} \Rightarrow \frac{du}{dx} &= 2x \quad \Rightarrow \frac{dv}{dx} = e^{\cos x} \cdot -\sin x \\ &= -\sin x e^{\cos x} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= x^2(-\sin x \cdot e^{\cos x}) + e^{\cos x}(2x) \\ &= -x^2 \sin x \cdot e^{\cos x} + 2x e^{\cos x} \\ &= x e^{\cos x}(-x \sin x + 2) \end{aligned}$$

Q5. $y = e^{3x} \sin(\pi x)$

Product Rule $\Rightarrow u = e^{3x}$ and $v = \sin(\pi x)$

$$\Rightarrow \frac{du}{dx} = 3e^{3x} \quad \Rightarrow \frac{dv}{dx} = \cos(\pi x) \cdot \pi$$

$$= \pi \cos(\pi x)$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = e^{3x} \cdot \pi \cos(\pi x) + \sin(\pi x) \cdot 3e^{3x}$$

$$\text{At } x = 1 \Rightarrow \frac{dy}{dx} = e^{3(1)} \cdot \pi \cos(\pi \cdot 1) + \sin(\pi \cdot 1) \cdot 3e^{3(1)}$$

$$= e^3 \cdot \pi(-1) + 0 \cdot 3e^3$$

$$= -\pi e^3$$

Q6. $y = e^{2x} \Rightarrow \frac{dy}{dx} = 2e^{2x}$

$$\Rightarrow \frac{d^2 y}{dx^2} = 2 \cdot 2e^{2x} = 4e^{2x}$$

Hence, $\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y$

$$= 4e^{2x} - 3(2e^{2x}) + 2(e^{2x})$$

$$= 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$$

Q7. $y = e^x (\cos x - \sin x)$

Product Rule $\Rightarrow u = e^x$ and $v = \cos x - \sin x$

$$\Rightarrow \frac{du}{dx} = e^x \quad \Rightarrow \frac{dv}{dx} = -\sin x - \cos x$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = e^x(-\sin x - \cos x) + (\cos x - \sin x)e^x$$

$$= -e^x \sin x - e^x \cos x + e^x \cos x - e^x \sin x$$

$$= -2e^x \sin x$$

Q8. $y = xe^x$

Product Rule $\Rightarrow u = x$ and $v = e^x$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = x \cdot e^x + e^x \cdot 1 = e^x(x+1)$$

$$\frac{d^2 y}{dx^2} = u \frac{dv}{dx} + v \frac{du}{dx} = e^x(1) + (x+1)e^x = e^x(1+x+1) = e^x(x+2)$$

$$\frac{d^2 y}{dx^2} + y = e^x(x+2) + xe^x = e^x(x+2+x) = e^x(2x+2) = 2e^x(x+1)$$

$$= 2 \frac{dy}{dx}$$

Q9.

$$f(x) = e^{2x} - ae^x$$

$$\Rightarrow f'(x) = 2e^{2x} - ae^x \quad \text{OR} \quad 2(e^x)^2 - ae^x$$

$$\begin{aligned} \text{When } e^x = \frac{a}{2} \Rightarrow f'(x) &= 2\left(\frac{a}{2}\right)^2 - a\left(\frac{a}{2}\right) \\ &= 2\frac{a^2}{4} - \frac{a^2}{2} = \frac{a^2}{2} - \frac{a^2}{2} = 0 \end{aligned}$$

Q10.

$$y = e^{mx} \Rightarrow \frac{dy}{dx} = e^{mx} \cdot m = me^{mx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = me^{mx} \cdot m = m^2e^{mx}$$

$$\text{Hence, } \frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$$

$$\Rightarrow m^2e^{mx} - 3me^{mx} - 4e^{mx} = 0$$

$$\Rightarrow e^{mx}(m^2 - 3m - 4) = 0$$

$$\Rightarrow (m+1)(m-4) = 0$$

$$\Rightarrow m = -1, m = 4$$

Q11.

$$f(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}(e^x + e^{-x})$$

$$\Rightarrow f'(x) = \frac{1}{2}(e^x + e^{-x} \cdot -1) = \frac{1}{2}(e^x - e^{-x})$$

$$\begin{aligned} \Rightarrow f''(x) &= \frac{1}{2}(e^x - e^{-x} \cdot -1) \\ &= \frac{1}{2}(e^x + e^{-x}) = f(x) \end{aligned}$$

Q12.

$$y = 3e^x - \sin x + 5$$

$$\Rightarrow \frac{dy}{dx} = 3e^x - \cos x$$

$$\begin{aligned} \text{At } x = 0 \Rightarrow \frac{dy}{dx} &= 3e^0 - \cos 0 \\ &= 3(1) - 1 = 2 \end{aligned}$$

$$\begin{aligned} \text{At } x = 0 \Rightarrow y &= 3e^0 - \sin 0 + 5 \\ &= 3(1) - 0 + 5 = 8 \end{aligned}$$

$$\text{Slope} = 2 \quad \text{Point} = (0, 8)$$

$$\Rightarrow \text{equation of tangent: } y - 8 = 2(x - 0)$$

$$\Rightarrow y - 8 = 2x$$

$$\Rightarrow y = 2x + 8$$

Q13. $\ell_1 : y = 2e^x - x$
 $\Rightarrow \frac{dy}{dx} = 2e^x - 1$
Point $(0, 2) \Rightarrow \frac{dy}{dx} = 2e^0 - 1 = 2(1) - 1 = 1$
 \Rightarrow equation of tangent : $y - 2 = 1(x - 0)$
 $\Rightarrow y = x + 2$
 $\ell_2 : y = \sin 2x - x^2$
 $\Rightarrow \frac{dy}{dx} = \cos 2x \cdot 2 - 2x = 2 \cos 2x - 2x$
Point $(0, 0) \Rightarrow \frac{dy}{dx} = 2 \cos 2(0) - 2(0)$
 $= 2 \cos 0 - 0 = 2 \cdot 1 = 2$
 \Rightarrow equation of tangent : $y - 0 = 2(x - 0)$
 $\Rightarrow y = 2x$
Point of intersection: $y = 2x \cap y = x + 2$
 $\Rightarrow 2x = x + 2$
 $\Rightarrow x = 2$
 $\Rightarrow y = 2(2) = 4 \Rightarrow \text{Point} = (2, 4)$

Exercise 2.9

Q1. $y = \log_e 5x \Rightarrow \frac{dy}{dx} = \frac{1}{5x} \cdot 5 = \frac{1}{x}$

Q2. $y = \log_e (2x + 3) \Rightarrow \frac{dy}{dx} = \frac{1}{2x + 3} \cdot 2 = \frac{2}{2x + 3}$

Q3. $y = \log_e (3x^2) \Rightarrow \frac{dy}{dx} = \frac{1}{3x^2} \cdot 6x = \frac{2}{x}$

Q4. $y = \log_e (\sin x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cotan x$

Q5. $y = \log_e (x^2 - 6x) \Rightarrow \frac{dy}{dx} = \frac{1}{x^2 - 6x} \cdot 2x - 6 = \frac{2(x - 3)}{x^2 - 6x}$

Q6. $y = \log_e (\cos 3x) \Rightarrow \frac{dy}{dx} = \frac{1}{\cos 3x} \cdot -\sin 3x \cdot 3$
 $= \frac{-3 \sin 3x}{\cos 3x} = -3 \tan 3x$

Q7.

$$y = x \log_e x$$

$$\text{Product Rule} \Rightarrow u = x \quad \text{and} \quad v = \log_e x$$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = x \cdot \frac{1}{x} + \log_e x \cdot 1 = \log_e x + 1$$

Q8.

$$y = x^2 \ln(3x)$$

$$\text{Product Rule} \Rightarrow u = x^2 \quad \text{and} \quad v = \ln(3x)$$

$$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dv}{dx} = \frac{1}{3x} \cdot 3 = \frac{1}{x}$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = x^2 \cdot \frac{1}{x} + \ln 3x \cdot 2x \\ &= x + 2x \ln 3x \end{aligned}$$

Q9.

$$y = \frac{\ln x}{x}$$

$$\text{Quotient Rule} \Rightarrow u = \ln x \quad \text{and} \quad v = x$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x} \quad \Rightarrow \frac{dv}{dx} = 1$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} \\ &= \frac{1 - \ln x}{x^2} \end{aligned}$$

$$\text{Q10. (i)} \quad y = \log_e (3x+1)^3 = 3 \log_e (3x+1)$$

$$\Rightarrow \frac{dy}{dx} = 3 \cdot \frac{1}{3x+1} \cdot 3 = \frac{9}{3x+1}$$

$$\text{(ii)} \quad y = \log_e \left(\frac{2x+1}{1-3x} \right) = \log_e (2x+1) - \log_e (1-3x)$$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \frac{1}{2x+1} \cdot 2 - \frac{1}{1-3x} \cdot -3 \\ &= \frac{2}{2x+1} + \frac{3}{1-3x} \\ &= \frac{2(1-3x) + 3(2x+1)}{(2x+1)(1-3x)} \\ &= \frac{2-6x+6x+3}{(2x+1)(1-3x)} = \frac{5}{(2x+1)(1-3x)} \end{aligned}$$

$$\text{(iii)} \quad y = \log_e \sqrt{1+x^2} = \log_e (1+x^2)^{\frac{1}{2}} = \frac{1}{2} \log_e (1+x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1+x^2} \cdot 2x = \frac{x}{1+x^2}$$

$$(iv) \quad y = \log_e \sqrt{\sin x} = \log_e (\sin x)^{\frac{1}{2}} = \frac{1}{2} \log_e (\sin x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\sin x} \cdot \cos x = \frac{1}{2} \frac{\cos x}{\sin x} = \frac{1}{2} \cotan x$$

$$(v) \quad y = \log_e (x^2 + 4)^2 = 2 \log_e (x^2 + 4)$$

$$\Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{x^2 + 4} \cdot 2x = \frac{4x}{x^2 + 4}$$

$$(vi) \quad y = \log_e \sqrt{\frac{x}{1+x}} = \log_e \left(\frac{x}{1+x} \right)^{\frac{1}{2}}$$

$$= \frac{1}{2} \log_e \left(\frac{x}{1+x} \right) = \frac{1}{2} [\log x - \log(1+x)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x} - \frac{1}{1+x} \right)$$

$$= \frac{1}{2} \left[\frac{1(1+x) - 1(x)}{x(1+x)} \right]$$

$$= \frac{1}{2} \left[\frac{1+x-x}{x(1+x)} \right] = \frac{1}{2x(x+1)}$$

Q11. $y = \ln 3x^4 \Rightarrow \frac{dy}{dx} = \frac{1}{3x^4} \cdot 12x^3 = \frac{4}{x} = 4x^{-1}$

$$\Rightarrow \frac{d^2 y}{dx^2} = -4x^{-2} = \frac{-4}{x^2}$$

Q12. $y = [\log_e (x+4)]^2$

$$\Rightarrow \frac{dy}{dx} = 2 [\log_e (x+4)] \cdot \frac{1}{x+4} \cdot 1 = \frac{2 \log_e (x+4)}{x+4}$$

Q13. $y = x \log_e x$

Product Rule $\Rightarrow u = x$ and $v = \log_e x$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = x \cdot \frac{1}{x} + \log_e x \cdot 1 = 1 + \log_e x$$

$$\frac{d^2 y}{dx^2} = 0 + \frac{1}{x} = \frac{1}{x}$$

Q14. $y = \log_e x - 2x + x^2$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} - 2 + 2x$$

When $x = 2 \Rightarrow \frac{dy}{dx} = \frac{1}{2} - 2 + 2(2) = \frac{1}{2} - 2 + 4 = \frac{5}{2}$

Q15.

$$y = (\ln x)^2$$

$$\Rightarrow \frac{dy}{dx} = 2(\ln x) \cdot \frac{1}{x} = \frac{2\ln x}{x}$$

$$\text{At } x = e \Rightarrow \frac{dy}{dx} = \frac{2\ln e}{e} = \frac{2 \cdot 1}{e} = \frac{2}{e}$$

Q16.

$$y = \ln(1 + \sin t) \Rightarrow \frac{dy}{dt} = \frac{1}{1 + \sin t} \cdot \cos t = \frac{\cos t}{1 + \sin t}$$

$$\frac{dy}{dt} = \frac{\cos t}{1 + \sin t}$$

$$\text{Quotient Rule} \Rightarrow u = \cos t \quad \text{and} \quad v = 1 + \sin t$$

$$\frac{du}{dt} = -\sin t \Rightarrow \frac{dv}{dt} = \cos t$$

$$\begin{aligned} \frac{d^2 y}{dt^2} &= \frac{v \frac{du}{dt} - u \frac{dv}{dt}}{v^2} = \frac{(1 + \sin t)(-\sin t) - (\cos t)(\cos t)}{(1 + \sin t)^2} \\ &= \frac{-\sin t - \sin^2 t - \cos^2 t}{(1 + \sin t)^2} \\ &= \frac{-(\sin t + \sin^2 t + \cos^2 t)}{(1 + \sin t)^2} = \frac{-(1 + \sin t)}{(1 + \sin t)^2} = \frac{-1}{1 + \sin t} \end{aligned}$$

$$\text{If } (1 + \sin t) \frac{d^2 y}{dt^2} + k = 0$$

$$\Rightarrow (1 + \sin t) \cdot \frac{-1}{(1 + \sin t)} + k = 0$$

$$\Rightarrow -1 + k = 0$$

$$\Rightarrow k = 1$$

Q17.

$$\begin{aligned} y &= \ln(e^x \cos x) = \ln e^x + \ln \cos x \\ &= x + \ln \cos x \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{1}{\cos x} \cdot (-\sin x) = 1 - \frac{\sin x}{\cos x} = 1 - \tan x$$

Revision Exercise 2 (Core)

$$\text{Q1. (i)} \quad y = x^2 + \frac{1}{x} = x^2 + x^{-1}$$

$$\Rightarrow \frac{dy}{dx} = 2x - x^{-2} = 2x - \frac{1}{x^2}$$

$$\text{(ii)} \quad y = (2x + 3)^3$$

$$\Rightarrow \frac{dy}{dx} = 3(2x + 3)^2 \cdot 2 = 6(2x + 3)^2$$

$$\text{(iii)} \quad y = \sqrt{1 + 3x} = (1 + 3x)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(1 + 3x)^{-\frac{1}{2}} \cdot 3 = \frac{3}{2\sqrt{1 + 3x}}$$

Q2. $f(x) = x^2 + 3x - 4$

$$\Rightarrow f(x+h) = (x+h)^2 + 3(x+h) - 4$$

$$= x^2 + 2xh + h^2 + 3x + 3h - 4$$

$$\Rightarrow f(x+h) - f(x) = x^2 + 2xh + h^2 + 3x + 3h - 4 - (x^2 + 3x - 4)$$

$$= x^2 + 2xh + h^2 + 3x + 3h - 4 - x^2 - 3x + 4$$

$$= 2xh + h^2 + 3h$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 + 3h}{h} = 2x + h + 3$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (2x + h + 3) = 2x + 3$$

$$\therefore f'(x) = 2x + 3$$

Q3. (i) $y = \frac{1}{3}(x+2)^3$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3} \cdot 3(x+2)^2 \cdot 1 = (x+2)^2$$

(ii) $y = \frac{2x}{x+1}$

Quotient Rule : $u = 2x$ and $v = x+1$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x+1)(2) - (2x)(1)}{(x+1)^2}$$

$$= \frac{2x + 2 - 2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Q4. (i) $f(x) = 2x^2 - \frac{3}{x^2} = 2x^2 - 3x^{-2}$

$$\Rightarrow f'(x) = 4x + 6x^{-3} = 4x + \frac{6}{x^3}$$

(ii) $y = 4 \sin 6x \Rightarrow \frac{dy}{dx} = 4 \cos 6x \cdot 6 = 24 \cos 6x$

(iii) $y = 3e^{x^2} \Rightarrow \frac{dy}{dx} = 3e^{x^2} \cdot 2x = 6xe^{x^2}$

Q5. $y = \frac{2x+3}{x-4}$
 Quotient Rule $\Rightarrow u = 2x+3$ and $v = x-4$
 $\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \frac{dv}{dx} = 1$
 $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x-4).2 - (2x+3).1}{(x-4)^2}$
 $= \frac{2x-8-2x-3}{(x-4)^2} = \frac{-11}{(x-4)^2}$
 $\Rightarrow k = -11$

Q6. (i) $y = 6x^2 - x^3$
 $\Rightarrow \frac{dy}{dx} = 12x - 3x^2 = 12$ (Gradient)
 $\Rightarrow 3x^2 - 12x + 12 = 0$
 $\Rightarrow x^2 - 4x + 4 = 0$
 $\Rightarrow (x-2)(x-2) = 0 \Rightarrow x = 2$
 (ii) $x = 2 \Rightarrow y = 6(2)^2 - (2)^3 = 24 - 8 = 16$
 Point = (2, 16) slope = 12
 Equation of Tangent : $y - 16 = 12(x - 2)$
 $\Rightarrow y - 16 = 12x - 24$
 $\Rightarrow 12x - y - 8 = 0$

Q7. (i) $y = 3x^2 - x + \frac{3}{x} = 3x^2 - x + 3x^{-1}$
 $\Rightarrow \frac{dy}{dx} = 6x - 1 - 3x^{-2} = 6x - 1 - \frac{3}{x^2}$
 (ii) $y = \frac{3x^2}{x-1}$
 Quotient Rule : $u = 3x^2$ and $v = x-1$
 $\Rightarrow \frac{du}{dx} = 6x \quad \Rightarrow \frac{dv}{dx} = 1$
 $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x-1).(6x) - (3x^2).(1)}{(x-1)^2}$
 $= \frac{6x^2 - 6x - 3x^2}{(x-1)^2} = \frac{3x^2 - 6x}{(x-1)^2}$
 (iii) $y = \cos^2 4x$
 $\Rightarrow \frac{dy}{dx} = 2 \cos 4x \cdot -\sin 4x \cdot 4$
 $= -8 \cos 4x \sin 4x$

Q8. $y = \frac{4x^2 + 6}{x} = \frac{4x^2}{x} + \frac{6}{x} = 4x + 6x^{-1}$
 $\frac{dy}{dx} = 4 - 6x^{-2} = 4 - \frac{6}{x^2}$

Q9. $f(x) = a \sin 3x$
 $\Rightarrow f'(x) = a \cos 3x \cdot 3 = 3a \cos 3x$
 $\Rightarrow f'(\pi) = 3a \cos 3(\pi) = 2$
 $\Rightarrow 3a \cos 3\pi = 2$
 $\Rightarrow 3a(-1) = 2$
 $\Rightarrow -3a = 2 \Rightarrow a = -\frac{2}{3}$

Q10. $y = x \sin 2x$
Product Rule $\Rightarrow u = x$ and $v = \sin 2x$
 $\Rightarrow \frac{du}{dx} = 1 \Rightarrow \frac{dv}{dx} = \cos 2x \cdot 2 = 2 \cos 2x$
 $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = x(2 \cos 2x) + (\sin 2x) \cdot (1)$
 $= 2x \cos 2x + \sin 2x$
 $x = \frac{\pi}{3} \Rightarrow \frac{dy}{dx} = 2\left(\frac{\pi}{3}\right) \cos 2\left(\frac{\pi}{3}\right) + \sin 2\left(\frac{\pi}{3}\right)$
 $= \frac{2\pi}{3} \cos \frac{2\pi}{3} + \sin \frac{2\pi}{3}$
 $= \frac{2\pi}{3} \left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2} = -\frac{\pi}{3} + \frac{\sqrt{3}}{2}$

Q11. $\frac{dy}{dx} = (x+1)(x-2)$
 $P(1, 2) \Rightarrow \frac{dy}{dx} = (1+1)(1-2) = (2)(-1) = -2$
 \Rightarrow Equation of Tangent $\Rightarrow y - 2 = -2(x - 1)$
 $\Rightarrow y - 2 = -2x + 2$
 $\Rightarrow 2x + y - 4 = 0$

Q12. $f(x) = \sqrt{x} + \frac{1}{x^2} = x^{\frac{1}{2}} + x^{-2}$
 $\Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-3} = \frac{1}{2\sqrt{x}} - \frac{2}{x^3}$
 $\Rightarrow f'(4) = \frac{1}{2\sqrt{4}} - \frac{2}{(4)^3} = \frac{1}{4} - \frac{2}{64} = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$

Q13. $y = 2x^2 - 1$

(i) $x = 1 \Rightarrow y = 2(1)^2 - 1 = 1$ Point (1, 1)
 $x = 4 \Rightarrow y = 2(4)^2 - 1 = 31$ Point (4, 31)
 \Rightarrow average rate of change $= \frac{31-1}{4-1} = \frac{30}{3} = 10$

(ii) $\frac{dy}{dx} = 4x$

When $x = 4 \Rightarrow \frac{dy}{dx} = 4(4) = 16$

Q14. $y = \tan^{-1}(5x)$

Chain Rule $\Rightarrow u = 5x$ and $y = \tan^{-1} u$

$\Rightarrow \frac{du}{dx} = 5 \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot 5 = \frac{5}{1+(5x)^2} = \frac{5}{1+25x^2}$

Q15. $y = 2x^2 - 2x + 3$

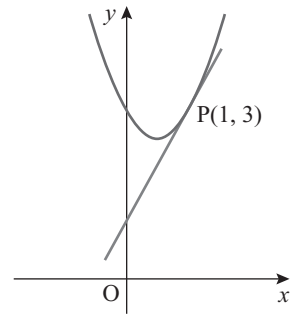
$\frac{dy}{dx} = 4x - 2$

P(1, 3) $\Rightarrow \frac{dy}{dx} = 4(1) - 2 = 2$

\Rightarrow Equation of Tangent : $y - 3 = 2(x - 1)$

$\Rightarrow y - 3 = 2x - 2$

$\Rightarrow 2x - y + 1 = 0$



Q16. $f(x) = 2x^{-3} + \frac{k}{2}x^{-2} - x$

$\Rightarrow f'(x) = -6x^{-4} + \frac{k}{2}(-2x^{-3}) - 1$

$= \frac{-6}{x^4} - \frac{k}{x^3} - 1$

$\Rightarrow f'(-2) = \frac{-6}{(-2)^4} - \frac{k}{(-2)^3} - 1 = 0$

$\Rightarrow \frac{-6}{16} - \frac{k}{-8} - 1 = 0$

$\Rightarrow \frac{-3}{8} + \frac{k}{8} - 1 = 0$

$\Rightarrow -3 + k - 8 = 0$

$\Rightarrow k = 11$

Revision Exercise 2 (Advanced)

Q1. $f(x) = \sin x - \cos x$
 $\Rightarrow f'(x) = \cos x - (-\sin x) = \cos x + \sin x$
At $x = \frac{\pi}{2} \Rightarrow f'\left(\frac{\pi}{2}\right) = \cos \frac{\pi}{2} + \sin \frac{\pi}{2}$
 $= 0 + 1 = 1$

Q2. $y = x^2 \sin x$
Product Rule $\Rightarrow u = x^2$ and $v = \sin x$
 $\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dv}{dx} = \cos x$
 $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = x^2(\cos x) + \sin x(2x)$
 $= x^2 \cos x + 2x \sin x$
At $x = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \left(\frac{\pi}{2}\right)^2 \left(\cos \frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) \sin \frac{\pi}{2}$
 $= \frac{\pi^2}{4} \cdot 0 + \pi \cdot 1 = \pi$

Q3. $y = x^2 + \ln x$
 $\Rightarrow \frac{dy}{dx} = 2x + \frac{1}{x} = 3$
 $\Rightarrow 2x^2 + 1 = 3x \Rightarrow 2x^2 - 3x + 1 = 0$
 $\Rightarrow (x-1)(2x-1) = 0$
 $\Rightarrow x = 1$ or $x = \frac{1}{2}$
 $\Rightarrow y = (1)^2 + \ln(1) = 1$ OR $y = \left(\frac{1}{2}\right)^2 + \ln \frac{1}{2}$
 $= \frac{1}{4} + \ln 1 - \ln 2$
 $= \frac{1}{4} - \ln 2$
Points are $(1, 1), \left(\frac{1}{2}, \frac{1}{4} - \ln 2\right)$

Q4. $y = x - 1 + \frac{1}{x-1} = x - 1 + (x-1)^{-1}$
 $\frac{dy}{dx} = 1 - 1(x-1)^{-2} = 1 - \frac{1}{(x-1)^2} = 0$
 $\Rightarrow (x-1)^2 - 1 = 0$
 $\Rightarrow x^2 - 2x + 1 - 1 = 0$
 $\Rightarrow x^2 - 2x = 0$
 $\Rightarrow x(x-2) = 0 \Rightarrow x = 0, 2$

Q5. (i) $y = \ln(3x^4) \Rightarrow \frac{dy}{dx} = \frac{1}{3x^4} \cdot 12x^3 = \frac{4}{x}$

(ii) $y = \ln\left(\frac{3}{\sqrt{x}}\right) = \ln 3 - \ln \sqrt{x} = \ln 3 - \ln x^{\frac{1}{2}}$
 $= \ln 3 - \frac{1}{2} \ln x$
 $\Rightarrow \frac{dy}{dx} = 0 - \frac{1}{2} \left(\frac{1}{x}\right) = -\frac{1}{2x}$

Q6. $y = e^{nx} \Rightarrow \frac{dy}{dx} = e^{nx} \cdot n = ne^{nx}$
 $\Rightarrow \frac{d^2y}{dx^2} = ne^{nx} \cdot n = n^2 e^{nx}$

Hence, $\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$
 $\Rightarrow n^2 e^{nx} - 5(ne^{nx}) + 6(e^{nx}) = 0$
 $\Rightarrow n^2 e^{nx} - 5ne^{nx} + 6e^{nx} = 0$
 $\Rightarrow e^{nx}(n^2 - 5n + 6) = 0$
 $\Rightarrow (n-2)(n-3) = 0 \Rightarrow n = 2, 3$

Q7. $y = x^3 - 3x^2 - 5x + 10$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 6x - 5$

Line $y = 4x - 7$ has slope = 4

Hence, $3x^2 - 6x - 5 = 4$

$\Rightarrow 3x^2 - 6x - 9 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x+1)(x-3) = 0$

$\Rightarrow x = -1, x = 3$

$\Rightarrow y = (-1)^3 - 3(-1)^2 - 5(-1) + 10$ and $y = (3)^3 - 3(3)^2 - 5(3) + 10$
 $= -1 - 3 + 5 + 10 = 11$ $= 27 - 27 - 15 + 10 = -5$

$\Rightarrow \text{Points} = (-1, 11), (3, -5)$

Q8. $y = a\sqrt{x} - 5 = ax^{\frac{1}{2}} - 5$

$\Rightarrow \frac{dy}{dx} = a \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{a}{2\sqrt{x}}$

At point (4, b) $\Rightarrow \frac{dy}{dx} = \frac{a}{2\sqrt{4}} = 2$

$\Rightarrow \frac{a}{4} = 2 \Rightarrow a = 8$

Curve: $y = 8\sqrt{x} - 5$

Point (4, b) $\Rightarrow b = 8\sqrt{4} - 5 = 16 - 5 = 11$

Q9. $V = 80(30 - t)^3$

(i) Point A $\Rightarrow t = 0 \Rightarrow V = 80(30 - 0)^3 = 2,160,000$
 $\Rightarrow A = (0, 2160000)$

Point B $\Rightarrow V = 0 \Rightarrow 80(30 - t)^3 = 0$
 $\Rightarrow t = 30$

$\Rightarrow B = (30, 0)$

$\Rightarrow \text{Full tank} = 2,160,000 \text{ m}^3$

and empty tank occurs after 30 mins

(ii) $t = 10 \Rightarrow V = 80(30 - 10)^3 = 640,000 \text{ m}^3$

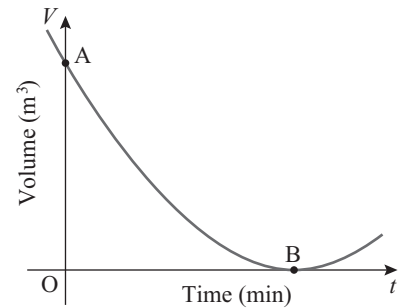
(iii) $(0, 2,160,000)$ and $(10, 640,000)$

$\Rightarrow \text{average rate} = \frac{640,000 - 2,160,000}{10 - 0} = \frac{1,520,000}{10}$
 $= 152,000 \text{ m}^3 / \text{min}$

(iv) $V = 80(30 - t)^3 \Rightarrow \frac{dV}{dt} = 240(30 - t)^2 \cdot -1 = -240(30 - t)^2$

$t = 10 \Rightarrow \frac{dV}{dt} = -240(30 - 10)^2 = -96,000$

$\Rightarrow \text{water is draining at rate} = 96,000 \text{ m}^3 / \text{min}$



Q10. $y = \frac{x}{\sqrt{1 - x^2}}$

Quotient Rule: $u = x$ and $v = \sqrt{1 - x^2} = (1 - x^2)^{\frac{1}{2}}$

$\Rightarrow \frac{du}{dx} = 1$ $\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} \cdot -2x$

$= \frac{-x}{\sqrt{1 - x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$= \frac{\sqrt{1 - x^2} \cdot 1 - x \cdot \frac{-x}{\sqrt{1 - x^2}}}{(\sqrt{1 - x^2})^2} = \frac{\sqrt{1 - x^2} \cdot \sqrt{1 - x^2} + x^2}{1 - x^2}$

$= \frac{1 - x^2 + x^2}{(1 - x^2)\sqrt{1 - x^2}} = \frac{1}{(1 - x^2)^{\frac{3}{2}}}$

$\Rightarrow k = \frac{3}{2}$

Q11.

$$y = \tan^{-1}\left(\frac{1}{x}\right)$$

Chain Rule: $u = \frac{1}{x} = x^{-1}$ and $y = \tan^{-1} u$

$$\Rightarrow \frac{du}{dx} = -1x^{-2} = \frac{-1}{x^2} \quad \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot \frac{-1}{x^2} = \frac{1}{1+\frac{1}{x^2}} \cdot \frac{-1}{x^2} = \frac{-1}{x^2+1}$$

At $x=1 \Rightarrow \frac{dy}{dx} = \frac{-1}{(1)^2+1} = -\frac{1}{2}$

Q12.

$$y = x^3 e^x$$

Product Rule: $u = x^3$ and $v = e^x$

$$\Rightarrow \frac{du}{dx} = 3x^2 \quad \Rightarrow \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = x^3 \cdot e^x + e^x \cdot 3x^2 = x^3 e^x + 3x^2 e^x$$

At $x=0 \Rightarrow \frac{dy}{dx} = (0)^3 e^0 + 3(0)^2 \cdot e^0 = 0+0=0 \Rightarrow \text{slope} = 0$

and $y = 0^2 \cdot e^0 = 0 \Rightarrow \text{Point } (0,0)$

$\Rightarrow \text{Equation of Tangent: } y-0 = 0(x-0) \Rightarrow y=0$

Q13.

$$y = kx^2 \Rightarrow \frac{dy}{dx} = k \cdot 2x = 2kx$$

$$x \frac{dy}{dx} + \frac{1}{2} \left(\frac{dy}{dx} \right)^2 + y = 0$$

$$\Rightarrow x \cdot 2kx + \frac{1}{2} (2kx)^2 + kx^2 = 0$$

$$\Rightarrow 2kx^2 + \frac{1}{2} \cdot 4k^2 x^2 + kx^2 = 0$$

$$\Rightarrow 2k^2 x^2 + 3kx^2 = 0$$

$$\Rightarrow 2k^2 + 3k = 0$$

$$\Rightarrow k(2k+3) = 0$$

$$\Rightarrow k=0 \quad \text{or} \quad k = \frac{-3}{2} \quad \Rightarrow \text{Ans: } k = \frac{-3}{2}$$

Q14.

$$f(x) = x^3 + x^2 - 1 \quad \Rightarrow \quad f'(x) = 3x^2 + 2x$$

$$x_1 = 1 \Rightarrow f(1) = (1)^3 + (1)^2 - 1 = 1 \quad \text{and} \quad f'(1) = 3(1)^2 + 2(1) = 5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{5} = \frac{4}{5}$$

Q15.

$$y = \ln(1+e^x) \Rightarrow \frac{dy}{dx} = \frac{1}{1+e^x} \cdot e^x = \frac{e^x}{1+e^x}$$

Quotient Rule: $u = e^x$ and $v = 1+e^x$

$$\Rightarrow \frac{du}{dx} = e^x \quad \Rightarrow \frac{dv}{dx} = e^x$$

$$\frac{d^2y}{dx^2} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(1+e^x) \cdot e^x - e^x \cdot e^x}{(1+e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2}$$

$$= \frac{e^x}{(1+e^x)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = \frac{e^x}{(1+e^x)^2} + \left(\frac{e^x}{1+e^x} \right)^2$$

$$= \frac{e^x}{(1+e^x)^2} + \frac{(e^x)^2}{(1+e^x)^2}$$

$$= \frac{e^x(1+e^x)}{(1+e^x)^2} = \frac{e^x}{1+e^x} = \frac{dy}{dx}$$

Q16.

$$y = x^3 - x + 1$$

(i) At B: $x = -1+h \Rightarrow y = (-1+h)^3 - (-1+h) + 1$

$$= -1 + 3h - 3h^2 + h^3 + 1 - h + 1$$

$$= 1 + 2h - 3h^2 + h^3$$

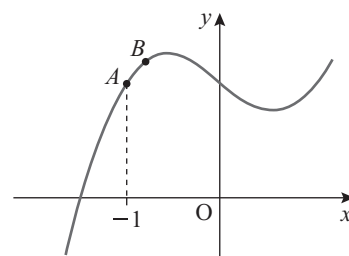
(ii) at A: $x = -1 \Rightarrow y = (-1)^3 - (-1) + 1 = -1 + 1 + 1 = 1$

A(-1, 1) B(-1+h, 1+2h-3h^2+h^3)

$$\text{Gradient } AB = \frac{1+2h-3h^2+h^3-1}{-1+h+1} = \frac{2h-3h^2+h^3}{h}$$

$$= 2-3h+h^2$$

(iii) As h becomes smaller \Rightarrow gradient $= 2-3(0)+(0)^2 = 2$



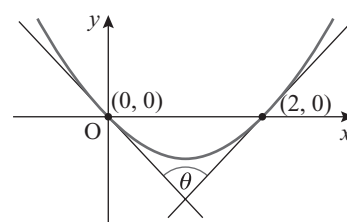
Revision Exercise 2 (Extended-Response)

Q1. (i) $y = x(x-2) = x^2 - 2x$

On x -axis, $y=0 \Rightarrow x(x-2)=0$

$$\Rightarrow x=0, x=2$$

\Rightarrow points are (0,0) and (2,0)



$$(ii) \quad \text{Slope} = \frac{dy}{dx} = 2x - 2$$

$$\text{At } x = 0 \Rightarrow \frac{dy}{dx} = 2(0) - 2 = -2$$

$$\text{At } x = 2 \Rightarrow \frac{dy}{dx} = 2(2) - 2 = +2$$

Curve is symmetrical; hence, the slopes of the tangents at any two points on the curve which have the same y -value will differ only in sign.

$$(iii) \quad \text{Point } (0,0), \text{ slope} = -2$$

$$\Rightarrow \text{equation of the tangent: } y - 0 = -2(x - 0)$$

$$\Rightarrow y = -2x$$

$$\Rightarrow 2x + y = 0$$

$$\text{Point } (2,0), \text{ slope} = +2$$

$$\Rightarrow \text{equation of the tangent: } y - 0 = 2(x - 2)$$

$$\Rightarrow y = 2x - 4$$

$$\Rightarrow 2x - y - 4 = 0$$

$$(iv) \quad \tan \theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2} = \pm \frac{2 - (-2)}{1 + 2(-2)} = \pm \frac{4}{-3} = \pm \frac{4}{3}$$

$$\text{acute angle} \Rightarrow \tan \theta = +\frac{4}{3}$$

$$\Rightarrow \theta = \tan^{-1} \frac{4}{3} = 53.13^\circ = 53^\circ$$

$$(v) \quad y = x(x-2)(x-5) = x^3 - 7x^2 + 10x$$

$$\text{On } x\text{-axis, } y = 0 \Rightarrow x(x-2)(x-5) = 0$$

$$\Rightarrow x = 0, x = 2 \text{ or } x = 5$$

$$\frac{dy}{dx} = 3x^2 - 14x + 10$$

$$\text{At } x = 0 \Rightarrow \frac{dy}{dx} = 3(0)^2 - 14(0) + 10 = 10 = p$$

$$\text{At } x = 2 \Rightarrow \frac{dy}{dx} = 3(2)^2 - 14(2) + 10 = -6 = q$$

$$\text{At } x = 5 \Rightarrow \frac{dy}{dx} = 3(5)^2 - 14(5) + 10 = 15 = r$$

$$\text{Hence, } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{10} + \frac{1}{-6} + \frac{1}{15}$$

$$= \frac{1}{10} - \frac{1}{6} + \frac{1}{15} = 0$$

Q2. (i) $y = 2\ln(x\sqrt{x^2+1})$

$$= 2\ln\left[x.(x^2+1)^{\frac{1}{2}}\right]$$

$$= 2\left[\ln x + \ln(x^2+1)^{\frac{1}{2}}\right]$$

$$= 2\left[\ln x + \frac{1}{2}\ln(x^2+1)\right]$$

$$= 2\ln x + \ln(x^2+1)$$

$$\frac{dy}{dx} = 2 \cdot \frac{1}{x} + \frac{1}{x^2+1} \cdot 2x = \frac{2}{x} + \frac{2x}{x^2+1}$$

$$= \frac{2(x^2+1) + 2x(x)}{x(x^2+1)} = \frac{2x^2+2+2x^2}{x(x^2+1)}$$

$$= \frac{4x^2+2}{x(x^2+1)}$$

$$= \frac{2(2x^2+1)}{x(x^2+1)} \Rightarrow k = 2$$

(ii) $y = 2\ln(x\sqrt{x^2+1})$

When $x = 1 \Rightarrow y = 2\ln(1\sqrt{1^2+1}) = 2\ln\sqrt{2} = \ln(\sqrt{2})^2 = \ln 2 \Rightarrow \text{point} = (1, \ln 2)$

$$\frac{dy}{dx} = \frac{2(2x^2+1)}{x(x^2+1)}$$

When $x = 1 \Rightarrow \text{slope} = \frac{dy}{dx} = \frac{2(2(1)^2+1)}{1((1)^2+1)} = 3$

\Rightarrow equation of the tangent: $y - y_1 = m(x - x_1)$

$$\Rightarrow y - \ln 2 = 3(x - 1)$$

$$\Rightarrow y - \ln 2 = 3x - 3$$

$$\Rightarrow y = 3x + \ln 2 - 3$$

Q3. $y = \frac{x^2}{4} - x = \frac{1}{4}x^2 - x$

(i) $\frac{dy}{dx} = \frac{1}{4}(2x) - 1 = \frac{x}{2} - 1$

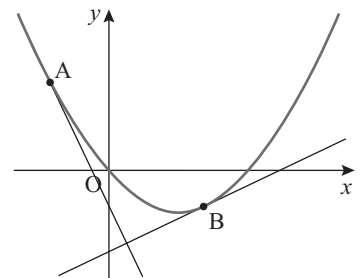
When $x = 3 \Rightarrow \text{slope} = \frac{dy}{dx} = \frac{3}{2} - 1 = \frac{1}{2}$ and point B $\left(3, -\frac{3}{4}\right)$

\Rightarrow equation of the tangent: $y + \frac{3}{4} = \frac{1}{2}(x - 3)$

$$\Rightarrow 2y + \frac{3}{2} = x - 3$$

$$\Rightarrow 4y + 3 = 2x - 6$$

$$\Rightarrow 2x - 4y - 9 = 0$$



$$(ii) \quad \text{slope} = \frac{1}{2} \Rightarrow \text{perpendicular slope} = -2$$

$$\Rightarrow \frac{x}{2} - 1 = -2 \Rightarrow x - 2 = -4 \Rightarrow x = -2$$

$$\text{When } x = -2 \Rightarrow y = \frac{(-2)^2}{4} - (-2) = 3 \Rightarrow A = (-2, 3)$$

$$(iii) \quad \text{Tangent at } A(-2, 3) \text{ with slope} = -2$$

$$\Rightarrow y - 3 = -2(x + 2)$$

$$\Rightarrow y - 3 = -2x - 4 \Rightarrow 2x + y = -1$$

$$\text{and } -2x + 4y = -9$$

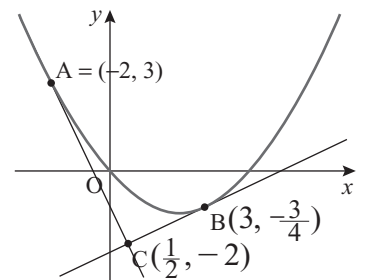
$$\text{add } \Rightarrow 5y = -10 \Rightarrow y = -2$$

$$\Rightarrow 2x - 2 = -1$$

$$\Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2} \Rightarrow \text{point } C = \left(\frac{1}{2}, -2\right)$$

$$(iv) \quad A(-2, 3), B\left(3, -\frac{3}{4}\right), C\left(\frac{1}{2}, -2\right)$$

$$\begin{array}{ccc} +2 \downarrow -3 & +2 \downarrow -3 & +2 \downarrow -3 \\ (0, 0) & \left(5, -3\frac{3}{4}\right) & \left(2\frac{1}{2}, -5\right) \end{array}$$



$$\text{Area Triangle ABC} = \frac{1}{2} |x_1 y_2 - x_2 y_1|$$

$$= \frac{1}{2} \left| (5)(-5) - \left(2\frac{1}{2}\right)\left(-3\frac{3}{4}\right) \right| = \frac{1}{2} \left| \frac{-125}{8} \right| = \frac{125}{16}$$

$$\text{Q4. (a)} \quad f(x) = x^3 - x$$

$$\text{at } Q, x = (2 + h) \Rightarrow f(2 + h) = (2 + h)^3 - (2 + h) = h^3 + 6h^2 + 12h + 8 - 2 - h$$

$$= h^3 + 6h^2 + 11h + 6$$

$$\Rightarrow Q = (2 + h, h^3 + 6h^2 + 11h + 6) \text{ and } P = (2, 6)$$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{h^3 + 6h^2 + 11h + 6 - 6}{2 + h - 2} = \frac{h^3 + 6h^2 + 11h}{h}$$

$$(b) (i) \quad h = 0.5 \Rightarrow \text{slope} = \frac{(0.5)^3 + 6(0.5)^2 + 11(0.5)}{0.5} = \frac{7.125}{0.5} = 14.25$$

$$(ii) \quad h = 0.1 \Rightarrow \text{slope} = \frac{(0.1)^3 + 6(0.1)^2 + 11(0.1)}{0.1} = \frac{1.161}{0.1} = 11.61$$

$$(iii) \quad h = 0.01 \Rightarrow \text{slope} = \frac{(0.01)^3 + 6(0.01)^2 + 11(0.01)}{0.01} = \frac{0.110601}{0.01} = 11.0601$$

$$(iv) \quad h = 0.001 \Rightarrow \text{slope} = \frac{(0.001)^3 + 6(0.001)^2 + 11(0.001)}{0.001} = \frac{0.011006001}{0.001} = 11.006001$$

$$(c) \quad \text{As } h \rightarrow 0, \text{ slope of PQ} = 11$$

$$(d) \quad \text{Hence, slope of the curve at } P(2, 6) = 11$$

- (e) At Q, $x = (a+h) \Rightarrow f(a+h) = (a+h)^3 - (a+h)$
 $= a^3 + 3a^2h + 3ah^2 + h^3 - a - h$
 $\Rightarrow Q = [a+h, a^3 + 3a^2h + 3ah^2 + h^3 - a - h]$ $P = [a, a^3 - a]$
 slope PQ = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a - h - a^3 + a}{a+h-a}$
 $= \frac{3a^2h + 3ah^2 + h^3 - h}{h}$
 $= 3a^2 + 3ah + h^2 - 1$
- (f) As $h \rightarrow 0$, slope = $3a^2 + 3a(0) + (0)^2 - 1 = 3a^2 - 1$

Q5. (a)

$$f(x) = e^{-\frac{1}{2}x^2}$$

$$\Rightarrow f'(x) = e^{-\frac{1}{2}x^2} \cdot -\frac{1}{2}(2x) = -x.e^{-\frac{1}{2}x^2}$$

Find $f''(x) \Rightarrow$ Product Rule: $u = -x$ and $v = e^{-\frac{1}{2}x^2}$

$$\Rightarrow \frac{du}{dx} = -1 \quad \Rightarrow \quad \frac{dv}{dx} = -xe^{-\frac{1}{2}x^2}$$

$$\Rightarrow f''(x) = u \frac{dv}{dx} + v \frac{du}{dx} = (-x) \cdot \left(-xe^{-\frac{1}{2}x^2}\right) + \left(e^{-\frac{1}{2}x^2}\right)(-1)$$

$$= (x^2 - 1)e^{-\frac{1}{2}x^2}$$

(b)

Point of inflection at P $\Rightarrow f''(x) = 0$

$$\Rightarrow (x^2 - 1).e^{-\frac{1}{2}x^2} = 0$$

$$\Rightarrow (x-1)(x+1) = 0$$

$$\Rightarrow x = 1 \quad \text{or} \quad x = -1 \quad (\text{Not in first quadrant})$$

When $x = 1 \Rightarrow$ slope = $f'(1) = (-1)e^{-\frac{1}{2}(1)^2} = (-1)e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}$

When $x = 1 \Rightarrow f(1) = e^{-\frac{1}{2}(1)^2} = e^{-\frac{1}{2}} \Rightarrow$ Point P $\left(1, e^{-\frac{1}{2}}\right)$

\Rightarrow Equation of Tangent: $y - y_1 = m(x - x_1)$

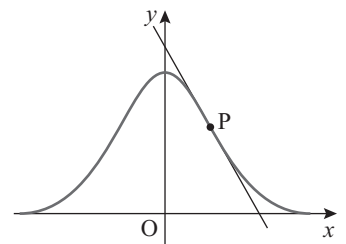
$$\Rightarrow y - e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}(x - 1)$$

$$\Rightarrow y - e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}x + e^{-\frac{1}{2}}$$

$$\Rightarrow e^{-\frac{1}{2}}x + y = 2e^{-\frac{1}{2}}$$

On x -axis, $y = 0 \Rightarrow e^{-\frac{1}{2}}x + 0 = 2e^{-\frac{1}{2}}$

$$\Rightarrow x = \frac{2e^{-\frac{1}{2}}}{e^{-\frac{1}{2}}} = 2 \Rightarrow \text{Point} = (2, 0)$$



Q6. $D = 50e^{kt}$

(a) $\frac{dD}{dt} = 50e^{kt} \cdot k = k \cdot 50e^{kt} = kD = \text{a constant times } D$

(b) $k = 0.2$ and $D = 100 \Rightarrow 50e^{0.2t} = 100$

$$\Rightarrow e^{0.2t} = \frac{100}{50} = 2$$

$$\Rightarrow \ln e^{0.2t} = \ln 2$$

$$\Rightarrow 0.2t(\ln e) = \ln 2$$

$$\Rightarrow t = \frac{\ln 2}{0.2} = 3.4657$$

$$\begin{aligned} \Rightarrow \frac{dD}{dt} &= (0.2)50e^{(0.2)(3.4657)} \\ &= 10e^{0.69314} = 19.99 = 20 \text{ cm/year} \end{aligned}$$

Q7. (i) $y = \ln \sqrt{1 + \sin 2x} = \ln(1 + \sin 2x)^{\frac{1}{2}}$

$$= \frac{1}{2} \ln(1 + \sin 2x)$$

$$\text{Slope} = \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + \sin 2x} \cdot \cos 2x \cdot 2 = \frac{\cos 2x}{1 + \sin 2x}$$

$$\text{When } x = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{\cos 2\left(\frac{\pi}{2}\right)}{1 + \sin 2\left(\frac{\pi}{2}\right)} = \frac{\cos \pi}{1 + \sin \pi} = \frac{-1}{1 + 0} = -1$$

(ii) $y = (x^2 - 1)^n \Rightarrow \frac{dy}{dx} = n(x^2 - 1)^{n-1} \cdot (2x) = 2nx(x^2 - 1)^{n-1}$

$$\begin{aligned} \text{Hence, } (x^2 - 1) \frac{dy}{dx} - 2nxy &= (x^2 - 1) \cdot 2nx(x^2 - 1)^{n-1} - 2nx(x^2 - 1)^n \\ &= 2nx \cdot (x^2 - 1) \cdot (x^2 - 1)^{n-1} - 2nx(x^2 - 1)^n \\ &= 2nx \cdot (x^2 - 1)^n - 2nx(x^2 - 1)^n \\ &= 0 \end{aligned}$$

(iii) $f(x) = x^3 - 6x^2 + 12x + 5$

$$\begin{aligned} \Rightarrow f'(x) &= 3x^2 - 12x + 12 \\ &= 3(x^2 - 4x + 4) \\ &= 3(x - 2)(x - 2) \\ &= 3(x - 2)^2, \text{ which is always positive} \end{aligned}$$

Q8. $f(x) = x(x - k)^2$

$$= x(x^2 - 2kx + k^2) = x^3 - 2kx^2 + k^2x$$

(i) $f'(x) = 3x^2 - 4kx + k^2$

$$= (x - k)(3x - k)$$

$$\begin{aligned}
\text{(ii)} \quad \text{Tangents are parallel to } x\text{-axis} &\Rightarrow f'(x) = 0 \\
&\Rightarrow (x-k)(3x-k) = 0 \\
&\Rightarrow x = k \quad \text{or} \quad 3x = k \\
&\Rightarrow x = \frac{k}{3}
\end{aligned}$$

$$\text{When } x = k \Rightarrow f(k) = k(k-k)^2 = k \cdot 0 = 0 \Rightarrow \text{Point } (k, 0)$$

$$\begin{aligned}
\text{When } x = \frac{k}{3} \Rightarrow f\left(\frac{k}{3}\right) &= \frac{k}{3} \left(\frac{k}{3} - k\right)^2 = \frac{k}{3} \left(\frac{-2k}{3}\right)^2 = \frac{k}{3} \cdot \frac{4k^2}{9} \\
&= \frac{4k^3}{27} \Rightarrow \text{Point } \left(\frac{k}{3}, \frac{4k^3}{27}\right)
\end{aligned}$$

$$\text{(iii)} \quad (k, 0), \left(\frac{k}{3}, \frac{4k^3}{27}\right) \Rightarrow \text{slope} = \frac{\frac{4k^3}{27} - 0}{\frac{k}{3} - k} = \frac{\frac{4k^3}{27}}{\frac{-2k}{3}} = -\frac{2k^2}{9}$$

$$\begin{aligned}
\text{Equation of the line: } y - 0 &= \frac{-2k^2}{9}(x - k) \\
&\Rightarrow y = \frac{-2k^2}{9}(x - k)
\end{aligned}$$

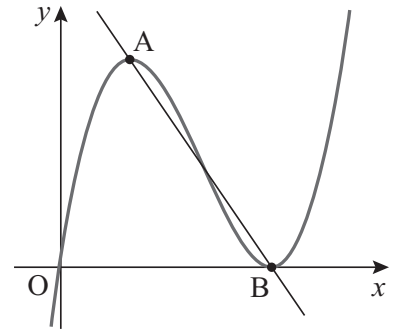
$$\text{(iv)} \quad A\left(\frac{k}{3}, \frac{4k^3}{27}\right), B(k, 0)$$

$$\text{Midpoint} = \left[\frac{\frac{k}{3} + k}{2}, \frac{\frac{4k^3}{27} + 0}{2} \right] = \left(\frac{2k}{3}, \frac{2k^3}{27} \right)$$

$$f(x) = x(x-k)^2$$

$$\begin{aligned}
\Rightarrow f\left(\frac{2k}{3}\right) &= \frac{2k}{3} \left(\frac{2k}{3} - k\right)^2 \\
&= \frac{2k}{3} \left(\frac{-k}{3}\right)^2 \\
&= \frac{2k}{3} \cdot \frac{k^2}{9} = \frac{2k^3}{27}
\end{aligned}$$

Hence, AB intersects the curve at the midpoint of [AB].



Chapter 3: Applications of Differential Calculus

Exercise 3.1

Q1. (i) $y = x^2 - 3x + 2$

$$\Rightarrow \frac{dy}{dx} = 2x - 3$$

$$\text{At } (1, 0) \Rightarrow \frac{dy}{dx} = 2(1) - 3 = -1$$

(ii) $y = x + \frac{1}{x} = x + x^{-1}$

$$\Rightarrow \frac{dy}{dx} = 1 - 1x^{-2} = 1 - \frac{1}{x^2}$$

$$\begin{aligned} \text{At } \left(\frac{1}{2}, \frac{5}{2}\right) \Rightarrow \frac{dy}{dx} &= 1 - \frac{1}{\left(\frac{1}{2}\right)^2} \\ &= 1 - \frac{1}{\frac{1}{4}} = 1 - 4 = -3 \end{aligned}$$

Q2. $f(x) = 2x^2 - 4x - 5$

$$\Rightarrow f'(x) = 4x - 4$$

$$\text{At } (3, 1) \Rightarrow f'(3) = 4(3) - 4 = 8$$

$$\Rightarrow \text{Equation of Tangent: } y - 1 = 8(x - 3)$$

$$\Rightarrow y - 1 = 8x - 24$$

$$\Rightarrow 8x - y - 23 = 0$$

Q3. $f(x) = x^2 - 6x$

$$\Rightarrow f'(x) = 2x - 6$$

$$\text{Where } x = 2 \Rightarrow f'(2) = 2(2) - 6 = -2 \text{ (slope)}$$

$$\text{Where } x = 2 \Rightarrow f(2) = (2)^2 - 6(2) = -8$$

$$\Rightarrow \text{Point } (2, -8)$$

$$\Rightarrow \text{Equation of Tangent: } y + 8 = -2(x - 2)$$

$$\Rightarrow y + 8 = -2x + 4$$

$$\Rightarrow 2x + y + 4 = 0$$

Q4. $y = x^3 + \frac{1}{2x^2} = x^3 + \frac{1}{2}x^{-2}$
 $\Rightarrow \frac{dy}{dx} = 3x^2 - 1x^{-3} = 3x^2 - \frac{1}{x^3}$
where $x = 1 \Rightarrow \frac{dy}{dx} = 3(1)^2 - \frac{1}{(1)^3} = 3 - 1 = 2$ (slope)
where $x = 1 \Rightarrow y = (1)^3 + \frac{1}{2(1)^2} = 1 + \frac{1}{2} = 1\frac{1}{2}$
 $\Rightarrow \text{Point} \left(1, 1\frac{1}{2} \right)$
 $\Rightarrow \text{Equation of Tangent: } y - 1\frac{1}{2} = 2(x - 1)$
 $\Rightarrow y - 1\frac{1}{2} = 2x - 2$
 $\Rightarrow 2y - 3 = 4x - 4$
 $\Rightarrow 4x - 2y - 1 = 0$

Q5. $y = x^2 + kx$
 $\Rightarrow \frac{dy}{dx} = 2x + k$
where $x = -1 \Rightarrow \frac{dy}{dx} = 2(-1) + k = 3$
 $\Rightarrow -2 + k = 3$
 $\Rightarrow k = 5$

Q6. $y = x^2 + 3x - 1$
 $\Rightarrow \frac{dy}{dx} = 2x + 3 = 5$
 $\Rightarrow 2x = 2$
 $\Rightarrow x = 1$
 $\Rightarrow y = (1)^2 + 3(1) - 1 = 1 + 3 - 1 = 3$
 $\Rightarrow \text{Point} = (1, 3)$

Q7. $y = x^2 + 4x + 6$
 $\Rightarrow \frac{dy}{dx} = 2x + 4 = -2$
 $\Rightarrow 2x = -6$
 $\Rightarrow x = -3$
 $\Rightarrow y = (-3)^2 + 4(-3) + 6$
 $= 9 - 12 + 6 = 3$
 $\Rightarrow \text{Point} = (-3, 3)$

Q8.

$$y = \frac{5x^2}{1+x^2}$$

Quotient Rule: $u = 5x^2$ and $v = 1+x^2$

$$\Rightarrow \frac{du}{dx} = 10x \quad \Rightarrow \frac{dv}{dx} = 2x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(1+x^2)(10x) - (5x^2)(2x)}{(1+x^2)^2} \\ &= \frac{10x + 10x^3 - 10x^3}{(1+x^2)^2} \\ &= \frac{10x}{(1+x^2)^2} \end{aligned}$$

$$\text{Point } (2, 4) \Rightarrow \frac{dy}{dx} = \frac{10(2)}{[1+(2)^2]^2} = \frac{20}{(5)^2} = \frac{20}{25} = \frac{4}{5}$$

$$\Rightarrow \text{Equation of Tangent: } y - 4 = \frac{4}{5}(x - 2)$$

$$\Rightarrow 5y - 20 = 4x - 8$$

$$\Rightarrow 4x - 5y + 12 = 0$$

Q9.

$$y = x^3 - 12x + 4$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 12 = 0$$

$$\Rightarrow x^2 - 4 = 0$$

$$\Rightarrow (x+2)(x-2) = 0$$

$$\Rightarrow x = -2 \quad \text{OR} \quad x = 2$$

$$\begin{aligned} \Rightarrow y &= (-2)^3 - 12(-2) + 4 \quad \text{and} \quad y = (2)^3 - 12(2) + 4 \\ &= -8 + 24 + 4 = 20 \quad \quad \quad = 8 - 24 + 4 = -12 \end{aligned}$$

$$\Rightarrow \text{Points} = (-2, 20) \quad \text{and} \quad (2, -12)$$

Q10.

$$y = ax^2 + bx + 5$$

$$\Rightarrow \frac{dy}{dx} = 2ax + b$$

$$\text{Point } (5, 0) \Rightarrow \frac{dy}{dx} = 2a(5) + b = 4$$

$$\Rightarrow 10a + b = 4$$

$$\text{Point } (5, 0) \Rightarrow 0 = a(5)^2 + b(5) + 5$$

$$\Rightarrow 0 = 25a + 5b + 5$$

$$\Rightarrow 5a + b = -1$$

$$\text{and} \quad \underline{-10a - b = -4}$$

$$\Rightarrow \quad \underline{-5a} \quad = -5$$

$$\Rightarrow a = 1$$

$$\Rightarrow 10(1) + b = 4 \Rightarrow b = -6$$

Q11.

$$y = ax^2 + b$$

$$\Rightarrow \frac{dy}{dx} = 2ax$$

$$\text{Point } (2, -2) \Rightarrow \frac{dy}{dx} = 2a(2) = 3$$

$$\Rightarrow 4a = 3 \Rightarrow a = \frac{3}{4}$$

$$\text{Hence, } y = \frac{3}{4}x^2 + b$$

$$\text{Point } (2, -2) \Rightarrow -2 = \frac{3}{4}(2)^2 + b$$

$$\Rightarrow -2 = 3 + b$$

$$\Rightarrow b = -5$$

Q12.

$$y = \ln x + x - 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} + 1$$

$$\text{where } x = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{1} + 1 = 2 \text{ (slope)}$$

$$\text{where } x = 1 \Rightarrow y = \ln(1) + 1 - 2 = 0 - 1 = -1$$

$$\Rightarrow \text{Point } (1, -1)$$

$$\Rightarrow \text{Equation of Tangent: } y + 1 = 2(x - 1)$$

$$\Rightarrow y + 1 = 2x - 2$$

$$\Rightarrow 2x - y - 3 = 0$$

Q13.

$$y = e^{3x} \Rightarrow \frac{dy}{dx} = e^{3x} \cdot 3 = 3e^{3x}$$

$$\text{Point } (0, 1) \Rightarrow \frac{dy}{dx} = 3e^{3(0)} = 3 \cdot e^0 = 3 \cdot 1 = 3$$

$$\Rightarrow \text{Equation of Tangent: } y - 1 = 3(x - 0)$$

$$\Rightarrow y - 1 = 3x - 0$$

$$\Rightarrow 3x - y + 1 = 0$$

Q14.

$$y = x^3 - 3x^2 - 5x + 10$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 6x - 5$$

$$\text{Line: } y = 4x - 7 \Rightarrow \text{slope} = 4$$

$$\text{Hence, } 3x^2 - 6x - 5 = 4$$

$$\Rightarrow 3x^2 - 6x - 9 = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x + 1)(x - 3) = 0$$

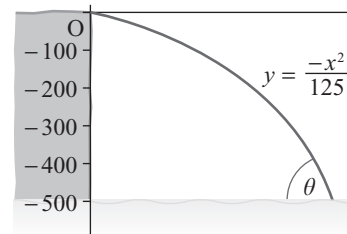
$$\Rightarrow x = -1 \quad \text{OR} \quad x = 3$$

$$\begin{aligned} x = -1 &\Rightarrow y = (-1)^3 - 3(-1)^2 - 5(-1) + 10 \\ &= -1 - 3 + 5 + 10 = 11 \Rightarrow \text{Point } (-1, 11) \end{aligned}$$

$$\begin{aligned} x = 3 &\Rightarrow y = (3)^3 - 3(3)^2 - 5(3) + 10 \\ &= 27 - 27 - 15 + 10 = -5 \end{aligned}$$

$$\Rightarrow \text{Point } (3, -5)$$

Q15. (i) $y = -\frac{x^2}{125} = -500$
 $\Rightarrow -x^2 = -62500$
 $\Rightarrow x^2 = 62500$
 $\Rightarrow x = \sqrt{62500} = 250 \text{ m}$



(ii) $y = -\frac{1}{125}x^2 \Rightarrow \frac{dy}{dx} = \frac{-2}{125}x$
at $x = 250 \Rightarrow \frac{dy}{dx} = \frac{-2}{125}(250) = -4$
 $\Rightarrow \tan \theta = -4$
 $\Rightarrow \theta = \tan^{-1}(-4) = 104.036^\circ = 104^\circ$

The angle at which the stone enters the water is $180^\circ - 104^\circ = 76^\circ$.

Q16. (i) Curve is increasing $\Rightarrow \frac{dy}{dx} > 0$ (Positive)

(ii) Curve is decreasing $\Rightarrow \frac{dy}{dx} < 0$ (Negative)

(a) $y = x^2 - x - 6$

$\Rightarrow \frac{dy}{dx} = 2x - 1$

Turning point $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 2x - 1 = 0$

$\Rightarrow x = \frac{1}{2}$

Function is decreasing $\Rightarrow 2x - 1 < 0$

$\Rightarrow 2x < 1$

$\Rightarrow x < \frac{1}{2}$

(b) $y = x^3 + 6x^2 - 2$

$\Rightarrow \frac{dy}{dx} = 3x^2 + 12x$

Turning points $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 3x^2 + 12x = 0$

$\Rightarrow x^2 + 4x = 0$

$\Rightarrow x(x + 4) = 0$

$\Rightarrow x = 0 \text{ OR } x = -4$

Function is decreasing $\Rightarrow 3x^2 + 12x < 0$

$\Rightarrow x^2 + 4x < 0$

$\Rightarrow -4 < x < 0$

Q17. $f(x) = 4x^2 + 4x + 7$

(i) $\Rightarrow f'(x) = 8x + 4$

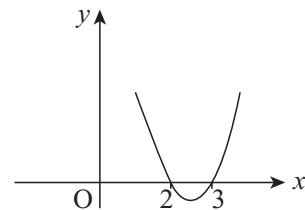
$$\begin{aligned}
 \text{(ii)(a)} \quad f(x) \text{ is increasing} &\Rightarrow f'(x) > 0 \\
 &\Rightarrow 8x + 4 > 0 \\
 &\Rightarrow 2x + 1 > 0 \\
 &\Rightarrow 2x > -1 \\
 &\Rightarrow x > -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad f(x) \text{ is decreasing} &\Rightarrow f'(x) < 0 \\
 &\Rightarrow 8x + 4 < 0 \\
 &\Rightarrow 2x + 1 < 0 \\
 &\Rightarrow 2x < -1 \\
 &\Rightarrow x < -\frac{1}{2}
 \end{aligned}$$

Q18. (i) $f(x) = 4x - 3x^2$
 $\Rightarrow f'(x) = 4 - 6x$
 $f(x)$ is increasing $\Rightarrow f'(x) > 0$
 $\Rightarrow 4 - 6x > 0$
 $\Rightarrow 2 - 3x > 0$
 $\Rightarrow -3x > -2$
 $\Rightarrow 3x < 2$
 $\Rightarrow x < \frac{2}{3}$

(ii) $f(x) = 3x^2 + 8x + 2$
 $\Rightarrow f'(x) = 6x + 8$
 $f(x)$ is increasing $\Rightarrow f'(x) > 0$
 $\Rightarrow 6x + 8 > 0$
 $\Rightarrow 3x + 4 > 0$
 $\Rightarrow 3x > -4$
 $\Rightarrow x > -\frac{4}{3}$

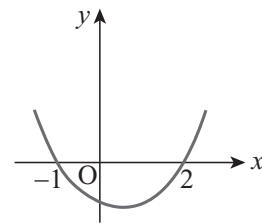
(iii) $f(x) = 2x^3 - 15x^2 + 36x$
 $\Rightarrow f'(x) = 6x^2 - 30x + 36$
 $f(x)$ is increasing $\Rightarrow f'(x) > 0$
 $\Rightarrow 6x^2 - 30x + 36 > 0$
 $\Rightarrow x^2 - 5x + 6 > 0$
 Factors $\Rightarrow (x - 2)(x - 3) = 0$
 Roots $\Rightarrow x = 2, 3$
 Solution: $x < 2$ OR $x > 3$



Q19. (i) $f(x) = 3x - 5x^2$
 $\Rightarrow f'(x) = 3 - 10x$
 $f(x)$ is decreasing $\Rightarrow f'(x) < 0$
 $\Rightarrow 3 - 10x < 0$
 $\Rightarrow -10x < -3$
 $\Rightarrow 10x > 3$
 $\Rightarrow x > \frac{3}{10}$ (OR $x > 0.3$)

(ii) $f(x) = 4 - 2x - x^2$
 $\Rightarrow f'(x) = -2 - 2x$
 $f(x)$ is decreasing $\Rightarrow f'(x) < 0$
 $\Rightarrow -2 - 2x < 0$
 $\Rightarrow -2x < 2$
 $\Rightarrow 2x > -2$
 $\Rightarrow x > -1$

(iii) $f(x) = 2x^3 - 3x^2 - 12x$
 $\Rightarrow f'(x) = 6x^2 - 6x - 12$
 $f(x)$ is decreasing $\Rightarrow f'(x) < 0$
 $\Rightarrow 6x^2 - 6x - 12 < 0$
 $\Rightarrow x^2 - x - 2 < 0$
Factors $\Rightarrow (x+1)(x-2) = 0$
Roots $\Rightarrow x = -1, 2$
Solution: $-1 < x < 2$



Q20. $f(x) = x^3 - 6x^2 + 18x + 4$
 $f'(x) = 3x^2 - 12x + 18$
 $f(x)$ is increasing $\Rightarrow f'(x) > 0$
 $\Rightarrow f'(x) = 3x^2 - 12x + 18 > 0$
 $\Rightarrow x^2 - 4x + 6 > 0$
 $\Rightarrow x^2 - 4x + 4 + 2 > 0$
 $\Rightarrow (x-2)^2 + 2 > 0$ True

Q21.

$$y = \frac{2x+1}{3x+6}$$

Quotient Rule : $u = 2x+1$ and $v = 3x+6$

$$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \quad \frac{dv}{dx} = 3$$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(3x+6).(2) - (2x+1).(3)}{(3x+6)^2} \\ &= \frac{6x+12-6x-3}{(3x+6)^2} \\ &= \frac{9}{(3x+6)^2} \end{aligned}$$

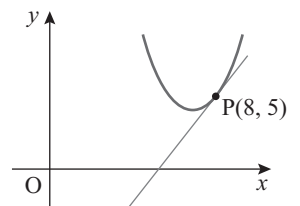
$$\begin{aligned} f(x) \text{ is increasing} &\Rightarrow f'(x) > 0 \\ &\Rightarrow \frac{9}{(3x+6)^2} > 0 \quad \text{True} \end{aligned}$$

Q22. (i) $y = x^2 - 14x + 53$

$$\Rightarrow \frac{dy}{dx} = 2x - 14$$

Point (8, 5) $\Rightarrow \frac{dy}{dx} = 2(8) - 14 = 2$

$$\begin{aligned} \Rightarrow \text{Equation of Tangent: } y - 5 &= 2(x - 8) \\ &\Rightarrow y - 5 = 2x - 16 \\ &\Rightarrow 2x - y - 11 = 0 \end{aligned}$$



(ii) $y = -x^2 + 10x - 27$

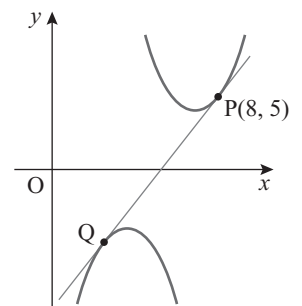
$$\Rightarrow \frac{dy}{dx} = -2x + 10 = 2$$

$$\Rightarrow -2x = -8$$

$$\Rightarrow x = 4$$

$$\begin{aligned} \Rightarrow y &= -(4)^2 + 10(4) - 27 \\ &= -16 + 40 - 27 = -3 \end{aligned}$$

$$\Rightarrow Q = (4, -3)$$

**Q23.**

$$y = 2 + 0.12x - 0.01x^3$$

(i) $\Rightarrow \frac{dy}{dx} = 0.12 - 0.03x^2$

At $x = 0 \Rightarrow \frac{dy}{dx} = 0.12 - 0.03(0)^2 = 0.12$

$$\begin{aligned} \text{At } x = 3 \Rightarrow \frac{dy}{dx} &= 0.12 - 0.03(3)^2 \\ &= 0.12 - 0.27 = -0.15 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \text{gradient is zero} &\Rightarrow 0.12 - 0.03x^2 = 0 \\
 &\Rightarrow 4 - x^2 = 0 \\
 &\Rightarrow (2+x)(2-x) = 0 \\
 &\Rightarrow x = -2, 2
 \end{aligned}$$

$$\text{For } 0 \leq x \leq 3 \Rightarrow x = 2$$

$$\begin{aligned}
 &\Rightarrow y = 2 + 0.12(2) - 0.01(2)^3 \\
 &= 2 + 0.24 - 0.08 \\
 &= 2.16 \text{ km} = \text{height}
 \end{aligned}$$

Q24.

$$y = \sqrt{x+2}$$

$$\begin{aligned}
 \text{(a)} \quad \text{On } x\text{-axis} &\Rightarrow y = 0 \Rightarrow \sqrt{x+2} = 0 \\
 &\Rightarrow x+2 = 0 \\
 &\Rightarrow x = -2 \Rightarrow A(-2, 0)
 \end{aligned}$$

$$\begin{aligned}
 \text{On } y\text{-axis} &\Rightarrow x = 0 \Rightarrow y = \sqrt{0+2} = \sqrt{2} \\
 &\Rightarrow B(0, \sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad y &= \sqrt{x+2} = (x+2)^{\frac{1}{2}} \\
 \Rightarrow \frac{dy}{dx} &= \frac{1}{2}(x+2)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+2}}
 \end{aligned}$$

$$\text{(c) (i) At } x = -1 \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{-1+2}} = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

$$\text{(ii) At } x = -1 \Rightarrow y = \sqrt{-1+2} = \sqrt{1} = 1 \Rightarrow \text{Point } (-1, 1)$$

$$\begin{aligned}
 &\Rightarrow \text{Equation of Tangent: } y - 1 = \frac{1}{2}(x + 1) \\
 &\Rightarrow 2y - 2 = x + 1 \\
 &\Rightarrow 2y - x = 3
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii) On } x\text{-axis} &\Rightarrow y = 0 \Rightarrow 0 - x = 3 \\
 &\Rightarrow x = -3 \Rightarrow C(-3, 0)
 \end{aligned}$$

$$\text{On } y\text{-axis} \Rightarrow x = 0 \Rightarrow 2y - 0 = 3$$

$$\Rightarrow y = 1\frac{1}{2} \Rightarrow D\left(0, 1\frac{1}{2}\right)$$

$$\Rightarrow |CD| = \sqrt{(-3-0)^2 + \left(0-1\frac{1}{2}\right)^2} = \sqrt{9 + 2\frac{1}{4}} = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2}$$

$$\begin{aligned}
 \text{(d)} \quad \frac{dy}{dx} < 1 &\Rightarrow \frac{1}{2\sqrt{x+2}} < 1 \\
 &\Rightarrow 2\sqrt{x+2} > 1 \\
 &\Rightarrow \sqrt{x+2} > \frac{1}{2} \\
 &\Rightarrow x+2 > \frac{1}{4} \\
 &\Rightarrow x > -2 + \frac{1}{4} = \frac{-7}{4} \\
 &\Rightarrow x > \frac{-7}{4}
 \end{aligned}$$

Exercise 3.2

Q1.

$$y = x^2 - 4x + 9$$

$$\Rightarrow \frac{dy}{dx} = 2x - 4 = 0$$

$$\Rightarrow 2x = 4 \Rightarrow x = 2$$

$$\Rightarrow y = (2)^2 - 4(2) + 9 = 4 - 8 + 9 = 5 \Rightarrow \text{Point } (2, 5)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 > 0 \Rightarrow \text{Point}(2, 5) \text{ is a minimum}$$

Q2.

$$y = 4 - 8x - 2x^2$$

$$\Rightarrow \frac{dy}{dx} = -8 - 4x = 0$$

$$\Rightarrow -4x = 8$$

$$\Rightarrow x = -2$$

$$\begin{aligned} \Rightarrow y &= 4 - 8(-2) - 2(-2)^2 \\ &= 4 + 16 - 8 = 12 \Rightarrow \text{Point } (-2, 12) \end{aligned}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -4 < 0 \Rightarrow \text{Point } (-2, 12) \text{ is a maximum}$$

Q3.

$$y = 3x^2 - 6x + 4$$

$$\Rightarrow \frac{dy}{dx} = 6x - 6 = 0$$

$$\Rightarrow 6x = 6 \Rightarrow x = 1$$

$$\Rightarrow y = 3(1)^2 - 6(1) + 4 = 3 - 6 + 4 = 1 \Rightarrow \text{Point } (1, 1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6 > 0 \Rightarrow \text{Point } (1, 1) \text{ is a minimum}$$

Q4.

$$y = x^3 - 9x^2 + 15x + 2$$

$$\frac{dy}{dx} = 3x^2 - 18x + 15 = 0$$

$$\Rightarrow x^2 - 6x + 5 = 0$$

$$\Rightarrow (x-1)(x-5) = 0$$

$$\Rightarrow x = 1 \quad \text{OR} \quad x = 5$$

$$\Rightarrow y = (1)^3 - 9(1)^2 + 15(1) + 2 \quad \text{and} \quad y = (5)^3 - 9(5)^2 + 15(5) + 2$$
$$= 1 - 9 + 15 + 2 = 9 \qquad \qquad \qquad = 125 - 225 + 75 + 2$$

$$\Rightarrow \text{Point } (1, 9)$$

$$= -23$$

$$\Rightarrow \text{Point } (5, -23)$$

$$\frac{d^2y}{dx^2} = 6x - 18$$

$$\text{Point } (1, 9) \Rightarrow \frac{d^2y}{dx^2} = 6(1) - 18 = -12 < 0 \Rightarrow (1, 9) \text{ Maximum}$$

$$\text{Point } (5, -23) \Rightarrow \frac{d^2y}{dx^2} = 6(5) - 18 = +12 > 0 \Rightarrow (5, -23) \text{ Minimum}$$

Q5. (i)

$$y = 2x^3 - 3x^2 - 12x + 5$$

$$\Rightarrow \frac{dy}{dx} = 6x^2 - 6x - 12 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = 2 \quad \text{OR} \quad x = -1$$

$$\Rightarrow y = 2(2)^3 - 3(2)^2 - 12(2) + 5 \quad \text{and} \quad y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5$$
$$= 16 - 12 - 24 + 5 = -15 \qquad \qquad \qquad = -2 - 3 + 12 + 5 = 12$$

$$\Rightarrow \text{Point } (2, -15)$$

$$\text{Point } (-1, 12)$$

$$\frac{d^2y}{dx^2} = 12x - 6$$

$$\text{Point } (2, -15) \Rightarrow \frac{d^2y}{dx^2} = 12(2) - 6 = 18 > 0 \Rightarrow (2, -15) \text{ Minimum}$$

$$\text{Point } (-1, 12) \Rightarrow \frac{d^2y}{dx^2} = 12(-1) - 6 = -18 < 0 \Rightarrow (-1, 12) \text{ Maximum}$$

$$(ii) \quad y = \frac{x^2}{x+2}$$

Quotient Rule: $u = x^2$ and $v = x + 2$

$$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dv}{dx} = 1$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x+2).(2x) - (x^2).(1)}{(x+2)^2} \\ &= \frac{2x^2 + 4x - x^2}{(x+2)^2} \\ &= \frac{x^2 + 4x}{(x+2)^2} = 0 \end{aligned}$$

$$\Rightarrow x(x+4) = 0$$

$$\Rightarrow x = 0 \quad \text{OR} \quad x = -4$$

$$\Rightarrow y = \frac{(0)^2}{0+2} = 0 \quad \text{and} \quad y = \frac{(-4)^2}{-4+2}$$

$$\text{Point } (0,0) \quad = \frac{16}{-2} = -8$$

Point $(-4, -8)$

$$\frac{dy}{dx} = \frac{x^2 + 4x}{(x+2)^2}$$

Quotient Rule: $u = x^2 + 4x$ and $v = (x+2)^2$

$$\Rightarrow \frac{du}{dx} = 2x + 4 \quad \Rightarrow \frac{dv}{dx} = 2(x+2)$$

$$\frac{d^2y}{dx^2} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x+2)^2(2x+4) - (x^2+4x).2(x+2)}{(x+2)^4}$$

$$\text{at } (0, 0) \Rightarrow \frac{d^2y}{dx^2} = \frac{(0+2)(0+4) - (0+0).2(0+2)}{(0+2)^4} = \frac{8}{16} = \frac{1}{2} > 0$$

$\Rightarrow (0,0)$ Minimum

$$\text{at } (-4, -8) \Rightarrow \frac{d^2y}{dx^2} = \frac{(-4+2)^2(-8+4) - (16-16).2(-4+2)}{(-4+2)^2} = \frac{-16}{16} = -1 < 0$$

$\Rightarrow (-4, -8)$ Maximum

Q6.

$$f(x) = 4x + \frac{4}{x} = 4x + 4x^{-1}$$

$$f'(x) = 4 - 4x^{-2} = 4 - \frac{4}{x^2} = 0$$

$$\Rightarrow 4x^2 - 4 = 0$$

$$\Rightarrow x^2 - 1 = 0$$

$$\Rightarrow (x+1)(x-1) = 0$$

$$\Rightarrow x = -1 \quad \text{OR} \quad x = 1$$

$$f(-1) = 4(-1) + \frac{4}{-1} = -8 \quad \text{and} \quad f(1) = 4(1) + \frac{4}{1} = 8$$

Point $(-1, -8)$

Point $(1, 8)$

$$f''(x) = 0 + 8x^{-3} = \frac{8}{x^3}$$

$$\text{At } (-1, -8) \Rightarrow f''(-1) = \frac{8}{(-1)^3} = \frac{8}{-1} = -8 < 0$$

$\Rightarrow (-1, -8)$ Maximum

$$\text{At } (1, 8) \Rightarrow f''(1) = \frac{8}{(1)^3} = 8 > 0$$

$\Rightarrow (1, 8)$ Minimum

Q7.

$$y = x^2 + \frac{250}{x} = x^2 + 250x^{-1}$$

$$\frac{dy}{dx} = 2x - 250x^{-2} = 2x - \frac{250}{x^2} = 0$$

$$\Rightarrow 2x^3 - 250 = 0$$

$$\Rightarrow x^3 - 125 = 0$$

$$\Rightarrow x = \sqrt[3]{125} = 5$$

$$\Rightarrow y = (5)^2 + \frac{250}{5} = 25 + 50 = 75 \Rightarrow \text{Point } (5, 75)$$

$$\frac{d^2y}{dx^2} = 2 + 500x^{-3} = 2 + \frac{500}{x^3}$$

$$\text{At } (5, 75) \Rightarrow \frac{d^2y}{dx^2} = 2 + \frac{500}{(5)^3} = 2 + 4 = 6 > 0$$

$\Rightarrow (5, 75)$ Minimum

Q8. $y = x - \sqrt{x} = x - x^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{2}x^{-\frac{1}{2}} = 1 - \frac{1}{2\sqrt{x}} = 0$$

$$\Rightarrow 2\sqrt{x} - 1 = 0$$

$$\Rightarrow 2\sqrt{x} = 1$$

$$\Rightarrow \sqrt{x} = \frac{1}{2} \Rightarrow x = \frac{1}{4}$$

$$\Rightarrow y = \frac{1}{4} - \sqrt{\frac{1}{4}} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \Rightarrow \text{Point} \left(\frac{1}{4}, -\frac{1}{4} \right)$$

$$\frac{d^2y}{dx^2} = 0 + \frac{1}{4}x^{-\frac{3}{2}} = \frac{1}{4x^{\frac{3}{2}}}$$

$$\text{At} \left(\frac{1}{4}, -\frac{1}{4} \right) \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{4\left(\frac{1}{4}\right)^{\frac{3}{2}}} = \frac{1}{\frac{1}{2}} = 2 > 0$$

$$\Rightarrow \left(\frac{1}{4}, -\frac{1}{4} \right) \text{Minimum}$$

Q9. (i) $y = x^3 + 3x^2 + 1$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + 6x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6x + 6 = 0$$

$$\Rightarrow x + 1 = 0$$

$$\Rightarrow x = -1 \Rightarrow y = (-1)^3 + 3(-1)^2 + 1 = 3$$

$$\Rightarrow \text{Point of inflection} = (-1, 3)$$

(ii) $y = x^3 - 6x^2 + 9x + 2$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 9$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6x - 12 = 0$$

$$\Rightarrow x - 2 = 0 \Rightarrow x = 2$$

$$\Rightarrow y = (2)^3 - 6(2)^2 + 9(2) = 4$$

$$\Rightarrow \text{Point of inflection} = (2, 4)$$

Q10. $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cos x$$

$$\text{At } x = \frac{\pi}{2} \Rightarrow \frac{d^2y}{dx^2} = -\cos \frac{\pi}{2} = -0 = 0$$

Hence, point of inflection occurs at $x = \frac{\pi}{2}$

Q11.

$$y = ax^3 + bx^2 + c$$

$$\Rightarrow \frac{dy}{dx} = 3ax^2 + 2bx$$

$$\text{Point } (-1, 5) \Rightarrow \frac{dy}{dx} = 3a(-1)^2 + 2b(-1) = 0$$

$$\Rightarrow 3a - 2b = 0$$

$$\text{Point } (0, 4) \Rightarrow 4 = a(0)^3 + b(0)^2 + c \Rightarrow c = 4$$

$$\text{Point } (-1, 5) \Rightarrow 5 = a(-1)^3 + b(-1)^2 + 4$$

$$\Rightarrow 5 = -a + b + 4$$

$$\Rightarrow a - b = -1 \Rightarrow 2a - 2b = -2$$

$$\text{and} \quad \begin{array}{r} -3a + 2b = 0 \\ -a = -2 \end{array}$$

$$\Rightarrow a = 2$$

$$\Rightarrow 2 - b = -1$$

$$-b = -3 \Rightarrow b = 3$$

$$\Rightarrow a = 2, b = 3, c = 4$$

Q12.

$$f(x) = \frac{x+1}{x-3}$$

$$\text{Quotient Rule} \Rightarrow u = x+1 \text{ and } v = x-3$$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = 1$$

$$\begin{aligned} f'(x) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x-3) \cdot 1 - (x+1) \cdot 1}{(x-3)^2} \\ &= \frac{x-3-x-1}{(x-3)^2} = \frac{-4}{(x-3)^2} = 0 \end{aligned}$$

$$\Rightarrow -4 = 0 \text{ False}$$

\Rightarrow graph has no turning points

Q13. (i)

$$y = 2x^2 - \ln x$$

$$\frac{dy}{dx} = 4x - \frac{1}{x} = 4x - x^{-1}$$

$$\text{At } x = 1 \Rightarrow \frac{dy}{dx} = 4(1) - \frac{1}{1} = 3$$

$$\begin{aligned}
\text{(ii)} \quad \frac{dy}{dx} &= 0 \Rightarrow 4x - \frac{1}{x} = 0 \\
&\Rightarrow 4x^2 - 1 = 0 \\
&\Rightarrow (2x+1)(2x-1) = 0 \\
&\Rightarrow x = \frac{-1}{2}, \quad x = \frac{1}{2} \\
\text{since } x > 0 &\Rightarrow x = \frac{1}{2} \Rightarrow y = 2\left(\frac{1}{2}\right)^2 - \ln \frac{1}{2} \\
&= \frac{1}{2} + \ln 2 \\
\text{Point } &\left(\frac{1}{2}, \frac{1}{2} + \ln 2\right) \\
\Rightarrow \frac{d^2y}{dx^2} &= 4 + x^{-2} = 4 + \frac{1}{x^2} \\
\Rightarrow \text{at } x = \frac{1}{2} &\Rightarrow \frac{d^2y}{dx^2} = 4 + \frac{1}{\left(\frac{1}{2}\right)^2} = 8 > 0 \\
&\Rightarrow \text{Minimum point}
\end{aligned}$$

Q14. (i) $y = e^x - x$

$$\begin{aligned}
\frac{dy}{dx} &= e^x - 1 = 0 \\
&\Rightarrow e^x = 1 \Rightarrow x = 0 \\
&\Rightarrow y = e^0 - 0 = 1 - 0 \Rightarrow \text{Point } (0, 1)
\end{aligned}$$

(ii) $\frac{d^2y}{dx^2} = e^x$

$$\begin{aligned}
\text{Point } (0, 1) &\Rightarrow \frac{d^2y}{dx^2} = e^0 = 1 > 0 \\
&\Rightarrow \text{Minimum point}
\end{aligned}$$

Q15. (a) $y = x^3 - 9x^2 + 24x - 20$

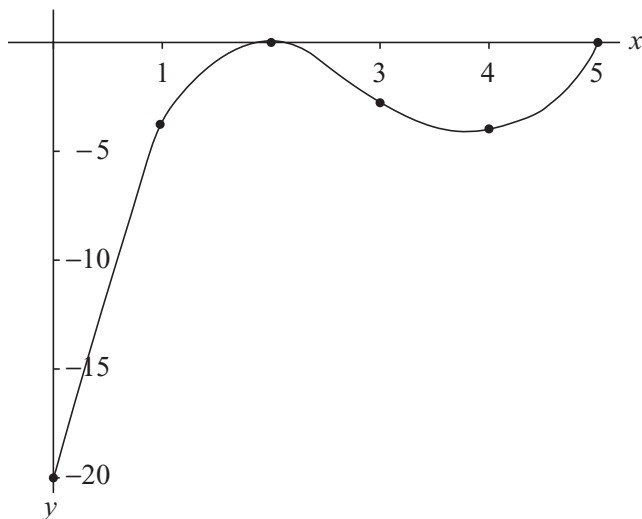
$$\begin{aligned}
&\Rightarrow \frac{dy}{dx} = 3x^2 - 18x + 24 = 0 \\
&\Rightarrow x^2 - 6x + 8 = 0 \\
&\Rightarrow (x-2)(x-4) = 0 \\
&\Rightarrow x = 2 \quad \text{OR} \quad x = 4 \\
&\Rightarrow y = (2)^3 - 9(2)^2 + 24(2) - 20 \quad \text{and} \quad y = (4)^3 - 9(4)^2 + 24(4) - 20 \\
&\quad = 8 - 36 + 48 - 20 = 0 \quad \quad \quad = 64 - 144 + 96 - 20 = -4 \\
&\quad \text{Point } (2, 0) \quad \quad \quad \text{Point } (4, -4) \\
&\Rightarrow \frac{d^2y}{dx^2} = 6x - 18 \\
&\text{Point } (2, 0) \Rightarrow \frac{d^2y}{dx^2} = 6(2) - 18 = -6 < 0 \Rightarrow (2, 0) \text{ Maximum} \\
&\text{Point } (4, -4) \Rightarrow \frac{d^2y}{dx^2} = 6(4) - 18 = 6 > 0 \Rightarrow (4, -4) \text{ Minimum}
\end{aligned}$$

$$\begin{aligned}
 \text{(b) (i)} \quad (x-2)^2(x-5) &= (x^2 - 4x + 4)(x-5) \\
 &= x^3 - 5x^2 - 4x^2 + 20x + 4x - 20 \\
 &= x^3 - 9x^2 + 24x - 20
 \end{aligned}$$

$$\text{(ii)} \quad f(x) = x^3 - 9x^2 + 24x - 20$$

$$f(0) = -20 \quad f(1) = -4 \quad f(2) = 0$$

$$f(3) = -2 \quad f(4) = -4 \quad f(5) = 0$$



Q16. (i) $f(x) = (1+x)\log_e(1+x)$

Product Rule: $u = 1+x$ and $v = \log_e(1+x)$

$$\Rightarrow \frac{du}{dx} = 1 \quad \frac{dv}{dx} = \frac{1}{1+x}$$

$$\Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = (1+x) \cdot \frac{1}{1+x} + \log_e(1+x) \cdot 1$$

$$= 1 + \log_e(1+x) = 0$$

$$\Rightarrow \log_e(1+x) = -1 \Rightarrow e^{-1} = 1+x$$

$$\Rightarrow \frac{1}{e} = x+1$$

$$\Rightarrow x = \frac{1}{e} - 1 = \frac{1-e}{e}$$

$$x = \frac{1-e}{e} \Rightarrow f\left(\frac{1-e}{e}\right) = \left(1 + \frac{1}{e} - 1\right) \left(\log_e\left(1 + \frac{1}{e} - 1\right)\right)$$

$$= \frac{1}{e} \cdot \log_e \frac{1}{e}$$

$$= \frac{1}{e} \cdot \log_e e^{-1}$$

$$= \frac{1}{e} \cdot -1 \cdot \log_e e = -\frac{1}{e}$$

$$\Rightarrow \text{Turning Point} = \left(\frac{1-e}{e}, -\frac{1}{e}\right)$$

$$\begin{aligned}
 \text{(ii)} \quad & \frac{dy}{dx} = 1 + \log_e(1+x) \\
 \Rightarrow & \frac{d^2y}{dx^2} = 0 + \frac{1}{1+x} = \frac{1}{1+x} \\
 \text{At } x = \frac{1}{e} - 1 \Rightarrow & \frac{d^2y}{dx^2} = \frac{1}{1 + \frac{1}{e} - 1} = \frac{1}{\frac{1}{e}} = e > 0 \\
 \Rightarrow & \left(\frac{1-e}{e}, \frac{-1}{e} \right) \text{ is a minimum}
 \end{aligned}$$

Q17.

$$\begin{aligned}
 f(x) &= ax^3 + bx^2 + cx + d \\
 \text{Point } (0,4) \Rightarrow f(0) &= a(0)^3 + b(0)^2 + c(0) + d = 4 \\
 &\Rightarrow d = 4 \\
 \text{Point } (1,0) \Rightarrow a(1)^3 + b(1)^2 + c(1) + 4 &= 0 \\
 &\Rightarrow a + b + c = -4 \\
 f'(x) &= 3ax^2 + 2bx + c \\
 \Rightarrow f'(0) &= 3a(0)^2 + 2b(0) + c = 0 \Rightarrow c = 0 \\
 f''(x) &= 6ax + 2b \\
 \Rightarrow f''(1) &= 6a(1) + 2b = 0 \Rightarrow 3a + b = 0 \\
 \text{and } a + b + 0 &= -4 \Rightarrow \underline{-a - b = +4} \\
 &\quad \quad \quad 2a = 4 \\
 &\quad \quad \Rightarrow a = 2 \\
 &\Rightarrow 2 + b = -4 \\
 &\Rightarrow b = -6
 \end{aligned}$$

Q18.

$$\begin{aligned}
 f(x) &= \frac{x}{x+2} \\
 \text{Quotient Rule: } u &= x \quad \text{and} \quad v = x+2 \\
 \Rightarrow \frac{du}{dx} &= 1 \quad \Rightarrow \frac{dv}{dx} = 1 \\
 \Rightarrow f'(x) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x+2)(1) - (x)(1)}{(x+2)^2} \\
 &= \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2} = 0 \\
 &\Rightarrow 2 = 0 \quad \text{False} \\
 \Rightarrow f(x) &\text{ has no turning points} \\
 f'(x) &= \frac{2}{(x+2)^2} = 2(x+2)^{-2} \\
 \Rightarrow f''(x) &= -4(x+2)^{-3} = \frac{-4}{(x+2)^3} = 0 \\
 &\Rightarrow -4 = 0 \quad \text{False} \\
 \Rightarrow f(x) &\text{ has no points of inflection}
 \end{aligned}$$

Q19. (i) $g(x) = x^2 + \frac{a}{x^2} = x^2 + ax^{-2}$

$$\Rightarrow g'(x) = 2x - 2ax^{-3} = 2x - \frac{2a}{x^3}$$

$$\Rightarrow g'(2) = 2(2) - \frac{2a}{(2)^3} = 4 - \frac{2a}{8} \Rightarrow 32 - 2a = 0$$

$$\Rightarrow a = 16$$

(ii) $g(x) = x^2 + \frac{16}{x^2} \Rightarrow g'(x) = 2x - \frac{32}{x^3} = 0$

$$\Rightarrow 2x^4 - 32 = 0$$

$$\Rightarrow x^4 - 16 = 0$$

$$\Rightarrow (x^2 - 4)(x^2 + 4) = 0$$

$$\Rightarrow (x + 2)(x - 2) = 0$$

$$\Rightarrow x = -2 \quad \text{OR} \quad x = 2$$

$$g'(x) = 2x - \frac{32}{x^3} = 2x - 32x^{-3}$$

$$\Rightarrow g''(x) = 2 + 96x^{-4} = 2 + \frac{96}{x^4}$$

$$\Rightarrow g''(-2) = 2 + \frac{96}{(-2)^4} = 2 + 6 = 8 > 0$$

$$\Rightarrow \text{Minimum at } x = -2$$

$$\Rightarrow g''(2) = 2 + \frac{96}{(2)^4} = 2 + 6 = 8 > 0$$

$$\Rightarrow \text{Minimum at } x = 2$$

$$\Rightarrow g(x) \text{ has no local maximum point}$$

Q20. (i) $C = \frac{1400}{v} + \frac{2v}{7} = 1400v^{-1} + \frac{2}{7}v$

$$\Rightarrow \frac{dC}{dv} = -1400v^{-2} + \frac{2}{7} = \frac{-1400}{v^2} + \frac{2}{7} = 0$$

$$\Rightarrow -9800 + 2v^2 = 0$$

$$\Rightarrow 2v^2 = 9800$$

$$\Rightarrow v^2 = 4900$$

$$\Rightarrow v = \sqrt{4900} = 70 \text{ km/hr}$$

(ii) $\frac{d^2C}{dv^2} = 2800v^{-3} = \frac{2800}{v^3}$

$$\text{At } v = 70 \Rightarrow \frac{d^2C}{dv^2} = \frac{2800}{(70)^3} = \frac{2800}{343000} > 0$$

$$\Rightarrow \text{Minimum at } v = 70$$

(iii) $v = 70 \Rightarrow C = \frac{1400}{70} + \frac{2(70)}{7} \Rightarrow C = \text{€ } 40$

Exercise 3.3

Q1. In (ii) $\Rightarrow \frac{dy}{dx}$ positive for all values of x .

Q2. In (i) and (iii) $\Rightarrow \frac{dy}{dx}$ negative for all values of x .

Q3. (i) Positive slope

(ii) $x < -2$ OR $x > 3$

(iii) $-2 < x < 3$

(iv) $f'(x) = 0 \Rightarrow x = -2, 3$

Q4.

- Positive slope for $x < -1$
- Turning point at $x = -1$
- Negative slope for $x > 1$

Q5. Answer Ⓒ because

- Positive slope for $x < 1\frac{1}{2}$
- Turning point at $x = 1\frac{1}{2}$
- Negative slope for $x > 1\frac{1}{2}$

Q6. Answer Ⓑ because

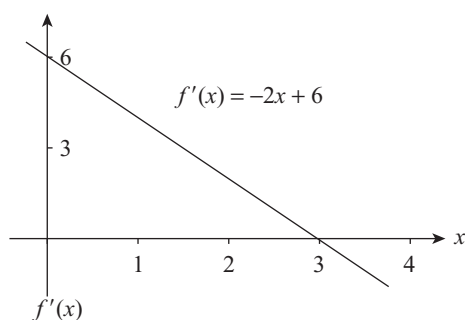
- Positive slope for $x < A$ OR $x > B$
- Turning points at $x = A$ and B
- Negative slope for $A < x < B$

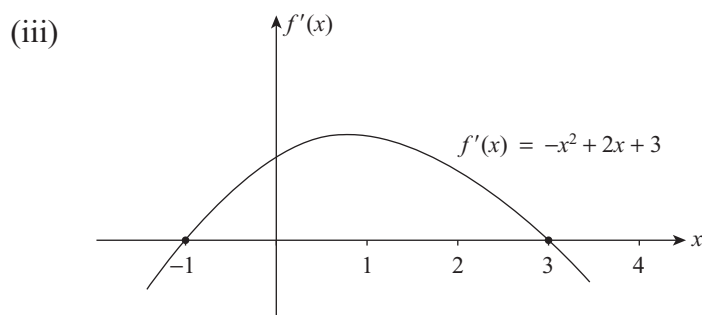
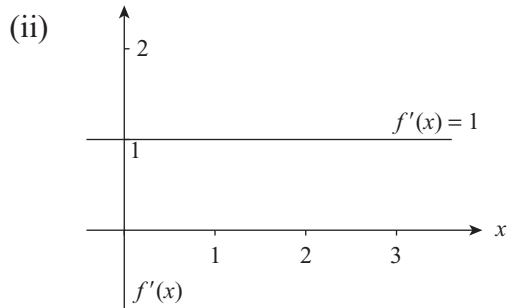
Q7. (i) $f'(x) > 0$ for $-2 < x < 1$

(ii) $f'(x) < 0$ for $x < -2$, OR $x > 1$

(iii) $f'(x) = 0$ at $x = -2, 1$

Q8. (i)





Q9. (i) A turning point at $x = -3$ for curve (a)

A turning point at $x = 4$ for curve (b)

(ii) Curve (a) is decreasing for $x < -3$

Curve (b) is decreasing for $x > 4$

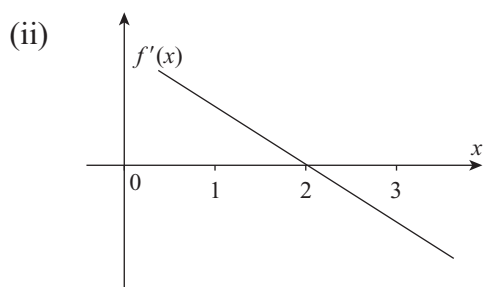
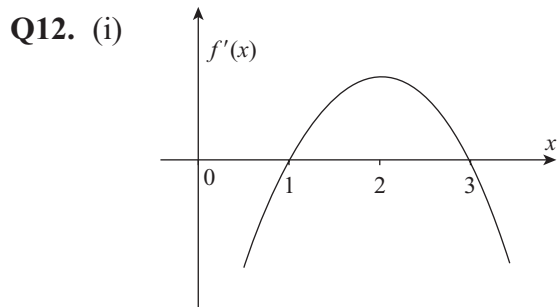
Q10. (i) Curve (a) has stationary points at $x = -1, 3$

Curve (b) has stationary points at $x = -4.5, 1$

(ii) Curve (a) is increasing for $x < -1$ OR $x > 3$

Curve (b) is increasing for $-4.5 < x < 1$

Q11. C. is true.



Q13. (i) $f'(x) = k(x-a)(x-b) = k(x-2)(x-4) = 0$
 $\Rightarrow x = 2, x = 4$
 $\Rightarrow a = 2, b = 4$

(ii) $f'(0) = 6 \Rightarrow k(0-2)(0-4) = 6$
 $\Rightarrow 8k = 6$
 $\Rightarrow k = \frac{6}{8} = \frac{3}{4}$

Exercise 3.4

Q1. $x + y = 6$ and $A = x^2y$
 $\Rightarrow y = 6 - x \Rightarrow A = x^2(6 - x)$
 $\Rightarrow A = 6x^2 - x^3$
 $\Rightarrow \frac{dA}{dx} = 12x - 3x^2 = 0$
 $\Rightarrow 3x(4 - x) = 0$
 $\Rightarrow x = 0$ OR $x = 4$
 $\Rightarrow \frac{d^2A}{dx^2} = 12 - 6x$
At $x = 0 \Rightarrow \frac{d^2A}{dx^2} = 12 - 6(0) = 12 \Rightarrow$ Minimum
At $x = 4 \Rightarrow \frac{d^2A}{dx^2} = 12 - 6(4) = -12 < 0$
 \Rightarrow Maximum at $x = 4$
 $\Rightarrow A = (4)^2(6 - 4) = 32$

Q2. (i) $P(x) = \frac{1152}{x} + 8x + 20 = 1152x^{-1} + 8x + 20$
 $\Rightarrow \frac{dP}{dx} = -1152x^{-2} + 8 = \frac{-1152}{x^2} + 8 = 0$
 $\Rightarrow -1152 + 8x^2 = 0$
 $\Rightarrow 8x^2 = 1152$
 $\Rightarrow x^2 = 144$
 $\Rightarrow x = 12$
 $\Rightarrow \frac{d^2P}{dx^2} = 2304x^{-3} = \frac{2304}{x^3}$
At $x = 12 \Rightarrow \frac{d^2P}{dx^2} = \frac{2304}{(12)^3} = 1\frac{1}{3} > 0 \Rightarrow$ Minimum

(ii) At $x = 12 \Rightarrow P = \frac{1152}{12} + 8(12) + 20 = 212 \text{ cm}$

Q3.

$$\text{Perimeter} = 2x + 2y = 100$$

$$\Rightarrow x + y = 50$$

$$\Rightarrow y = 50 - x$$

$$\text{Area} = (x)(y) = (x)(50 - x)$$

$$\Rightarrow A = 50x - x^2$$

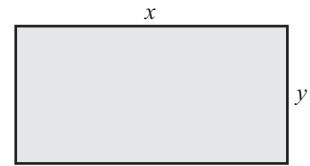
$$\Rightarrow \frac{dA}{dx} = 50 - 2x = 0$$

$$\Rightarrow -2x = -50$$

$$\Rightarrow x = 25$$

$$\frac{d^2 A}{dx^2} = -2 < 0 \Rightarrow \text{Maximum at } x = 25$$

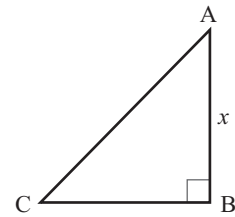
$$\Rightarrow \text{Area} = (25)(25) = 625 \text{ m}^2$$

**Q4. (i)**

$$|AB| + |BC| = 8$$

$$\text{If } |AB| = x \Rightarrow x + |BC| = 8$$

$$\Rightarrow |BC| = 8 - x$$



$$(ii) \quad \text{Area Triangle ABC} = \frac{1}{2} |AB| \cdot |BC|$$

$$\Rightarrow A = \frac{1}{2} x(8 - x)$$

$$\Rightarrow A = \frac{1}{2} [8x - x^2]$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{2} [8 - 2x] = 4 - x = 0$$

$$\Rightarrow x = 4$$

$$\Rightarrow \frac{d^2 A}{dx^2} = \frac{1}{2} [0 - 2] = -1 < 0 \Rightarrow \text{Maximum}$$

$$\text{At } x = 4 \Rightarrow \text{Area} = \frac{1}{2} (4)(8 - 4)$$

$$= (2)(4) = 8 \text{ cm}^2$$

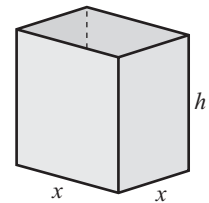
Q5. (i)

$$\text{Volume} = x \cdot x \cdot h = x^2 h = 108$$

$$\Rightarrow h = \frac{108}{x^2}$$

$$(ii) \quad \text{Surface Area, } S = x \cdot x + 4x \cdot h$$

$$\Rightarrow S = x^2 + 4x \cdot \frac{108}{x^2} = x^2 + \frac{432}{x}$$

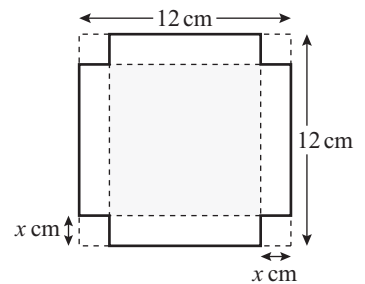


$$\begin{aligned}
\text{(iii)} \quad S &= x^2 + 432x^{-1} \\
\Rightarrow \frac{dS}{dx} &= 2x - 432x^{-2} \\
&= 2x - \frac{432}{x^2} = 0 \\
\Rightarrow 2x^3 - 432 &= 0 \\
\Rightarrow x^3 - 216 &= 0 \\
\Rightarrow x^3 &= 216 \\
\Rightarrow x = \sqrt[3]{216} = 6 &\Rightarrow h = \frac{108}{(6)^2} = 3 \\
\Rightarrow \frac{d^2S}{dx^2} &= 2 + 864x^{-3} = 2 + \frac{864}{x^3} \\
\text{At } x = 6 \Rightarrow \frac{d^2S}{dx^2} &= 2 + \frac{864}{(6)^3} = 6 > 0 \Rightarrow \text{Minimum} \\
&\Rightarrow \text{dimensions: 6 m by 6 m by 3 m.}
\end{aligned}$$

Q6. (i) Length of side of the box = $12 - 2x$
 $\Rightarrow \text{Volume } (V) = (12 - 2x)(12 - 2x).x = 4x^3 - 48x^2 + 144x$

(ii) $\frac{dV}{dx} = 12x^2 - 96x + 144 = 0$
 $\Rightarrow x^2 - 8x + 12 = 0$
 $\Rightarrow (x - 2)(x - 6) = 0$
 $\Rightarrow x = 2$ (valid) OR $x = 6$ (invalid)
 $\Rightarrow \frac{d^2V}{dx^2} = 24x - 96$

At $x = 2 \Rightarrow \frac{d^2V}{dx^2} = 24(2) - 96 = -48 < 0$
 $\Rightarrow \text{maximum when } x = 2 \text{ cm}$



Q7. (i) Total Surface Area = 54 cm^2
 $\Rightarrow 2x.x + 4x.h = 54$
 $\Rightarrow x^2 + 2xh = 27$
 $\Rightarrow 2xh = 27 - x^2$
 $\Rightarrow h = \frac{27 - x^2}{2x}$

(ii) Volume $(V) = x.x.h = x^2h$
 $\Rightarrow V = x^2 \frac{(27 - x^2)}{2x}$
 $\Rightarrow V = \frac{1}{2}(27x - x^3)$

$$\begin{aligned}
 \text{(iii)} \quad \frac{dV}{dx} &= \frac{1}{2}(27 - 3x^2) = 0 \\
 &\Rightarrow -3x^2 = -27 \\
 &\Rightarrow x^2 = 9 \\
 &\Rightarrow x = 3 \Rightarrow h = \frac{27 - (3)^2}{2(3)} = 3 \\
 &\Rightarrow \frac{d^2V}{dx^2} = \frac{1}{2}(0 - 6x) = -3x \\
 \text{At } x = 3 &\Rightarrow \frac{d^2V}{dx^2} = -3(2) = -6 < 0 \Rightarrow \text{Maximum} \\
 \text{Hence, volume} &= (3)(3)(3) = 27 \text{ cm}^3
 \end{aligned}$$

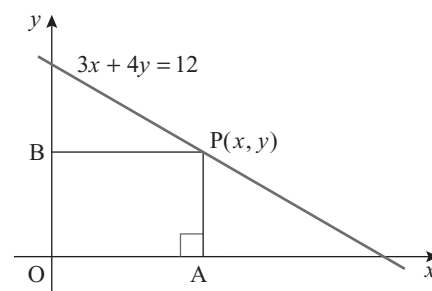
Q8. (i) $3x + 4y = 12$

$$\Rightarrow 4y = 12 - 3x \Rightarrow y = \frac{12 - 3x}{4}$$

$$\Rightarrow P = \left(x, \frac{12 - 3x}{4} \right)$$

(ii) $\text{Area (A)} = (x) \frac{(12 - 3x)}{4} = \frac{1}{4}(12x - 3x^2)$

$$\begin{aligned}
 \text{(iii)} \quad \frac{dA}{dx} &= \frac{1}{4}(12 - 6x) = 0 \\
 &\Rightarrow -6x = -12 \Rightarrow x = 2 \\
 &\Rightarrow \frac{d^2A}{dx^2} = \frac{1}{4}(0 - 6) = \frac{-3}{2} < 0 \Rightarrow \text{Maximum} \\
 &\Rightarrow \text{Area} = \frac{1}{4}(12(2) - 3(2)^2) = 3 \text{ sq.units}
 \end{aligned}$$



Q9. Total Surface Area = $2\pi rh + 2\pi r^2 = 24\pi$

$$\Rightarrow rh + r^2 = 12$$

$$\Rightarrow rh = 12 - r^2$$

$$\Rightarrow h = \frac{12 - r^2}{r}$$

$$\Rightarrow \text{Volume} = \pi r^2 h = \pi r^2 \left(\frac{12 - r^2}{r} \right) = \pi r(12 - r^2)$$

$$V = 12\pi r - \pi r^3$$

$$\Rightarrow \frac{dV}{dr} = 12\pi - 3\pi r^2 = 0$$

$$\Rightarrow 4 - r^2 = 0$$

$$\Rightarrow r^2 = 4 \Rightarrow r = \sqrt{4} = 2$$

$$\Rightarrow h = \frac{12 - (2)^2}{2} = 4$$

$$\Rightarrow \frac{d^2V}{dr^2} = 0 - 6\pi r$$

$$\text{At } r = 2 \Rightarrow \frac{d^2V}{dr^2} = -6\pi(2) = -12\pi < 0 \Rightarrow \text{Maximum}$$

Q10. (i) $r + h = 20 \Rightarrow h = (20 - r) \text{ cm}$

(ii) Volume (V) = $\frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r^2 (20 - r)$

$$\Rightarrow V = \frac{\pi}{3} (20r^2 - r^3)$$

$$\Rightarrow \frac{dV}{dr} = \frac{\pi}{3} (40r - 3r^2) = 0$$

$$\Rightarrow r(40 - 3r) = 0$$

$$\Rightarrow r = 0 \text{ (invalid) OR } r = \frac{40}{3} \text{ (valid)}$$

$$\Rightarrow \frac{d^2V}{dr^2} = \frac{\pi}{3} (40 - 6r)$$

$$\text{At } r = \frac{40}{3} \Rightarrow \frac{d^2V}{dr^2} = \frac{\pi}{3} \left(40 - 6 \left(\frac{40}{3} \right) \right) = \frac{-40\pi}{3} < 0$$

$$\Rightarrow \text{Volume is maximum when } r = \frac{40}{3} \text{ cm}$$

Q11. (i) Perimeter = $r\theta + 2r = 8$

$$\Rightarrow r\theta = 8 - 2r$$

$$\Rightarrow \theta = \frac{8 - 2r}{r} = \frac{8}{r} - 2$$

(ii) Area (A) = $\frac{1}{2} r^2 \theta = \frac{1}{2} r^2 \left(\frac{8}{r} - 2 \right)$

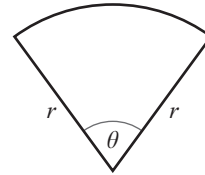
$$\Rightarrow A = 4r - r^2$$

(iii) $\Rightarrow \frac{dA}{dr} = 4 - 2r = 0$

$$\Rightarrow -2r = -4 \Rightarrow r = 2$$

$$\Rightarrow \frac{d^2A}{dr^2} = -2 < 0 \Rightarrow \text{Maximum}$$

$$\text{Hence, at } r = 2 \Rightarrow A = 4(2) - (2)^2 = 4 \text{ m}^2$$

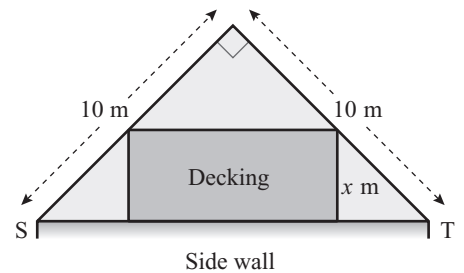


Q12. (a) (i) $|ST|^2 = (10)^2 + (10)^2 = 200$

$$\Rightarrow |ST| = \sqrt{200} = 10\sqrt{2}$$

(ii) \Rightarrow length of rectangle = $10\sqrt{2} - 2x$, width = x

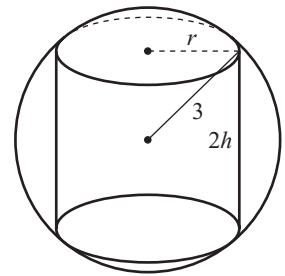
$$\Rightarrow \text{Area } (A) = (10\sqrt{2} - 2x)x = (10\sqrt{2})x - 2x^2$$



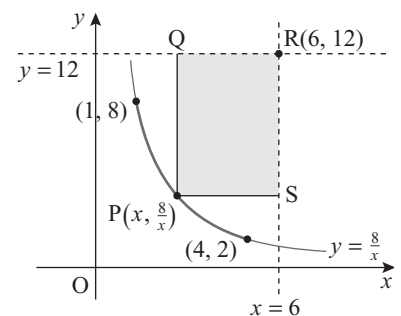
$$\begin{aligned}
 \text{(b)} \quad \frac{dA}{dx} &= 10\sqrt{2} - 4x = 0 \\
 &\Rightarrow -4x = -10\sqrt{2} \\
 &\Rightarrow x = \frac{10\sqrt{2}}{4} = \frac{5\sqrt{2}}{2} \text{ (width)} \\
 &\Rightarrow \text{length} = 10\sqrt{2} - 2\left(\frac{5\sqrt{2}}{2}\right) = 5\sqrt{2} \\
 &\Rightarrow \frac{d^2A}{dx^2} = 0 - 4 = -4 < 0 \Rightarrow \text{Maximum} \\
 &\Rightarrow \text{dimensions: length} = 5\sqrt{2} \text{ m, width} = \frac{5\sqrt{2}}{2} \text{ m}
 \end{aligned}$$

Q13.

$$\begin{aligned}
 r^2 + h^2 &= (3)^2 = 9 \\
 &\Rightarrow r^2 = 9 - h^2 \\
 &\Rightarrow r = \sqrt{9 - h^2} \\
 \text{Hence, Volume } (V) &= \pi r^2 h \\
 &= \pi(9 - h^2) \cdot (2h) \\
 &= 2\pi h(9 - h^2) = 18\pi h - 2\pi h^3 \\
 &\Rightarrow \frac{dV}{dh} = 18\pi - 6\pi h^2 = 0 \\
 &\Rightarrow 3 - h^2 = 0 \\
 &\Rightarrow h^2 = 3 \Rightarrow h = \sqrt{3} \\
 &\Rightarrow r = \sqrt{9 - 3} = \sqrt{6} \\
 &\Rightarrow \frac{d^2V}{dh^2} = 0 - 12\pi h \\
 \text{At } h = \sqrt{3} &\Rightarrow \frac{d^2V}{dh^2} = -12\pi(\sqrt{3}) = -12\sqrt{3}\pi < 0 \\
 &\Rightarrow \text{maximum when } h = \sqrt{3} \\
 &\Rightarrow \text{volume} = \pi(\sqrt{6})^2 \cdot (2\sqrt{3}) = 12\pi\sqrt{3} \text{ cm}^3
 \end{aligned}$$



$$\begin{aligned}
 \text{Q14. (a) (i)} \quad |PS| &= 6 - x \\
 |RS| &= 12 - \frac{8}{x} \\
 \text{(ii) Area } (A) &= (6 - x)\left(12 - \frac{8}{x}\right) \\
 &= 72 - \frac{48}{x} - 12x + 8 = 80 - 12x - \frac{48}{x}
 \end{aligned}$$



(b) $A = 80 - 12x - 48x^{-1}$

$$\Rightarrow \frac{dA}{dx} = -12 + 48x^{-2} = -12 + \frac{48}{x^2} = 0$$

$$\Rightarrow -12x^2 + 48 = 0$$

$$\Rightarrow +x^2 - 4 = 0$$

$$\Rightarrow x = \sqrt{4} = 2$$

$$\Rightarrow \frac{d^2A}{dx^2} = 0 - 96x^{-3} = \frac{-96}{x^3}$$

$$\text{At } x = 2 \Rightarrow \frac{d^2A}{dx^2} = \frac{-96}{(2)^3} = -12 < 0 \Rightarrow \text{Maximum}$$

$$\begin{aligned} \text{When } x = 2 \Rightarrow A &= 80 - 12(2) - \frac{48}{2} \\ &= 80 - 48 = 32 \text{ sq. units} \end{aligned}$$

P lies between $x = 1$ and $x = 4$

$$\Rightarrow \text{at } x = 1 \Rightarrow A = 80 - 12(1) - \frac{48}{1} = 20 \text{ (minimum)}$$

$$\text{and at } x = 4 \Rightarrow A = 80 - 12(4) - \frac{48}{4} = 20 \text{ (minimum)}$$

Q15. (i) $y = -x^2 + 6x$

$$\Rightarrow P(x, y) = (x, -x^2 + 6x)$$

(ii) at B, $y = 0 \Rightarrow -x^2 + 6x = 0$

$$\Rightarrow -x(x - 6) = 0$$

$$\Rightarrow x = 0, x = 6$$

$$O = (0, 0), B = (6, 0)$$

$$\Rightarrow |OA| = x \quad |OB| = 6 \Rightarrow |AB| = 6 - x$$

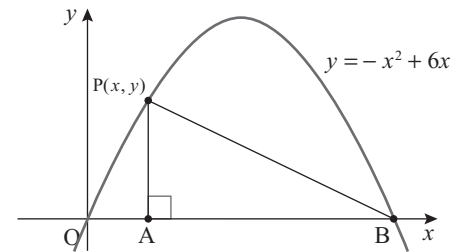
$$|AP| = y = -x^2 + 6x$$

$$\Rightarrow \text{Area}(A) = \frac{1}{2} |AB| \cdot |AP|$$

$$= \frac{1}{2} (6 - x)(-x^2 + 6x)$$

$$= \frac{1}{2} [-6x^2 + 36x + x^3 - 6x^2]$$

$$= \frac{1}{2} (x^3 - 12x^2 + 36x)$$



$$\begin{aligned}
\text{(iii)} \quad \frac{dA}{dx} &= \frac{1}{2}(3x^2 - 24x + 36) = 0 \\
&\Rightarrow x^2 - 8x + 12 = 0 \\
&\Rightarrow (x-2)(x-6) = 0 \\
&\Rightarrow x = 2, x = 6 \text{ (invalid)} \\
&\Rightarrow \frac{d^2A}{dx^2} = \frac{1}{2}(6x - 24) = 3x - 12 \\
\text{At } x = 2 &\Rightarrow \frac{d^2A}{dx^2} = 3(2) - 12 = -6 < 0 \text{ (Maximum)} \\
&\Rightarrow \text{Area} = \frac{1}{2}(6-2)((-2)^2 + 6(2)) \\
&= \frac{1}{2}(4)(8) = 16 \text{ sq.units}
\end{aligned}$$

Q16. (i) Perimeter = $2x + 2y = 120$

$$\Rightarrow x + y = 60$$

$$\Rightarrow y = 60 - x$$

(ii) $S = 5x^2y = 5x^2(60 - x)$

(iii) Possible values for $x \Rightarrow 5x^2(60 - x) > 0$
 $\Rightarrow 0 < x < 60$

(iv) $S = 300x^2 - 5x^3$

$$\Rightarrow \frac{dS}{dx} = 600x - 15x^2 = 0$$

$$\Rightarrow 40x - x^2 = 0$$

$$\Rightarrow x(40 - x) = 0$$

$$\Rightarrow x = 0 \text{ (invalid)} \text{ OR } x = 40 \text{ (valid)}$$

$$\Rightarrow \frac{d^2S}{dx^2} = 600 - 30x$$

$$\text{At } x = 40 \Rightarrow \frac{d^2S}{dx^2} = 600 - 30(40) = -600 < 0$$

$$\Rightarrow \text{Maximum (strongest) at } x = 40$$

$$\Rightarrow y = 60 - 40 = 20$$

(v) $x < 19 \text{ cm} \Rightarrow \text{Maximum strength} = 5(19)^2(60 - 19)$
 $= 5(19)^2(41) = 74005$

Exercise 3.5

Q1. $p = 2q^3 + q$

$$\Rightarrow \frac{dp}{dq} = 6q^2 + 1$$

$$\text{At } q = 4 \Rightarrow \frac{dp}{dq} = 6(4)^2 + 1 = 97$$

Q2. (i) $y = 2x^2 + x$

$$\Rightarrow \frac{dy}{dx} = 4x + 1$$

When $x = 4 \Rightarrow \frac{dy}{dx} = 4(4) + 1 = 17$

(ii) Rate of change is 9 $\Rightarrow 4x + 1 = 9$

$$\Rightarrow 4x = 8$$

$$\Rightarrow x = 2$$

Q3. $A = \pi r^2$

$$\Rightarrow \frac{dA}{dr} = 2\pi r$$

(i) $r = 5 \text{ cm} \Rightarrow \frac{dA}{dr} = 2\pi(5) = 10\pi \text{ cm}^2$

(ii) $r = 10 \text{ cm} \Rightarrow \frac{dA}{dr} = 2\pi(10) = 20\pi \text{ cm}^2$

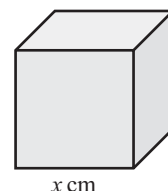
Q4. $V = x \cdot x \cdot x = x^3$

$$\Rightarrow \frac{dV}{dx} = 3x^2$$

(i) $x = 10 \text{ cm} \Rightarrow \frac{dV}{dx} = 3(10)^2 = 300 \text{ cm}^3$

(ii) $V = 125 \Rightarrow x^3 = 125$
 $\Rightarrow x = \sqrt[3]{125} = 5 \text{ cm}$

Hence, $\frac{dV}{dx} = 3(5)^2 = 75 \text{ cm}^3$



Q5. $P = 100(5 + t - 0.25t^2)$

$$\Rightarrow \frac{dP}{dt} = 100(0 + 1 - 0.5t) = 100 - 50t$$

When $t = 3 \Rightarrow \frac{dP}{dt} = 100 - 50(3) = -50$ people per year.

Thus the population is declining by 50 people after 3 years.

Q6. (i) $M = 200,000 + 600t^2 - \frac{200}{3}t^3$

$$\begin{aligned} \frac{dM}{dt} &= 0 + 1200t - \frac{200}{3} \cdot 3t^2 \\ &= 1200t - 200t^2 \end{aligned}$$

(ii) When $t = 3 \Rightarrow \frac{dM}{dt} = 1200(3) - 200(3)^2$
 $= 3600 - 1800$
 $= €1800 \text{ per month}$

$$\begin{aligned}
 \text{(iii) Rate of growth} &= 0 \\
 \Rightarrow 1200t - 200t^2 &= 0 \\
 \Rightarrow 200t(6 - t) &= 0 \\
 \Rightarrow t = 0 \text{ and } t = 6
 \end{aligned}$$

Q7. $s = t^3 - 2t^2 + 3t - 4$

(i) Speed $= \frac{ds}{dt} = 3t^2 - 4t + 3$

$$\begin{aligned}
 \text{When } t = 4 \Rightarrow \frac{ds}{dt} &= 3(4)^2 - 4(4) + 3 \\
 &= 48 - 16 + 3 = 35 \text{ m/sec.}
 \end{aligned}$$

(ii) Acceleration $= \frac{d^2s}{dt^2} = 6t - 4$

$$\text{When } t = 4 \Rightarrow \frac{d^2s}{dt^2} = 6(4) - 4 = 20 \text{ m/sec}^2.$$

Q8. $s = t^3 - 4t^2 + 4t$

(i) Speed $= \frac{ds}{dt} = 3t^2 - 8t + 4$

$$\begin{aligned}
 \text{When } t = 3 \Rightarrow \frac{ds}{dt} &= 3(3)^2 - 8(3) + 4 \\
 &= 27 - 24 + 4 = 7 \text{ m/sec}
 \end{aligned}$$

(ii) Acceleration $= \frac{d^2s}{dt^2} = 6t - 8$

$$\text{When } t = 1 \Rightarrow \frac{d^2s}{dt^2} = 6(1) - 8 = -2 \text{ m/sec}^2$$

(iii) Body momentarily at rest $\Rightarrow \frac{ds}{dt} = 0$

$$\begin{aligned}
 \Rightarrow 3t^2 - 8t + 4 &= 0 \\
 \Rightarrow (3t - 2)(t - 2) &= 0 \\
 \Rightarrow t = \frac{2}{3} \text{ sec, } t = 2 \text{ sec}
 \end{aligned}$$

Q9. $h = 600t - 5t^2$

(i) $\Rightarrow \frac{dh}{dt} = 600 - 10t = 0$

$$\Rightarrow -10t = -600$$

$$\Rightarrow t = 60 \text{ secs}$$

(ii) When $t = 60 \text{ secs} \Rightarrow h = 600(60) - 5(60)^2$

$$\begin{aligned}
 &= 36000 - 18000 \\
 &= 18000 \text{ m} = 18 \text{ km}
 \end{aligned}$$

Q10. $s = t^3 - 2t^2 + 4t$

(i) At $t = 2 \Rightarrow s = (2)^3 - 2(2)^2 + 4(2)$
 $= 8 - 8 + 8 = 8 \text{ m}$

(ii) Velocity $= \frac{ds}{dt} = 3t^2 - 4t + 4 = 0$
 $\Rightarrow 3t^2 - 4t = 0$
 $\Rightarrow t(3t - 4) = 0$
 $\Rightarrow t = 0 \text{ OR } t = \frac{4}{3}$

Q11. $s = 2t^3 - 5t^2 + 4t - 5$

(i) Velocity $= \frac{ds}{dt} = 6t^2 - 10t + 4 = 0$
 $\Rightarrow 3t^2 - 5t + 2 = 0$
 $\Rightarrow (3t - 2)(t - 1) = 0$
 $\Rightarrow t = \frac{2}{3} \text{ OR } t = 1$

Acceleration $= \frac{d^2s}{dt^2} = 12t - 10$

At $t = \frac{2}{3} \Rightarrow \frac{d^2s}{dt^2} = 12\left(\frac{2}{3}\right) - 10 = -2$

At $t = 1 \Rightarrow \frac{d^2s}{dt^2} = 12(1) - 10 = 2$

(ii) Acceleration $= 0 \Rightarrow 12t - 10 = 0$
 $\Rightarrow 12t = 10 \Rightarrow t = \frac{5}{6}$

$\Rightarrow \text{velocity} = \frac{ds}{dt} = 6\left(\frac{5}{6}\right)^2 - 10\left(\frac{5}{6}\right) + 4$
 $= \frac{25}{6} - \frac{25}{3} + 4 = -\frac{1}{6}$

Q12. $x = t^3 - 11t^2 + 24t - 3$

(i) $t = 0 \Rightarrow x = (0)^3 - 11(0)^2 + 24(0) - 3 = -3$

$\frac{dx}{dt} = 3t^2 - 22t + 24$

at $t = 0 \Rightarrow \text{Velocity} = \frac{dx}{dt} = 3(0)^2 - 22(0) + 24 = 24$

$\Rightarrow 3 \text{ cm to the left of O, moving to the right at } 24 \text{ cm/sec.}$

(ii) Velocity $= \frac{dx}{dt} = 3t^2 - 22t + 24$

$$\begin{aligned}
\text{(iii) Particle is stationary} &\Rightarrow \frac{dx}{dt} = 0 \\
&\Rightarrow 3t^2 - 22t + 24 = 0 \\
&\Rightarrow (3t - 4)(t - 6) = 0 \\
&\Rightarrow t = \frac{4}{3} \text{ secs OR } t = 6 \text{ secs} \\
\text{(iv) } t = \frac{4}{3} &\Rightarrow x = \left(\frac{4}{3}\right)^3 - 11\left(\frac{4}{3}\right)^2 + 24\left(\frac{4}{3}\right) - 3 \\
&= \frac{64}{27} - \frac{176}{9} + 32 - 3 = 11\frac{22}{27} \\
t = 6 &\Rightarrow x = (6)^3 - 11(6)^2 + 24(6) - 3 \\
&= 216 - 396 + 144 - 3 = -39 \\
&\Rightarrow 11\frac{22}{27} \text{ cm to the right of O and 39 cm to the left of O}
\end{aligned}$$

$$\text{(v) Velocity negative} = 6 - \frac{4}{3} = 4\frac{2}{3} \text{ secs}$$

$$\text{(vi) Acceleration} = \frac{d^2x}{dt^2} = 6t - 22$$

$$\begin{aligned}
\text{(vii) Acceleration is zero} &\Rightarrow 6t - 22 = 0 \\
&\Rightarrow 6t = 22 \\
&\Rightarrow t = \frac{22}{6} = \frac{11}{3} \text{ secs}
\end{aligned}$$

$$\begin{aligned}
\text{When } t = \frac{11}{3} &\Rightarrow s = \left(\frac{11}{3}\right)^3 - 11\left(\frac{11}{3}\right)^2 + 24\left(\frac{11}{3}\right) - 3 \\
&= \frac{1331}{27} - \frac{1331}{9} + 88 - 3 = -13\frac{16}{27} \text{ cm}
\end{aligned}$$

$$\begin{aligned}
\text{When } t = \frac{11}{3} &\Rightarrow \frac{ds}{dt} = 3\left(\frac{11}{3}\right)^2 - 22\left(\frac{11}{3}\right) + 24 \\
&= \frac{121}{3} - \frac{242}{3} + 24 = -16\frac{1}{3} \text{ cm/sec}
\end{aligned}$$

Hence, when $t = \frac{11}{3}$ secs, the particle is $13\frac{16}{27}$ cm left of O and moving to the left at $16\frac{1}{3}$ cm/sec.

Q13. $n = n_0 e^{0.2t}$

$$\text{When } n_0 = 5 \Rightarrow n = 5e^{0.2t}$$

$$\text{(i) } t = 0 \Rightarrow n = 5e^{0.2(0)} = 5e^0 = 5.1 = 5$$

$$t = 10 \Rightarrow n = 5e^{0.2(10)} = 5e^2 = 36.945 = 37$$

$$\begin{aligned}
\text{(ii) } (0, 5), (10, 37) &\Rightarrow \text{average rate of growth} = \frac{37-5}{10-0} \\
&= \frac{32}{10} = 3.2
\end{aligned}$$

$$(iii) \quad \frac{dn}{dt} = 5e^{0.2t} \cdot (0.2) = (1)e^{0.2t} = e^{0.2t}$$

$$\text{When } t = 5 \Rightarrow \frac{dn}{dt} = e^{0.2(5)} = e^1 = e$$

Exercise 3.6

$$\text{Q1. (i) } \frac{dr}{dt} \quad (ii) \frac{dt}{dr} \quad (iii) \frac{ds}{dt}$$

$$\begin{aligned} \text{Q2. (i)} \quad \frac{dA}{dt} &= \frac{dA}{dr} \cdot \frac{dr}{dt} \\ \Rightarrow 8 &= 4 \cdot \frac{dr}{dt} \\ \Rightarrow \frac{dr}{dt} &= \frac{8}{4} = 2 \end{aligned}$$

$$\begin{aligned} (ii) \quad \frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\ \Rightarrow 8 &= \frac{dV}{dr} \cdot 2 \\ \Rightarrow \frac{dV}{dr} &= \frac{8}{2} = 4 \end{aligned}$$

$$\begin{aligned} \text{Q3.} \quad \frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\ &= (10) \cdot (2) = 20 \end{aligned}$$

$$\begin{aligned} \text{Q4.} \quad A &= \pi r^2 \text{ and } \frac{dr}{dt} = 1 \\ \Rightarrow \frac{dA}{dr} &= 2\pi r \\ \Rightarrow \frac{dA}{dt} &= \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r \cdot 1 = 2\pi r \\ \text{When } r &= 5 \Rightarrow \frac{dA}{dt} = 2\pi(5) = 10\pi \end{aligned}$$

$$\begin{aligned} \text{Q5.} \quad \text{Given } \frac{dr}{dt} &= 3 \text{ cm/sec, find } \frac{dA}{dt} \\ \Rightarrow A &= \pi r^2 \\ \Rightarrow \frac{dA}{dr} &= 2\pi r \\ \Rightarrow \frac{dA}{dt} &= \frac{dA}{dr} \cdot \frac{dr}{dt} = (2\pi r) \cdot (3) = 6\pi r \\ \text{When } r &= 9 \text{ cm} \Rightarrow \frac{dA}{dt} = 6\pi(9) = 54\pi \text{ cm}^2/\text{sec} \end{aligned}$$

Q6. Given $\frac{dx}{dt} = 5 \text{ cm/sec}$, find $\frac{dA}{dt}$

$$\Rightarrow A = x \cdot x = x^2$$

$$\Rightarrow \frac{dA}{dx} = 2x$$

$$\Rightarrow \frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = 2x(5) = 10x$$

When $x = 10 \text{ cm} \Rightarrow \frac{dA}{dt} = 10(10) = 100 \text{ cm}^2/\text{sec}.$

Q7. Given $\frac{dp}{dt} = 2$, find $\frac{dM}{dt}$

$$M = (2p + 3)^4 \Rightarrow \frac{dM}{dp} = 4(2p + 3)^3 \cdot 2 = 8(2p + 3)^3$$

$$\frac{dM}{dt} = \frac{dM}{dp} \cdot \frac{dp}{dt} = 8(2p + 3)^3 \cdot 2 = 16(2p + 3)^3$$

When $p = 1 \Rightarrow \frac{dM}{dt} = 16[2(1) + 3]^3$

$$= 16(125) = 2000$$

Q8. Given $\frac{dV}{dt} = 6 \text{ cm}^3/\text{sec}$, find $\frac{dr}{dt}$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$\Rightarrow 6 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{6}{4\pi r^2} = \frac{3}{2\pi r^2}$$

When $r = 3 \Rightarrow \frac{dr}{dt} = \frac{3}{2\pi(3)^2} = \frac{3}{18\pi} = \frac{1}{6\pi} \text{ cm/sec}$

Q9. Given $\frac{dV}{dt} = 24\pi \text{ cm}^3/\text{sec}$, find $\frac{dr}{dt}$

$$\Rightarrow V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$\Rightarrow 24\pi = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{24\pi}{4\pi r^2} = \frac{6}{r^2}$$

At $r = 6 \Rightarrow \frac{dr}{dt} = \frac{6}{(6)^2} = \frac{1}{6} \text{ cm/sec}$

Q10. Rectangle: length = x cm and width = w cm

$$\text{Perimeter} = 40 \text{ cm} \Rightarrow 2x + 2w = 40$$

$$\Rightarrow x + w = 20$$

$$\Rightarrow w = 20 - x$$

(i) Area of rectangle = $x.w$

$$\Rightarrow A = x(20 - x) = (20x - x^2) \text{ cm}^2$$

$$(ii) \quad \frac{dA}{dx} = 20 - 2x$$

$$\text{Hence, } \frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = (20 - 2x) \cdot (0.5) = 10 - x$$

$$\text{When } x = 3 \Rightarrow \frac{dA}{dt} = 10 - 3 = 7 \text{ cm}^2/\text{sec}$$

Q11. Given $\frac{dx}{dt} = 10\sqrt{2}$, find $\frac{dy}{dt}$

$$y = x - \frac{x^2}{40} = x - \frac{1}{40} \cdot x^2$$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{40} \cdot 2x = 1 - \frac{1}{20}x$$

$$\text{Hence, } \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= \left(1 - \frac{1}{20}x\right) 10\sqrt{2}$$

$$\text{When } x = 10 \Rightarrow \frac{dy}{dt} = \left[1 - \frac{1}{20}(10)\right] \cdot 10\sqrt{2}$$

$$= \left(1 - \frac{1}{2}\right) \cdot 10\sqrt{2}$$

$$= \frac{1}{2} \cdot 10\sqrt{2} = 5\sqrt{2}$$

Q12. (i) Given $\frac{dr}{dt} = 1 \text{ cm/sec}$, find $\frac{dV}{dt}$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

$$\text{Hence, } \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot 1 = 4\pi r^2$$

$$\text{When } r = 2 \text{ m} = 200 \text{ cm} \Rightarrow \frac{dV}{dt} = 4\pi(200)^2 = 160000\pi \text{ cm}^3/\text{sec}$$

$$\begin{aligned}
 \text{(ii)} \quad \frac{dV}{dt} &= 160000 \pi \text{ cm}^3 / \text{sec} & V &= \frac{4}{3} \pi r^3 \\
 \frac{dr}{dt} &= \frac{dV}{dt} \cdot \frac{dr}{dV} & \Rightarrow \frac{dV}{dr} &= 4\pi r^2 \\
 &= 160000 \pi \times \frac{1}{4\pi r^2} & \Rightarrow \frac{dr}{dV} &= \frac{1}{4\pi r^2} \\
 &= 160000 \pi \times \frac{1}{4\pi (500)^2} \\
 &= \frac{4}{25} = 0.16 \text{ cm/sec}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \text{Area } (A) &= 4\pi r^2 & A &= 4\pi r^2 \\
 \frac{dA}{dt} &= \frac{dA}{dr} \cdot \frac{dr}{dt} & \Rightarrow \frac{dA}{dr} &= 8\pi r \\
 &= 8\pi r(0.16) \dots \text{from (ii) above} \\
 &= 8\pi \cdot 500(0.16) \\
 &= 640 \pi \text{ cm}^2 / \text{sec}
 \end{aligned}$$

Q13.

Ground to the top of the ladder = $(8 - y)$ m

Bottom of the ladder to the wall = $(6 + x)$ m

Right-angled triangle: $(8 - y)^2 + (6 + x)^2 = (10)^2$

$$\Rightarrow (6 + x)^2 = 100 - (8 - y)^2$$

$$\Rightarrow 6 + x = \sqrt{100 - (8 - y)^2}$$

$$\Rightarrow x = \left(100 - (8 - y)^2\right)^{\frac{1}{2}} - 6$$

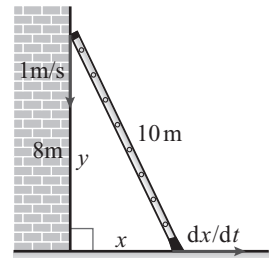
$$\Rightarrow \frac{dx}{dy} = \frac{1}{2} \left[100 - (8 - y)^2\right]^{\frac{1}{2}} \cdot -2(8 - y) \cdot -1$$

$$\Rightarrow \frac{dx}{dy} = \frac{8 - y}{\sqrt{100 - (8 - y)^2}}$$

$$\begin{aligned}
 \text{Given } \frac{dy}{dt} &= 1 \Rightarrow \frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} \\
 &= \frac{8 - y}{\sqrt{100 - (8 - y)^2}} \cdot 1
 \end{aligned}$$

$$\text{Since } 8 - y = 8 \Rightarrow y = 0$$

$$\begin{aligned}
 \text{hence, } \frac{dx}{dt} &= \frac{8 - 0}{\sqrt{100 - (8 - 0)^2}} = \frac{8}{\sqrt{100 - 64}} \\
 &= \frac{8}{\sqrt{36}} = \frac{8}{6} = 1\frac{1}{3} \text{ m/sec}
 \end{aligned}$$



Q14.Cylinder: Radius = r , height = $4r$

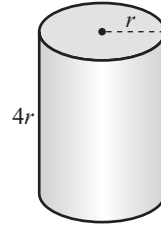
$$\text{Volume} = \pi r^2 h = \pi r^2 (4r) = 4\pi r^3$$

Given $\frac{dr}{dt} = 0.5 \text{ cm/sec}$, find $\frac{dV}{dt}$

$$\Rightarrow \frac{dV}{dr} = 4\pi \cdot 3r^2 = 12\pi r^2$$

hence, $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = (12\pi r^2) \cdot (0.5) = 6\pi r^2$

when $r = 6 \text{ cm} \Rightarrow \frac{dV}{dt} = 6\pi(6)^2 = 216\pi \text{ cm}^3 / \text{sec}$

**Q15.**Circumference $C = 2\pi r$

$$\Rightarrow \frac{dC}{dr} = 2\pi$$

$$\text{Area} = \pi r^2$$

$$\Rightarrow \frac{dA}{dr} = 2\pi r$$

hence, $\frac{dC}{dA} = \frac{dC}{dr} \cdot \frac{dr}{dA} = 2\pi \cdot \frac{1}{2\pi r} = \frac{1}{r}$

Given $\frac{dA}{dt} = 2 \text{ cm}^2 / \text{sec}$, find $\frac{dC}{dt}$

Hence, $\frac{dC}{dt} = \frac{dC}{dA} \cdot \frac{dA}{dt}$

$$= \frac{1}{r} \cdot 2 = \frac{2}{r}$$

When $r = 3 \text{ cm} \Rightarrow \frac{dC}{dt} = \frac{2}{3} \text{ cm/sec}$

Revision Exercise 3 (Core)

Q1.

$$y = x^2 - \frac{9}{x} = x^2 - 9x^{-1}$$

$$\Rightarrow \frac{dy}{dx} = 2x + 9x^{-2} = 2x + \frac{9}{x^2}$$

When $x = 3 \Rightarrow \frac{dy}{dx} = 2(3) + \frac{9}{(3)^2} = 6 + 1 = 7$

When $x = 3 \Rightarrow y = (3)^2 - \frac{9}{3} = 6 \Rightarrow \text{point } (3, 6)$

\Rightarrow Equation of Tangent: $y - 6 = 7(x - 3)$

$$\Rightarrow y - 6 = 7x - 21$$

$$\Rightarrow 7x - y - 15 = 0$$

Q2.

$$y = x^3 - 12x + 5$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 12 = 0$$

$$\Rightarrow x^2 - 4 = 0$$

$$\Rightarrow (x+2)(x-2) = 0$$

$$\Rightarrow x = -2 \text{ OR } x = 2$$

$$\begin{aligned} \text{At } x = -2 \Rightarrow y &= (-2)^3 - 12(-2) + 5 \\ &= -8 + 24 + 5 = 21 \Rightarrow \text{point } (-2, 21) \end{aligned}$$

$$\begin{aligned} \text{At } x = 2 \Rightarrow y &= (2)^3 - 12(2) + 5 \\ &= 8 - 24 + 5 = -11 \Rightarrow \text{point } (2, -11) \end{aligned}$$

$$\frac{d^2y}{dx^2} = 6x$$

$$\text{At } (-2, 21) \Rightarrow \frac{d^2y}{dx^2} = 6(-2) = -12 < 0 \Rightarrow \text{maximum}$$

$$\text{At } (2, -11) \Rightarrow \frac{d^2y}{dx^2} = 6(2) = 12 > 0 \Rightarrow \text{minimum}$$

Q3.

$$f(x) = x^3 - bx^2 - 9x + 7$$

$$\Rightarrow f'(x) = 3x^2 - 2bx - 9$$

$$\text{At } x = -1 \Rightarrow f'(-1) = 3(-1)^2 - 2b(-1) - 9 = 0$$

$$\Rightarrow 3 + 2b - 9 = 0$$

$$\Rightarrow 2b = 6$$

$$\Rightarrow b = 3$$

Q4.

$$f(x) = x^3 + 3x^2 - 9x$$

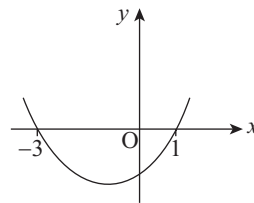
$$\Rightarrow f'(x) = 3x^2 + 6x - 9 < 0$$

$$\Rightarrow x^2 + 2x - 3 < 0$$

$$\text{Factors } \Rightarrow (x+3)(x-1) = 0$$

$$\text{Roots } \Rightarrow x = -3, x = 1$$

$$\text{Solution: } -3 < x < 1$$

**Q5.**

$$y = 6x^2 - x^3$$

$$(i) \quad \frac{dy}{dx} = 12x - 3x^2 = 12$$

$$\Rightarrow 3x^2 - 12x + 12 = 0$$

$$\Rightarrow x^2 - 4x + 4 = 0$$

$$\Rightarrow (x-2)(x-2) = 0$$

$$\Rightarrow x = 2$$

$$\begin{aligned} (ii) \quad x = 2 \Rightarrow y &= 6(2)^2 - (2)^3 \\ &= 24 - 8 = 16 \Rightarrow \text{point } (2, 16) \end{aligned}$$

$$\text{Equation of Tangent: } y - 16 = 12(x - 2)$$

$$\Rightarrow y - 16 = 12x - 24$$

$$\Rightarrow 12x - y - 8 = 0$$

Q6. $s = 2t^3 - 24t$

(i) $\Rightarrow \frac{ds}{dt} = 6t^2 - 24$

At $t = 4 \Rightarrow \text{speed} = \frac{ds}{dt} = 6(4)^2 - 24$
 $= 96 - 24 = 72 \text{ m/sec}$

(ii) Particle at rest $\Rightarrow \frac{ds}{dt} = 0$
 $\Rightarrow 6t^2 - 24 = 0$
 $\Rightarrow t^2 - 4 = 0$
 $\Rightarrow (t+2)(t-2) = 0$
 $\Rightarrow t = -2 \text{ (invalid) OR } t = 2 \text{ (valid)}$
 $\Rightarrow \text{Ans} = 2 \text{ secs}$

Q7. $y = x \sin 2x$

Product rule: $u = x$ and $v = \sin 2x$

$\Rightarrow \frac{du}{dx} = 1 \Rightarrow \frac{dv}{dx} = \cos 2x \cdot 2 = 2 \cos 2x$

$\Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$
 $= x \cdot 2 \cos 2x + \sin 2x \cdot 1$
 $= 2x \cos 2x + \sin 2x$

At $x = \frac{\pi}{3} \Rightarrow \frac{dy}{dx} = 2 \left(\frac{\pi}{3} \right) \cdot \cos 2 \left(\frac{\pi}{3} \right) + \sin 2 \left(\frac{\pi}{3} \right)$
 $= \frac{2\pi}{3} \cos \frac{2\pi}{3} + \sin \frac{2\pi}{3}$
 $= \frac{2\pi}{3} \cdot \left(-\frac{1}{2} \right) + \frac{\sqrt{3}}{2}$
 $= \frac{\sqrt{3}}{2} - \frac{\pi}{3}$

Q8. $x + y = 100$

$\Rightarrow y = 100 - x$

$\Rightarrow P = x \cdot y = x(100 - x) = 100x - x^2$

$\Rightarrow \frac{dP}{dx} = 100 - 2x = 0$

$\Rightarrow 2x = 100$

$\Rightarrow x = 50 \Rightarrow y = 100 - 50 = 50$

$\Rightarrow \frac{d^2P}{dx^2} = -2 < 0 \Rightarrow \text{maximum}$

Hence, $P = (50)(50) = 2500$

Q9. $s(x) = -x^3 + 3x^2 + 360x + 5000$
 $\Rightarrow s'(x) = -3x^2 + 6x + 360 = 0$
 $\Rightarrow x^2 - 2x - 120 = 0$
 $\Rightarrow (x+10)(x-12) = 0$
 $\Rightarrow x = -10 \text{ OR } x = 12$
 $\Rightarrow s''(x) = -6x + 6$
At $x = 12 \Rightarrow s''(12) = -6(12) + 6 = -66 < 0 \Rightarrow \text{maximum}$
Since $6 \leq x \leq 20 \Rightarrow x = 12^\circ \text{C}$

Q10. Given $\frac{dV}{dt} = 12 \text{ cm}^3 / \text{sec}$, find $\frac{dx}{dt}$
Volume (V) = $x.x.x = x^3$ Volume = 125
 $\Rightarrow \frac{dV}{dx} = 3x^2 \qquad \qquad \qquad \Rightarrow x^3 = 125$
 $\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \Rightarrow x = \sqrt[3]{125} = 5$

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$

$$\Rightarrow 12 = 3x^2 \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{12}{3x^2} = \frac{4}{x^2}$$

when $x = 5 \Rightarrow \frac{dx}{dt} = \frac{4}{(5)^2} = \frac{4}{25} \text{ cm/sec}$

Q11. $y = x + \frac{4}{x} = x + 4x^{-1}$
 $\Rightarrow \frac{dy}{dx} = 1 - 4x^{-2} = 1 - \frac{4}{x^2} = 0$
 $\Rightarrow x^2 - 4 = 0$
 $\Rightarrow (x+2)(x-2) = 0$
 $\Rightarrow x = -2 \text{ OR } x = 2$
When $x = -2 \Rightarrow y = -2 + \frac{4}{-2} = -2 - 2 = -4 \Rightarrow \text{point } (-2, -4)$
When $x = 2 \Rightarrow y = 2 + \frac{4}{2} = 2 + 2 = 4 \Rightarrow \text{point } (2, 4)$
Hence, $\frac{d^2y}{dx^2} = 0 + 8x^{-3} = \frac{8}{x^3}$
At $(-2, -4) \Rightarrow \frac{d^2y}{dx^2} = \frac{8}{(-2)^3} = \frac{8}{-8} = -1 < 0 \Rightarrow \text{maximum}$
At $(2, 4) \Rightarrow \frac{d^2y}{dx^2} = \frac{8}{(2)^3} = \frac{8}{8} = 1 > 0 \Rightarrow \text{minimum}$
For $x > 0 \Rightarrow \text{curve is increasing after the minimum point } (2, 4); \text{ hence } x > 2.$

Q12. (i) $f(x) = ax^2 + bx + c$
 $\Rightarrow f'(x) = 2ax + b$
 $\Rightarrow f''(x) = 2a$
 $f''(x) = 6 \Rightarrow 2a = 6 \Rightarrow a = 3$
 Gradient at (2, 24) is 22 $\Rightarrow f'(2) = 2a(2) + b = 22$
 $\qquad\qquad\qquad = 4a + b = 22$
 $\qquad\qquad\qquad a = 3 \Rightarrow 4(3) + b = 22$
 $\qquad\qquad\qquad \Rightarrow 12 + b = 22$
 $\qquad\qquad\qquad \Rightarrow b = 10$
 Point (2, 24) $\Rightarrow f(2) = 3(2)^2 + 10(2) + c = 24$
 $\qquad\qquad\qquad \Rightarrow 12 + 20 + c = 24$
 $\qquad\qquad\qquad \Rightarrow c = 24 - 32 = -8$

(ii) $f(x) = 3x^2 + 10x - 8$
 On x -axis $\Rightarrow f(x) = 0 \Rightarrow 3x^2 + 10x - 8 = 0$
 $\qquad\qquad\qquad \Rightarrow (x+4)(3x-2) = 0$
 $\qquad\qquad\qquad \Rightarrow x = -4, x = \frac{2}{3}$
 \Rightarrow Points on x -axis $= (-4, 0), \left(\frac{2}{3}, 0\right)$
 On y -axis $\Rightarrow x = 0 \Rightarrow f(0) = 3(0)^2 + 10(0) - 8 = -8$
 \Rightarrow Point on y -axis $= (0, -8)$

(iii) Turning point $\Rightarrow f'(x) = 6x + 10 = 0$
 $\qquad\qquad\qquad \Rightarrow 6x = -10$
 $\qquad\qquad\qquad \Rightarrow x = \frac{-10}{6} = \frac{-5}{3} = -1\frac{2}{3}$
 $\Rightarrow f\left(-1\frac{2}{3}\right) = 3\left(-1\frac{2}{3}\right)^2 + 10\left(-1\frac{2}{3}\right) - 8$
 $\qquad\qquad\qquad = \frac{25}{3} - \frac{50}{3} - 8 = -16\frac{1}{3} \Rightarrow \text{Point } \left(-1\frac{2}{3}, -16\frac{1}{3}\right)$
 $f''(x) = 6 > 0 \Rightarrow \text{minimum point} = \left(-1\frac{2}{3}, -16\frac{1}{3}\right)$

Q13. Given $\frac{dV}{dt} = 10\pi \text{ cm}^3 / \text{min}$, find $\frac{dr}{dt}$
 Volume $(V) = \frac{4}{3}\pi r^3$
 $\Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$
 Hence, $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$
 $\Rightarrow 10\pi = 4\pi r^2 \cdot \frac{dr}{dt}$
 $\Rightarrow \frac{dr}{dt} = \frac{10\pi}{4\pi r^2} = \frac{5}{2r^2}$
 At $r = 5 \Rightarrow \frac{dr}{dt} = \frac{5}{2(5)^2} = \frac{5}{50} = \frac{1}{10} \text{ cm/sec}$

Q14. $s = 196t - 4.9t^2$

$$\frac{ds}{dt} = 196 - 9.8t = 0$$

$$\Rightarrow 9.8t = 196$$

$$\Rightarrow t = \frac{196}{9.8} = 20 \text{ secs}$$

$$\frac{d^2s}{dt^2} = 0 - 9.8 = -9.8 < 0 \Rightarrow \text{maximum at } t = 20$$

$$\Rightarrow s = 196(20) - 4.9(20)^2$$

$$= 3920 - 1960 = 1960 \text{ m}$$

- Q15.** Answer = Graph Ⓒ
- $f(x)$ has no turning points \Rightarrow Graph Ⓒ is above x -axis for all values of x
 - $f(x)$ is strictly increasing \Rightarrow Graph Ⓒ is positive for all values of x
 - $f(x)$ has a point of inflection at the turning point of graph Ⓒ

Revision Exercise 3 (Advanced)

Q1. $f(x) = \sqrt{x^2 - 3} = (x^2 - 3)^{\frac{1}{2}}$

$$\Rightarrow f'(x) = \frac{1}{2}(x^2 - 3)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2 - 3}}$$

Point (2,1) $\Rightarrow f'(2) = \frac{2}{\sqrt{(2)^2 - 3}} = \frac{2}{\sqrt{1}} = \frac{2}{1} = 2$ (slope)

Equation of Tangent: $y - 1 = 2(x - 2)$

$$\Rightarrow y - 1 = 2x - 4$$

$$\Rightarrow 2x - y - 3 = 0$$

Q2. $y = x^2 + \ln x$

$$\Rightarrow \frac{dy}{dx} = 2x + \frac{1}{x} = 3$$

$$\Rightarrow 2x^2 + 1 = 3x$$

$$\Rightarrow 2x^2 - 3x + 1 = 0$$

$$\Rightarrow (2x - 1)(x - 1) = 0$$

$$\Rightarrow x = \frac{1}{2}, \quad x = 1$$

$$x = \frac{1}{2} \Rightarrow y = \left(\frac{1}{2}\right)^2 + \ln\left(\frac{1}{2}\right) = \frac{1}{4} - \ln 2 \Rightarrow \text{point}\left(\frac{1}{2}, \frac{1}{4} - \ln 2\right)$$

$$x = 1 \Rightarrow y = (1)^2 + \ln(1) = 1 + 0 = 1 \Rightarrow \text{point}(1, 1)$$

Q3. (i) Given $\frac{dV}{dt} = 2 \text{ m}^3 / \text{min}$, find $\frac{dr}{dt}$

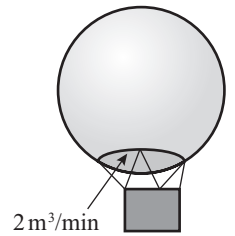
$$\text{Volume } (V) = \frac{4}{3} \pi r^3$$

$$\Rightarrow \frac{dV}{dr} = \frac{4}{3} \pi (3r^2) = 4\pi r^2$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \Rightarrow 2 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{2}{4\pi r^2} = \frac{1}{2\pi r^2}$$

$$\text{At } r = 2.5 \text{ m} \Rightarrow \frac{dr}{dt} = \frac{1}{2\pi (2.5)^2} = \frac{2}{25\pi} \text{ m/min}$$



(ii) Given $\frac{dr}{dt} = \frac{2}{25\pi}$, find $\frac{dA}{dt}$

$$\text{Area } (A) = 4\pi r^2$$

$$\Rightarrow \frac{dA}{dr} = 8\pi r$$

$$\text{Hence, } \frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{2}{25\pi} = \frac{16}{25} r$$

$$\text{At } r = 2.5 \Rightarrow \frac{dA}{dt} = \frac{16}{25} (2.5) = 1.6 \text{ m}^2 / \text{min}$$

Q4. (a)

$$y = x^3 - 6x^2 + 8x$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 8 = -1$$

$$\Rightarrow 3x^2 - 12x + 9 = 0$$

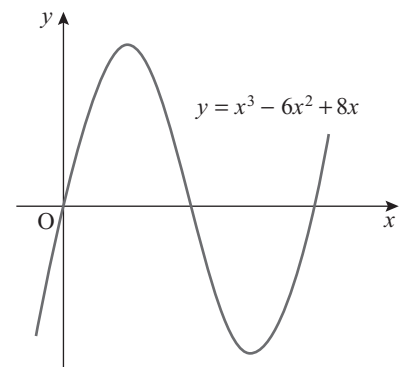
$$\Rightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow (x-1)(x-3) = 0$$

$$\Rightarrow x = 1 \text{ OR } x = 3$$

$$\Rightarrow y = (1)^3 - 6(1)^2 + 8(1) = 3 \text{ OR } y = (3)^3 - 6(3)^2 + 8(3) = -3$$

$$\text{points } (1, 3) \quad \text{OR} \quad (3, -3)$$



(b) Line $y = 4 - x$

$$\Rightarrow \frac{dy}{dx} = 0 - 1 = -1 \Rightarrow A = (1, 3) \text{ First Quadrant}$$

(c) Graph of $\frac{dy}{dx}$:

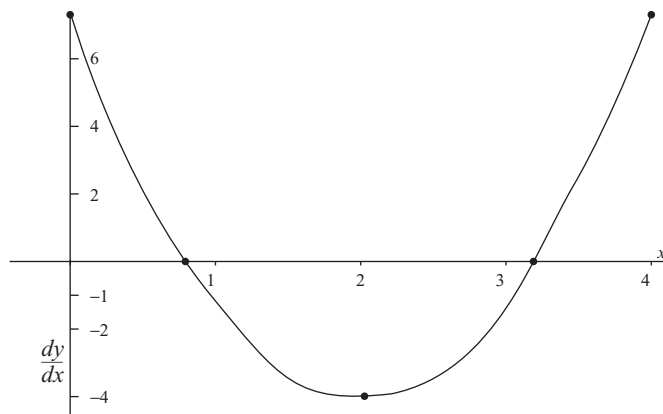
$$\frac{dy}{dx} = 3x^2 - 12 + 8 = 0$$

$$\Rightarrow x = 0.85, 3.15$$

$$\frac{d^2y}{dx^2} = 6x - 12 = 0$$

$$\Rightarrow x = 2$$

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= 3(2)^2 - 12(2) + 8 \\ &= -4\end{aligned}$$



In the graph of $\frac{dy}{dx}$, there is a turning point at $(2, -4)$.

This shows that there is a point of inflection at $x = 2$ in the graph of $y = x^3 - 6x^2 + 8x$.

Q5. (i) Volume = $x \cdot x \cdot h = 500 \text{ cm}^3$

$$\Rightarrow V = x^2 h = 500$$

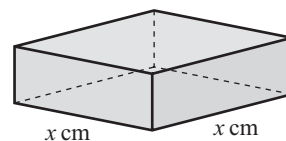
$$\Rightarrow h = \frac{500}{x^2}$$

$$\text{Area } (A) = x \cdot x + 4x \cdot h$$

$$= x^2 + 4xh$$

$$= x^2 + 4x \left(\frac{500}{x^2} \right)$$

$$\Rightarrow A = x^2 + \frac{2000}{x}$$



(ii) $A = x^2 + 2000x^{-1}$

$$\Rightarrow \frac{dA}{dx} = 2x - 2000x^{-2} = 0$$

$$\Rightarrow 2x - \frac{2000}{x^2} = 0$$

$$\Rightarrow 2x^3 - 2000 = 0$$

$$\Rightarrow x^3 = 1000$$

$$\Rightarrow x = \sqrt[3]{1000} = 10 \text{ cm} \Rightarrow h = \frac{500}{(10)^2} = 5 \text{ cm}$$

$$\frac{d^2A}{dx^2} = 2 + 4000x^{-3} = 2 + \frac{4000}{x^3}$$

$$\text{At } x = 10 \Rightarrow \frac{d^2A}{dx^2} = 2 + \frac{4000}{(10)^3} = 6 > 0 \Rightarrow \text{minimum}$$

$$\Rightarrow \text{Area} = (10)^2 + 4(10)(5) = 300 \text{ cm}^2$$

Q6. (i) $C = \frac{16}{t^3} + \frac{3t^2}{4} = 16t^{-3} + \frac{3}{4}t^2$

$$\Rightarrow \frac{dC}{dt} = -48t^{-4} + \frac{3}{4}(2t) = \frac{-48}{t^4} + \frac{3}{2}t$$

When $t = 4$ hours $\Rightarrow \frac{dC}{dt} = \frac{-48}{(4)^4} + \frac{3}{2}(4)$

$$= \frac{-3}{16} + 6 = \text{€ } 5\frac{13}{16} \text{ per hour}$$

(ii) Minimum $\Rightarrow \frac{dC}{dt} = 0$

$$\Rightarrow \frac{-48}{t^4} + \frac{3}{2}t = 0$$

$$\Rightarrow -96 + 3t^5 = 0$$

$$\Rightarrow 3t^5 = 96$$

$$\Rightarrow t^5 = 32$$

$$\Rightarrow t = \sqrt[5]{32} = 2$$

$$\Rightarrow \frac{d^2C}{dt^2} = 192t^{-5} + \frac{3}{2} = \frac{192}{t^5} + \frac{3}{2}$$

At $t = 2 \Rightarrow \frac{d^2C}{dt^2} = \frac{192}{(2)^5} + 1\frac{1}{2} = 7\frac{1}{2} > 0 \Rightarrow \text{Minimum}$

Hence, $C = \frac{16}{(2)^3} + \frac{3(2)^2}{4} = 2 + 3 = \text{€ } 5$

Q7.

$$y = x.e^x$$

Product Rule: $u = x$ and $v = e^x$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = x.e^x + e^x.1$$

$$= e^x(x+1)$$

Turning point $\Rightarrow \frac{dy}{dx} = 0$

$$\Rightarrow e^x(x+1) = 0$$

$$\Rightarrow x = -1$$

$$\Rightarrow y = (-1)e^{-1} = \frac{-1}{e} \Rightarrow \text{Point} \left(-1, \frac{-1}{e} \right)$$

$$\frac{dy}{dx} = e^x(x+1)$$

Product Rule: $u = e^x$ and $v = x+1$

$$\Rightarrow \frac{du}{dx} = e^x \quad \Rightarrow \frac{dv}{dx} = 1$$

Hence, $\frac{d^2y}{dx^2} = u \frac{dv}{dx} + v \frac{du}{dx} = e^x.(1) + (x+1).e^x$

$$= e^x(x+2)$$

At $x = -1 \Rightarrow \frac{d^2y}{dx^2} = e^{-1}(-1+2) = \frac{1}{e} > 0 \Rightarrow \text{Minimum}$

Q8. (A, 2), (B, 4), (C, 1), (D, 3)

Q9. $y = x^3 + ax^2 + bx + c$

$$\frac{dy}{dx} = 3x^2 + 2ax + b$$

$$\text{At } x = -1 \Rightarrow \frac{dy}{dx} = 3(-1)^2 + 2a(-1) + b = 0$$

$$\Rightarrow 3 - 2a + b = 0$$

$$\Rightarrow -2a + b = -3$$

$$\text{At } x = 3 \Rightarrow \frac{dy}{dx} = 3(3)^2 + 2a(3) + b = 0$$

$$\Rightarrow 27 + 6a + b = 0$$

$$\Rightarrow 6a + b = -27$$

$$\Rightarrow \underline{2a - b = 3}$$

$$\Rightarrow 8a = -24$$

$$\Rightarrow a = -3$$

$$\Rightarrow -2(-3) + b = -3$$

$$\Rightarrow 6 + b = -3$$

$$\Rightarrow b = -9$$

$$\text{Point } (1, 1) \Rightarrow 1 = (1)^3 - 3(1)^2 - 9(1) + c$$

$$\Rightarrow 1 = 1 - 3 - 9 + c$$

$$\Rightarrow 12 = c$$

$$\Rightarrow a = -3, b = -9, c = 12$$

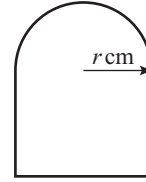
Q10.

Rectangular base \Rightarrow length $= 2r$ and width $= w$

$$\Rightarrow \text{Perimeter} = \pi r + 2r + 2w = 40$$

$$\Rightarrow 2w = 40 - \pi r - 2r$$

$$\Rightarrow w = \frac{1}{2}(40 - \pi r - 2r)$$



$$\text{Hence, Area} = \frac{\pi r^2}{2} + 2r \cdot w$$

$$= \frac{\pi r^2}{2} + 2r \cdot \frac{1}{2}(40 - \pi r - 2r)$$

$$= \frac{\pi r^2}{2} + 40r - \pi r^2 - 2r^2$$

$$\Rightarrow A = 40r - 2r^2 - \frac{\pi r^2}{2}$$

$$\frac{dA}{dr} = 40 - 4r - \frac{\pi}{2}(2r)$$

$$\Rightarrow 40 - 4r - \pi r = 0$$

$$\Rightarrow 4r + \pi r = 40$$

$$\Rightarrow r(4 + \pi) = 40$$

$$\Rightarrow r = \frac{40}{4 + \pi}$$

$$\frac{d^2 A}{dr^2} = 0 - 4 - \pi = -4 - \pi < 0 \Rightarrow \text{Maximum}$$

$$\text{Hence, Area} = 40 \left(\frac{40}{4 + \pi} \right) - 2 \left(\frac{40}{4 + \pi} \right)^2 - \frac{\pi}{2} \left(\frac{40}{4 + \pi} \right)^2$$

$$= \frac{1600}{4 + \pi} - \frac{3200}{(4 + \pi)^2} - \frac{800\pi}{(4 + \pi)^2}$$

$$= \frac{1600(4 + \pi) - 3200 - 800\pi}{(4 + \pi)^2}$$

$$= \frac{6400 + 1600\pi - 3200 - 800\pi}{(4 + \pi)^2}$$

$$= \frac{3200 + 800\pi}{(4 + \pi)^2} = \frac{800(4 + \pi)}{(4 + \pi)^2} = \frac{800}{4 + \pi}$$

Q11. (i) Point $(-1, 2)$ Slope $= m$

$$\Rightarrow \text{Equation of line: } y - 2 = m(x + 1)$$

$$\Rightarrow y - 2 = mx + m$$

$$\Rightarrow y = mx + m + 2$$

$$\text{x-axis } \Rightarrow y = 0 \Rightarrow mx + m + 2 = 0$$

$$\Rightarrow mx = -m - 2$$

$$\Rightarrow x = \frac{-m-2}{m} \Rightarrow A = \left(\frac{-m-2}{m}, 0 \right)$$

$$\text{y-axis } \Rightarrow x = 0 \Rightarrow y = m(0) + m + 2$$

$$= m + 2 \Rightarrow B = (0, m + 2)$$

$$\text{Hence, Area} = \frac{1}{2} \left| \left(\frac{-m-2}{m} \right) (m+2) - (0)(0) \right|$$

$$\Rightarrow A = \frac{(m+2)^2}{2m}$$

(ii) Quotient Rule: $u = (m+2)^2$ and $v = 2m$

$$\Rightarrow \frac{du}{dm} = 2(m+2) \Rightarrow \frac{dv}{dm} = 2$$

$$\text{Hence, } \frac{dA}{dm} = \frac{u \frac{dv}{dm} - v \frac{du}{dm}}{v^2}$$

$$= \frac{(m+2)^2 \cdot 2 - 2m \cdot 2(m+2)}{(2m)^2} = 0$$

$$= 2m^2 + 8m + 8 - 4m^2 - 8m = 0$$

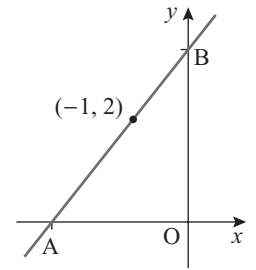
$$\Rightarrow -2m^2 + 8 = 0$$

$$\Rightarrow m^2 - 4 = 0$$

$$\Rightarrow (m+2)(m-2) = 0$$

$$\Rightarrow m = -2 \text{ (not valid) OR } m = 2 \text{ (valid)}$$

$$\text{When } m = 2 \Rightarrow A = \frac{(2+2)^2}{2(2)} = \frac{16}{4} = 4 \text{ sq.units}$$



Q12. Answer D

Q13.

$$f(x) = x^3 + 3kx^2 + 32$$

$$\Rightarrow f'(x) = 3x^2 + 6kx = 0$$

$$\Rightarrow 3x(x + 2k) = 0$$

$$\Rightarrow x = 0, x = -2k$$

$$\text{When } x = 0 \Rightarrow f(0) = (0)^3 + 3k(0)^2 + 32 = 32 \Rightarrow \text{point} = (0, 32)$$

$$\begin{aligned}\text{When } x = -2k \Rightarrow f(-2k) &= (-2k)^3 + 3k(-2k)^2 + 32 \\ &= -8k^3 + 12k^3 + 32 = 4k^3 + 32 \\ &\Rightarrow \text{point} = (-2k, 4k^3 + 32)\end{aligned}$$

Two roots are equal \Rightarrow Point $(-2k, 4k^3 + 32)$ lies on the x -axis $\Rightarrow 4k^3 + 32 = 0$

$$\Rightarrow 4k^3 = -32$$

$$\Rightarrow k^3 = -8$$

$$\Rightarrow k = \sqrt[3]{-8} = -2$$

Q14.

$$\text{Length of pipe} = 4 - x + \sqrt{9 + x^2}$$

$$\text{Cost} = 10(4 - x) + 25\sqrt{9 + x^2}$$

$$\Rightarrow C = 40 - 10x + 25(9 + x^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dC}{dx} = -10 + 25 \cdot \frac{1}{2}(9 + x^2)^{-\frac{1}{2}} \cdot 2x = 25x(9 + x^2)^{-\frac{1}{2}} - 10$$

$$= -10 + \frac{25x}{\sqrt{9 + x^2}} = 0$$

$$\Rightarrow \frac{25x}{\sqrt{9 + x^2}} = 10$$

$$\Rightarrow 25x = 10\sqrt{9 + x^2}$$

$$\Rightarrow 5x = 2\sqrt{9 + x^2}$$

$$\Rightarrow 25x^2 = 4(9 + x^2)$$

$$\Rightarrow 25x^2 = 36 + 4x^2$$

$$\Rightarrow 21x^2 = 36$$

$$\Rightarrow x^2 = \frac{36}{21} = 1.7143$$

$$\Rightarrow x = \sqrt{1.7143} = 1.309 = 1.3$$

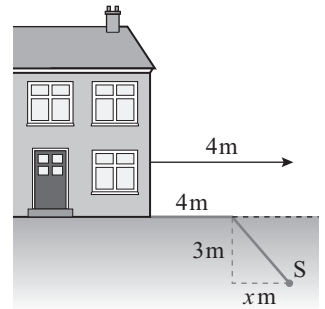
$$\frac{d^2C}{dx^2} = 25x \cdot \frac{-1}{2}(9 + x^2)^{-\frac{3}{2}} \cdot 2x + (9 + x^2)^{-\frac{1}{2}} \cdot 25$$

$$= \frac{-25x^2}{(9 + x^2)^{\frac{3}{2}}} + \frac{25}{\sqrt{9 + x^2}}$$

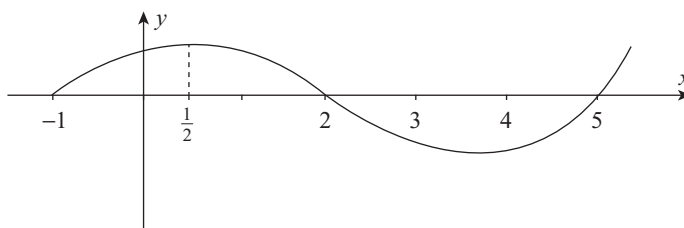
$$\text{At } x = 1.3 \Rightarrow \frac{d^2C}{dx^2} = \frac{-25(1.3)^2}{[9 + (1.3)^2]^{\frac{3}{2}}} + \frac{25}{\sqrt{9 + (1.3)^2}} = 6.4 > 0$$

\Rightarrow Minimum cost

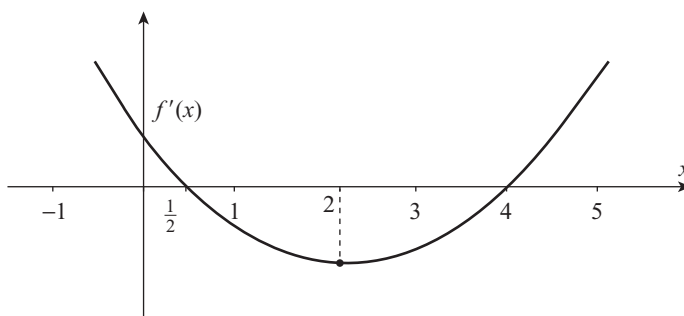
Hence, length $= 4 - 1.3 = 2.7$ m



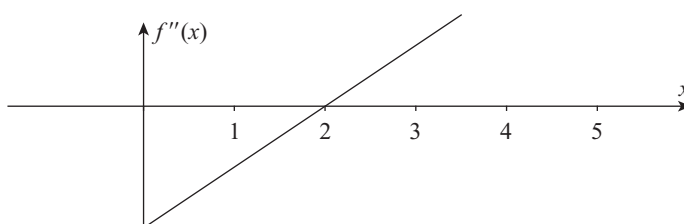
Q15. (i) Graph of $f(x)$



Graph of $f'(x)$



(ii) Graph of $f''(x)$



Point of inflection occurs where $x = 2$

Revision Exercise 3 (Extended-Response)

Q1. (a) Given $\frac{dA}{dt} = 0.032 \text{ cm}^2 / \text{sec}$, find $\frac{dx}{dt}$

Area (A) of cross-section $= \pi x^2$

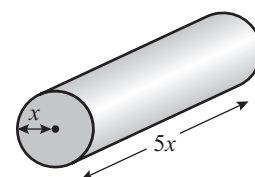
$$\Rightarrow \frac{dA}{dx} = 2\pi x$$

$$\text{Hence, } \frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$$

$$\Rightarrow 0.032 = 2\pi x \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{0.032}{2\pi x}$$

$$\begin{aligned} \text{When } x = 2 \text{ cm } \Rightarrow \frac{dx}{dt} &= \frac{0.032}{2\pi(2)} = \frac{0.032}{4\pi} \\ &= 0.00254 = 0.003 \text{ cm/sec} \end{aligned}$$



$$\begin{aligned}
 \text{(b)} \quad \text{Volume } (V) &= \pi x^2 \cdot 5x = 5\pi x^3 \\
 \Rightarrow \frac{dV}{dx} &= 15\pi x^2 \\
 \frac{dV}{dt} &= \frac{dV}{dx} \cdot \frac{dx}{dt} = 15\pi x^2 \cdot (0.003) \\
 &= 0.045\pi x^2
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = 2 \text{ cm} \Rightarrow \frac{dV}{dt} &= 0.045\pi \cdot (2)^2 \\
 &= 0.56548 = 0.5655 \text{ cm}^3/\text{sec}
 \end{aligned}$$

$$\begin{aligned}
 \text{Q2. (a)} \quad 8x + 8x + 4h &= 20 \\
 \Rightarrow 16x + 4h &= 20 \\
 \Rightarrow 4x + h = 5 \Rightarrow h &= 5 - 4x
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \text{Volume } (V) &= (x) \cdot (3x) \cdot (h) \\
 \Rightarrow V &= 3x^2(5 - 4x) = 15x^2 - 12x^3
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad V = 0 \Rightarrow 15x^2 - 12x^3 &= 0 \\
 \Rightarrow 5x^2 - 4x^3 &= 0 \\
 \Rightarrow x^2(5 - 4x) &= 0 \\
 \Rightarrow x = 0, x = \frac{5}{4}
 \end{aligned}$$

Hence, domain is $0 < x < \frac{5}{4}$

$$\text{(d)} \quad \frac{dV}{dx} = 30x - 36x^2$$

$$\begin{aligned}
 \text{(e)} \quad \frac{dV}{dx} = 0 \Rightarrow 30x - 36x^2 &= 0 \\
 \Rightarrow 5x - 6x^2 &= 0 \\
 \Rightarrow x(5 - 6x) &= 0 \\
 \Rightarrow x = 0 \text{ OR } x = \frac{5}{6}
 \end{aligned}$$

$$\text{Hence, } \frac{d^2V}{dx^2} = 30 - 72x$$

$$\text{At } x = 0 \Rightarrow \frac{d^2V}{dx^2} = 30 - 72(0) = 30 > 0 \Rightarrow \text{minimum}$$

$$\text{At } x = \frac{5}{6} \Rightarrow \frac{d^2V}{dx^2} = 30 - 72\left(\frac{5}{6}\right) = -30 < 0 \Rightarrow \text{maximum}$$

$$\text{At } x = \frac{5}{6} \Rightarrow V = 15\left(\frac{5}{6}\right)^2 - 12\left(\frac{5}{6}\right)^3 = 3\frac{17}{36} \text{ cm}^3$$

Q3. (i) $h = 2 + 40t - 5t^2 \Rightarrow \frac{dh}{dt} = 40 - 10t$

(a) When $t = 2 \Rightarrow \frac{dh}{dt} = 40 - 10(2)$
 $= 40 - 20 = 20 \text{ m/sec}$

(b) When $t = 2.5 \Rightarrow \frac{dh}{dt} = 40 - 10(2.5)$
 $= 40 - 25 = 15 \text{ m/sec.}$

(ii) $\frac{dh}{dt} = 0 \Rightarrow 40 - 10t = 0$
 $\Rightarrow 10t = 40 \Rightarrow t = 4 \text{ secs}$

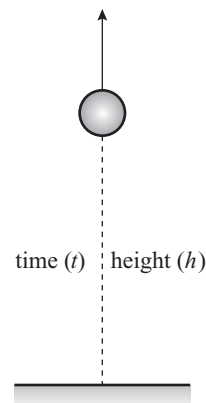
(iii) When $t = 4 \Rightarrow h = 2 + 40(4) - 5(4)^2$
 $= 162 - 80 = 82 \text{ m}$

(iv) When $t = 6 \Rightarrow \frac{dh}{dt} = 40 - 10(6) = -20$
 \Rightarrow Ball is falling towards the ground at 20 m/sec after 6 seconds

(v) When $t = 0 \Rightarrow \frac{dh}{dt} = 40 - 10(0) = 40 \text{ m/sec}$

(vi) When the ball hits the ground, then $h = 0$
 $\Rightarrow 2 + 40t - 5t^2 = 0$
 $\Rightarrow t = \frac{-40 \pm \sqrt{(40)^2 - 4(2)(-5)}}{2(-5)}$
 $= \frac{-40 \pm \sqrt{1640}}{-10}$
 $= \frac{-40 - 40.497}{-10}, \frac{-40 + 40.497}{-10} \text{ (not valid)}$
 $= 8.0497 = 8.05$
 $\Rightarrow \frac{dh}{dt} = 40 - 10(8.05) = 40 - 80.5 = -40.5$

Hence, speed of ball is 40.5 m/sec as it hits the ground after 8.05 secs.

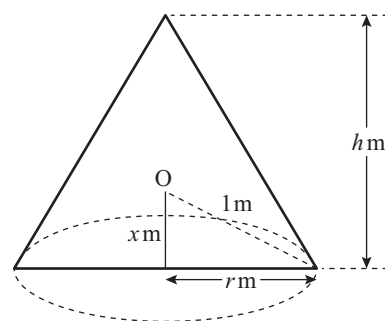


Q4. (a) (i) $x^2 + r^2 = 1^2$
 $\Rightarrow r^2 = 1 - x^2$
 $\Rightarrow r = \sqrt{1 - x^2}$

(ii) $h = 1 + x$

(b) $V = \frac{1}{3} \pi r^2 h$
 $= \frac{1}{3} \pi (\sqrt{1 - x^2})^2 \cdot (1 + x)$
 $= \frac{1}{3} \pi (1 - x^2)(1 + x) = \frac{\pi}{3} (1 + x - x^2 - x^3)$

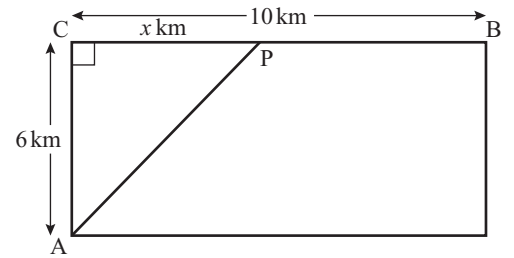
(c) $0 < x < 1$



$$\begin{aligned}
\text{(d) (i)} \quad \frac{dV}{dx} &= \frac{\pi}{3}(1-2x-3x^2) \\
\text{(ii)} \quad \frac{dV}{dx} &= 0 \Rightarrow \frac{\pi}{3}(1-2x-3x^2) = 0 \\
&\Rightarrow (1+x)(1-3x) = 0 \\
&\Rightarrow x = -1 \text{ (not valid) OR } x = \frac{1}{3} \text{ (valid)} \\
\text{(iii)} \quad \frac{d^2V}{dx^2} &= \frac{\pi}{3}(0-2-6x) \\
\text{At } x = \frac{1}{3} &\Rightarrow \frac{\pi}{3}\left(-2-6\left(\frac{1}{3}\right)\right) = \frac{-4\pi}{3} < 0 \\
&\Rightarrow \text{Maximum at } x = \frac{1}{3} \\
&\Rightarrow \text{Volume} = \frac{\pi}{3}\left(1+\frac{1}{3}-\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^3\right) \\
&= \frac{\pi}{3}\left(\frac{4}{3}-\frac{1}{9}-\frac{1}{27}\right) = \frac{32\pi}{81} \text{ m}^3
\end{aligned}$$

Q5.

$$\begin{aligned}
\text{(i)} \quad |AP|^2 &= (6)^2 + x^2 = 36 + x^2 \\
&\Rightarrow |AP| = \sqrt{36 + x^2} \text{ km} \\
&\text{and } |PB| = (10 - x) \text{ km} \\
\text{(ii)} \quad \text{Time}(T) &= \frac{|AP|}{5} + \frac{|PB|}{13} \\
&\Rightarrow T = \frac{1}{5}\sqrt{36 + x^2} + \frac{1}{13}(10 - x) \\
&\Rightarrow T = \frac{1}{5}(36 + x^2)^{\frac{1}{2}} + \frac{10}{13} - \frac{1}{13}x \\
\frac{dT}{dx} &= \frac{1}{5} \cdot \frac{1}{2}(36 + x^2)^{-\frac{1}{2}} \cdot 2x + 0 - \frac{1}{13} = 0 \\
&\Rightarrow \frac{x}{5\sqrt{36 + x^2}} = \frac{1}{13} \\
&\Rightarrow 13x = 5\sqrt{36 + x^2} \\
&\Rightarrow 169x^2 = 25(36 + x^2) \\
&\Rightarrow 169x^2 = 900 + 25x^2 \\
&\Rightarrow 144x^2 = 900 \\
&\Rightarrow x^2 = \frac{900}{144} = \frac{25}{4} \\
&\Rightarrow x = \sqrt{\frac{25}{4}} = \frac{5}{2} = 2.5 \text{ km}
\end{aligned}$$



- (iii) Time taken to go from A to B is a minimum at $x = 2.5$

$$\text{Hence, } T = \frac{1}{5}\sqrt{36 + (2.5)^2} + \frac{1}{13}(10 - 2.5)$$

$$= \frac{1}{5}\sqrt{40.25} + \frac{1}{13}(7.5)$$

$$= \frac{6.5}{5} + \frac{7.5}{13} = 1.8769 \text{ hours}$$

$$= 1 \text{ hour } 52.615 \text{ mins}$$

$$= 1 \text{ hour } 53 \text{ mins}$$

Q6.

$$P = 10 + 40r - 20r^2$$

(a) $r = 0 \Rightarrow P = 10 + 40(0) - 20(0)^2 = 10$

$$\text{Ans} = 10,000$$

(b) $P = 0 \Rightarrow 10 + 40r - 20r^2 = 0 \Rightarrow 1 + 4r - 2r^2 = 0$

$$\Rightarrow r = \frac{-4 \pm \sqrt{(4)^2 - 4(-2)(1)}}{2(-2)}$$

$$= \frac{-4 \pm \sqrt{24}}{-4}$$

$$= \frac{-2 \pm 2\sqrt{6}}{-2} = 1 \pm \sqrt{6}$$

$$\Rightarrow \text{domain : } 0 \leq x \leq 1 + \frac{\sqrt{6}}{2}$$

(c) Points for graph $P = f(r)$

$$f(0) = 10 \Rightarrow (0, 10)$$

$$f(1) = 10 + 40(1) - 20(1)^2 = 30 \Rightarrow (1, 30)$$

$$f(2) = 10 + 40(2) - 20(2)^2 = 10 \Rightarrow (2, 10)$$

$$f\left(1 + \frac{\sqrt{6}}{2}\right) = f(2.2) = 0 \Rightarrow (2.2, 0)$$

(d) $\frac{dP}{dr} = 40 - 40r$

(e) $r = 0.5 \Rightarrow \frac{dP}{dr} = 40 - 40(0.5) = 20$

$$r = 1 \Rightarrow \frac{dP}{dr} = 40 - 40(1) = 0$$

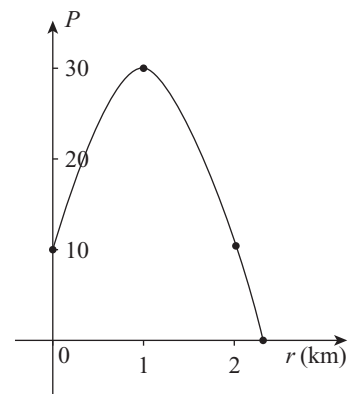
$$r = 2 \Rightarrow \frac{dP}{dr} = 40 - 40(2) = -40$$

(f) $\frac{dP}{dr} = 0 \Rightarrow 40 - 40r = 0$ Population = $10 + 40(1) - 20(1)^2$

$$\Rightarrow 40r = 40 \Rightarrow r = 1 \qquad = 10 + 40 - 20$$

$$\text{Greatest since } \frac{dP}{dr} = -40 \text{ i.e. } < 0 \qquad = 30$$

$$= 30,000$$



Q7. (a)

$$h = \left(10 - \frac{t}{200}\right)^2$$

$$\text{At } t = 0 \Rightarrow h = \left(10 - \frac{0}{200}\right)^2 = (10)^2 = 100 \text{ cm}$$

(b)

$$h = 64 \text{ cm} \Rightarrow \left(10 - \frac{t}{200}\right)^2 = 64$$

$$\Rightarrow 10 - \frac{t}{200} = \sqrt{64} = 8$$

$$\Rightarrow -\frac{t}{200} = -2$$

$$\Rightarrow t = 200(2) = 400 \text{ secs}$$

(c)

$$\text{Volume } (V) = \pi r^2 h = \pi (52)^2 \cdot h = 2704\pi h$$

$$\Rightarrow \frac{dV}{dh} = 2704\pi$$

$$h = \left(10 - \frac{t}{200}\right)^2$$

$$\Rightarrow \frac{dh}{dt} = 2 \left(10 - \frac{t}{200}\right)^1 \cdot \frac{-1}{200} = \frac{-1}{100} \left(10 - \frac{t}{200}\right)$$

$$\begin{aligned} \text{When } h = 64 \Rightarrow t = 400 \Rightarrow \frac{dh}{dt} &= \frac{-1}{100} \left(10 - \frac{400}{200}\right) \\ &= -\frac{1}{100} (8) = \frac{-2}{25} \end{aligned}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = 2704\pi \cdot \frac{-2}{25} = -216.32\pi = -679.58 = -680$$

Hence, volume of water is decreasing at the rate of $680 \text{ cm}^3 \text{ s}^{-1}$

(d)

$$\text{Hole is a circle, } r = 1 \Rightarrow \text{Area}(A) = \pi(1)^2 = \pi$$

$$\frac{dV}{dt} = \text{Area} \times \text{speed}$$

$$\Rightarrow \text{speed} = \frac{\frac{dV}{dt}}{\text{Area}} = \frac{216.32\pi}{\pi} = 216.32 \text{ cm s}^{-1}$$

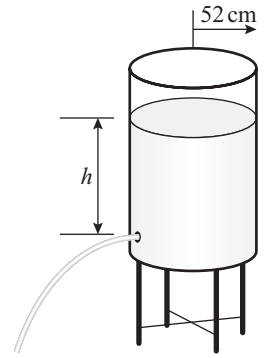
(e)

$$h = \left(10 - \frac{t}{200}\right)^2$$

$$\Rightarrow \sqrt{h} = 10 - \frac{t}{200}$$

Speed of the water = Rate at which volume of water is decreasing
divided by the area of the hole

$$\begin{aligned} &= \frac{\frac{dV}{dt}}{\pi} \\ &= \frac{2704\pi \cdot \frac{1}{100} \left(10 - \frac{t}{200}\right)}{\pi} \\ &= 27.04 \cdot \sqrt{h} \text{ cm/sec} \end{aligned}$$



$$\begin{aligned}
 \text{(f)} \quad V &= c\sqrt{1962h} \\
 \Rightarrow 27.04\sqrt{h} &= c\sqrt{1962} \cdot \sqrt{h} \\
 \Rightarrow c &= \frac{27.04}{\sqrt{1962}} = 0.61 = 0.6
 \end{aligned}$$

$$\text{Q8. (a) Volume } (V) = \frac{1}{3}\pi r^2 h$$

$$\begin{aligned}
 \text{(b)} \quad \frac{r}{h} &= \frac{1}{10} \Rightarrow 10r = h \\
 \Rightarrow r &= \frac{h}{10}
 \end{aligned}$$

$$\text{(c)} \quad V = \frac{1}{3}\pi \left(\frac{h}{10}\right)^2 \cdot h = \frac{\pi h^3}{300}$$

$$\text{(d)} \quad \frac{dV}{dt} = 0.1 \text{ cm}^3 \text{ s}^{-1}$$

$$\text{(e)} \quad V = \frac{\pi h^3}{300}$$

$$\Rightarrow \frac{dV}{dh} = \frac{3\pi h^2}{300} = \frac{\pi h^2}{100}$$

$$\text{At } h = \frac{1}{2}(10) = 5 \Rightarrow \frac{dV}{dh} = \frac{\pi 5^2}{100} = \frac{1}{4}\pi$$

$$\text{Hence, } \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$\Rightarrow 0.1 = \frac{1}{4}\pi \cdot \frac{dh}{dt}$$

$$\Rightarrow 0.4 = \pi \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{0.4}{\pi} = \frac{2}{5\pi} \text{ cm s}^{-1}$$

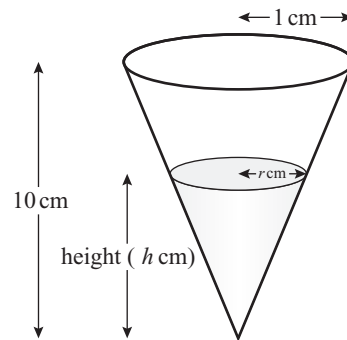
$$\text{(f)} \quad \text{Area } (A) = \pi r^2 = \pi \left(\frac{h}{10}\right)^2 = \frac{\pi h^2}{100}$$

$$\Rightarrow \frac{dA}{dh} = \frac{2\pi h}{100} = \frac{\pi h}{50}$$

$$\text{When } h = 5 \Rightarrow \frac{dA}{dh} = \frac{\pi(5)}{50} = \frac{\pi}{10}$$

$$\text{Hence, } \frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt}$$

$$= \frac{\pi}{10} \cdot \frac{2}{5\pi} = \frac{1}{25} \text{ cm}^2 \text{ s}^{-1}$$



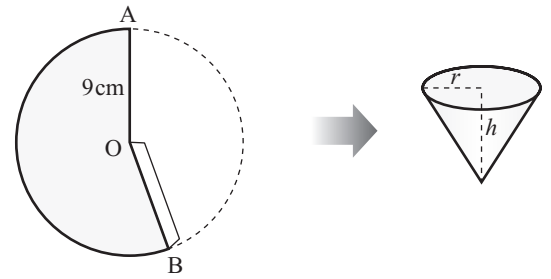
Q9. (a)

Slant height (h) = 9 cm

$$\Rightarrow r^2 + h^2 = 9^2 = 81$$

$$\Rightarrow r^2 = 81 - h^2$$

$$\begin{aligned} \text{Hence, Volume } (V) &= \frac{1}{3} \pi r^2 h \\ &= \frac{\pi}{3} h (81 - h^2) \end{aligned}$$



(b)

$$\text{Capacity} = \text{Volume } (V) = \frac{154\pi}{3}$$

$$\Rightarrow \frac{\pi}{3} h (81 - h^2) = \frac{154\pi}{3}$$

$$\Rightarrow 81h - h^3 = 154$$

$$\Rightarrow h^3 - 81h + 154 = 0$$

$$\begin{aligned} \text{When } h = 2 \Rightarrow f(2) &= (2)^3 - 81(2) + 154 \\ &= 8 - 162 + 154 = 0 \end{aligned}$$

Hence, $h = 2$ is an integer root

$$\Rightarrow (h - 2) \text{ is a factor}$$

$$\begin{array}{r} \Rightarrow \quad h - 2 \overline{) \begin{array}{r} h^3 + 2h - 77 \\ h^3 - 2h^2 \\ \hline 2h^2 - 81h \\ 2h^2 - 4h \\ \hline -77h + 154 \\ -77h + 154 \\ \hline 0 \end{array}} \end{array}$$

$$\text{Solve } h^2 + 2h - 77 = 0$$

$$\begin{aligned} \Rightarrow h &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-77)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{312}}{2} \\ &= \frac{-2 + \sqrt{312}}{2}, \frac{-2 - \sqrt{312}}{2} \\ &= 7.831, -9.831 \text{ (not valid)} \\ &= 7.83 \text{ non-integer root} \end{aligned}$$

(c)

$$V = \frac{\pi}{3} h(81 - h^2)$$

$$V = \frac{\pi}{3} [81h - h^3]$$

$$\Rightarrow \frac{dV}{dh} = \frac{\pi}{3} [81 - 3h^2] = 0$$

$$\Rightarrow 81 - 3h^2 = 0$$

$$\Rightarrow 27 - h^2 = 0$$

$$\Rightarrow h^2 = 27$$

$$\Rightarrow h = \sqrt{27} = 3\sqrt{3} = 5.20 \text{ cm}$$

$$\Rightarrow \frac{d^2V}{dh^2} = \frac{\pi}{3} (0 - 6h) = -2\pi h$$

At $h = 3\sqrt{3} \Rightarrow \frac{d^2V}{dh^2} = -2\pi 3\sqrt{3} = -6\pi\sqrt{3} < 0$

\Rightarrow Maximum volume

Hence, Volume = $\frac{\pi}{3} [81(3\sqrt{3}) - (3\sqrt{3})^3]$

$$= \frac{\pi}{3} [243\sqrt{3} - 81\sqrt{3}]$$

$$= \frac{\pi}{3} 162\sqrt{3} = 54\sqrt{3}\pi \text{ cm}^3 = 294 \text{ cm}^3$$

(d)

	Cups in part (b)		Cup in part (c)
Radius (r)	8.77 cm	4.44 cm	7.35 cm
height (h)	2 cm	7.83 cm	5.20 cm
Capacity (V)	$\frac{154\pi}{3} = 161 \text{ cm}^3$	$\frac{154\pi}{3} = 161 \text{ cm}^3$	294 cm^3

(e) A conical cup with radius = 4.44 cm and height = 7.83 cm is the most reasonable shape because it is well proportioned and it is easy to handle.

(f) In part (e), $r = 7.35 \text{ cm}$ and $l = 9 \text{ cm}$

$$\text{Curved Surface Area} = \pi r l$$

$$= \pi(4.44)(9)$$

$$= 125.538 \text{ cm}^2$$

$$\text{Area of sector of a circle} = \pi r^2 \frac{\theta}{360^\circ}$$

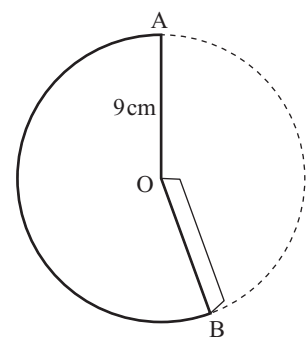
$$\Rightarrow \pi(9)^2 \frac{\theta}{360}$$

$$= (0.70686)\theta = 125.538 \text{ cm}^2$$

$$\Rightarrow \theta = \frac{125.538}{0.70686}$$

$$= 177.5995$$

$$= 178^\circ$$



Q10. (a) $P = 150 + 300e^{-0.05t}$
 $t = 0 \Rightarrow P = 150 + 300e^{-0.05(0)}$
 $= 150 + 300e^0$
 $= 150 + 300(1) = 450 \text{ birds}$

(b) $\frac{dP}{dt} = 0 + 300e^{-0.05t} \cdot (-0.05)$
 $= -15e^{-0.05t}$
 When $t = 10 \Rightarrow \frac{dP}{dt} = -15e^{-0.05(10)}$
 $= -15e^{-0.5}$
 $= -9.098$

Hence, a decreasing rate of change

(c) Limiting value of the bird population occurs as t approaches infinity
 Hence, as $t \rightarrow \infty$ then $\lim_{t \rightarrow \infty} e^{-0.05t} = 0$
 \Rightarrow limiting value for $P = 150 + 300(0)$
 $= 150 \text{ birds}$

(d) $150 + 300e^{-0.05t} < 200$
 $\Rightarrow 300e^{-0.05t} < 50$
 $\Rightarrow e^{-0.05t} < \frac{50}{300} = \frac{1}{6} = 0.16666667$
 Hence, $\ln e^{-0.05t} < \ln(0.16666667)$
 $\Rightarrow -0.05t < -1.79176$
 $\Rightarrow t > \frac{1.79176}{0.05} = 35.8$

Hence, species will be eligible after 36 years.

Chapter 4: Integration

Exercise 4.1

- Q1.** (i) $\int x dx = \frac{x^2}{2} + c$
- (ii) $\int x^2 dx = \frac{x^3}{3} + c$
- (iii) $\int (3x^2 + 4x) dx = \frac{3x^3}{3} + \frac{4x^2}{2} + c = x^3 + 2x^2 + c$
- (iv) $\int -2x^2 dx = \frac{-2x^3}{3} + c$
- (v) $\int 3 dx = 3x + c$
- (vi) $\int (-x^2 + 3) dx = -\frac{x^3}{3} + 3x + c$
- (vii) $\int (4x^3 + 6x) dx = \frac{4x^4}{4} + \frac{6x^2}{2} + c = x^4 + 3x^2 + c$
- (viii) $\int (2x^2 - 3x - 1) dx = \frac{2x^3}{3} - \frac{3x^2}{2} - x + c$
- (ix) $\int 12y^2 dy = \frac{12y^3}{3} + c = 4y^3 + c$

- Q2.** (i) $\int x^{-2} dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$
- (ii) $\int 2x^{-3} dx = \frac{2x^{-2}}{-2} + c = -\frac{1}{x^2} + c$
- (iii) $\int \frac{3}{x^2} dx = \int 3x^{-2} dx = \frac{3x^{-1}}{-1} + c = -\frac{3}{x} + c$
- (iv) $\int -\frac{2}{x^3} dx = \int -2x^{-3} dx = \frac{-2x^{-2}}{-2} + c = \frac{1}{x^2} + c$
- (v) $\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3} \sqrt{x^3} + c$
- (vi) $\int 3x^{\frac{1}{2}} dx = 3 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = 2\sqrt{x^3} + c$
- (vii) $\int \frac{1}{\sqrt{x}} dx = \int \frac{1}{x^{\frac{1}{2}}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{x} + c$
- (viii) $\int \sqrt[3]{x} dx = \int x^{\frac{1}{3}} dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{3}{4} \sqrt[3]{x^4} + c$
- (ix) $\int 4\pi r^2 dr = \frac{4\pi r^3}{3} + c = \frac{4\pi r^3}{3} + c$

$$\begin{aligned}\text{Q3. (i)} \quad \int \left(2x^3 + \frac{3}{x^2} \right) dx &= \int (2x^3 + 3x^{-2}) dx = 2 \frac{x^4}{4} + 3 \frac{x^{-1}}{-1} + c \\ &= \frac{x^4}{2} - \frac{3}{x} + c\end{aligned}$$

$$\begin{aligned}\text{(ii)} \quad \int \left(\frac{4}{x^2} - 2 + x^3 \right) dx &= \int (4x^{-2} - 2 + x^3) dx \\ &= 4 \frac{x^{-1}}{-1} - 2x + \frac{x^4}{4} + c \\ &= -\frac{4}{x} - 2x + \frac{x^4}{4} + c\end{aligned}$$

$$\begin{aligned}\text{(iii)} \quad \int (4\sqrt{x} - 3) dx &= \int (4x^{\frac{1}{2}} - 3) dx \\ &= 4 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3x + c = \frac{8}{3} \sqrt{x^3} - 3x + c\end{aligned}$$

$$\begin{aligned}\text{(iv)} \quad \int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx &= \int \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}} \right) dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{2}{3} \sqrt{x^3} + 2\sqrt{x} + c\end{aligned}$$

$$\begin{aligned}\text{(v)} \quad \int \left(2\sqrt{x} - \frac{2}{x^2} \right) dx &= \int \left(2x^{\frac{1}{2}} - 2x^{-2} \right) dx \\ &= 2 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2 \frac{x^{-1}}{-1} + c = \frac{4}{3} \sqrt{x^3} + \frac{2}{x} + c\end{aligned}$$

$$\begin{aligned}\text{(vi)} \quad \int \left(\frac{1}{x^2} - \frac{x}{\sqrt{x}} \right) dx &= \int \left(x^{-2} - x^{\frac{1}{2}} \right) dx \\ &= \frac{x^{-1}}{-1} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = -\frac{1}{x} - \frac{2}{3} \sqrt{x^3} + c\end{aligned}$$

$$\begin{aligned}\text{Q4. (i)} \quad \frac{dy}{dx} &= x^2 + 3x \\ \Rightarrow y &= \int (x^2 + 3x) dx = \frac{x^3}{3} + \frac{3x^2}{2} + c\end{aligned}$$

$$\begin{aligned}\text{(ii)} \quad \frac{dy}{dx} &= 6x^3 - 4x^2 + x - 5 \\ \Rightarrow y &= \int (6x^3 - 4x^2 + x - 5) dx \\ &= \frac{6x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} - 5x + c \\ &= \frac{3x^4}{2} - \frac{4x^3}{3} + \frac{x^2}{2} - 5x + c\end{aligned}$$

$$\begin{aligned}
 \text{Q5. (i)} \quad \int (x-3)^2 dx &= \int (x^2 - 6x + 9) dx \\
 &= \frac{x^3}{3} - \frac{6x^2}{2} + 9x + c \\
 &= \frac{x^3}{3} - 3x^2 + 9x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int \left(x - \frac{1}{x}\right)^2 dx &= \int \left(x^2 - 2 + \frac{1}{x^2}\right) dx \\
 &= \int (x^2 - 2 + x^{-2}) dx \\
 &= \frac{x^3}{3} - 2x + \frac{x^{-1}}{-1} + c \\
 &= \frac{x^3}{3} - 2x - \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \int \sqrt{x}(x-3) dx &= \int \left(x^{\frac{3}{2}} - 3x^{\frac{1}{2}}\right) dx \\
 &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 3 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{2}{5} \sqrt{x^5} - 2\sqrt{x^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Q6. (i)} \quad \int \frac{x^4 - 3x^3 + 4x}{x} dx &= \int (x^3 - 3x^2 + 4) dx \\
 &= \frac{x^4}{4} - \frac{3x^3}{3} + 4x + c \\
 &= \frac{x^4}{4} - x^3 + 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int \frac{3x^3 - x^2 + 6}{x^2} dx &= \int (3x - 1 + 6x^{-2}) dx \\
 &= \frac{3x^2}{2} - x + 6 \frac{x^{-1}}{-1} + c \\
 &= \frac{3x^2}{2} - x - \frac{6}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \int \frac{x^2 - 2x + 6}{\sqrt{x}} dx &= \int \left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}\right) dx \\
 &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{6x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{2}{5} x^{\frac{5}{2}} - \frac{4x^{\frac{3}{2}}}{3} + 12\sqrt{x} + c
 \end{aligned}$$

Q7. $f'(x) = 2x$
 $\Rightarrow f(x) = \int 2x dx = x^2 + c$
 Point $(-1, 4) \Rightarrow f(-1) = (-1)^2 + c = 4$
 $\Rightarrow 1 + c = 4 \Rightarrow c = 3$
 $\Rightarrow f(x) = x^2 + 3$

Q8. $f'(x) = 2x - 5$
 $\Rightarrow f(x) = \int (2x - 5) dx$
 $= x^2 - 5x + c$
 Point $(1, 7) \Rightarrow f(1) = (1)^2 - 5(1) + c = 7$
 $\Rightarrow 1 - 5 + c = 7$
 $\Rightarrow c = 7 + 4 = 11$
 $\Rightarrow f(x) = x^2 - 5x + 11$

Q9. $y = \int (6x + 5) dx = \frac{6x^2}{2} + 5x + c = 3x^2 + 5x + c$
 When $x = 2 \Rightarrow y = 3(2)^2 + 5(2) + c = 19$
 $\Rightarrow 12 + 10 + c = 19$
 $\Rightarrow c = 19 - 22 = -3$

Q10. $y = \int (6x^2 - 8x + 5) dx = \frac{6x^3}{3} - \frac{8x^2}{2} + 5x + c$
 $= 2x^3 - 4x^2 + 5x + c$
 When $x = 2 \Rightarrow y = 2(2)^3 - 4(2)^2 + 5(2) + c = 7$
 $\Rightarrow 16 - 16 + 10 + c = 7$
 $\Rightarrow c = 7 - 10 = -3$

Q11. (i) $\frac{dy}{dx} = x^2 + 2x$
 $\Rightarrow y = \int (x^2 + 2x) dx = \frac{x^3}{3} + x^2 + c$
 When $x = 0 \Rightarrow y = 2(0)^3 + (0)^2 + c = 2$
 $\Rightarrow 0 + 0 + c = 2 \Rightarrow c = 2$
 $\Rightarrow y = \frac{x^3}{3} + x^2 + 2$

(ii) $\frac{dy}{dx} = 3 - x^2$
 $\Rightarrow y = \int (3 - x^2) dx = 3x - \frac{x^3}{3} + c$
 When $x = 3 \Rightarrow y = 3(3) - \frac{(3)^3}{3} + c = 2$
 $\Rightarrow 9 - 9 + c = 2 \Rightarrow c = 2$
 $\Rightarrow y = 3x - \frac{x^3}{3} + 2$

Q12. (i) $\frac{dV}{dt} = t^2 - t$

$$\Rightarrow V = \int (t^2 - t) dt = \frac{t^3}{3} - \frac{t^2}{2} + c$$

When $t = 3 \Rightarrow V = \frac{(3)^3}{3} - \frac{(3)^2}{2} + c = 9$

$$\Rightarrow 9 - \frac{9}{2} + c = 9 \Rightarrow c = 4\frac{1}{2}$$

$$\Rightarrow V = \frac{t^3}{3} - \frac{t^2}{2} + 4\frac{1}{2}$$

(ii) $t = 10 \Rightarrow V = \frac{(10)^3}{3} - \frac{(10)^2}{2} + 4\frac{1}{2} = 287\frac{5}{6}$

Q13. (i) $f'(x) = 4x + k$

Turning point at $(-2, -1) \Rightarrow f'(-2) = 4(-2) + k = 0$

$$\Rightarrow -8 + k = 0 \Rightarrow k = 8$$

(ii) $f'(x) = 4x + 8$

$$\Rightarrow f(x) = \int (4x + 8) dx = \frac{4x^2}{2} + 8x + c$$

$$\Rightarrow f(x) = 2x^2 + 8x + c$$

Point $(-2, -1) \Rightarrow f(-2) = 2(-2)^2 + 8(-2) + c = -1$

$$\Rightarrow 8 - 16 + c = -1 \Rightarrow c = 7$$

$$\Rightarrow f(x) = 2x^2 + 8x + 7$$

On y -axis, $x = 0 \Rightarrow f(0) = 2(0)^2 + 8(0) + 7 = 7$

$$\Rightarrow \text{point on } y\text{-axis} = (0, 7)$$

Q14. Tangent at $(3, 6)$ passes through the origin $(0, 0)$

$$\Rightarrow \text{slope } m = \frac{6-0}{3-0} = \frac{6}{3} = 2$$

(i) When $x = 3 \Rightarrow \text{slope} = \frac{dy}{dx} = 2(3) + k = 2$

$$\Rightarrow 6 + k = 2 \Rightarrow k = -4$$

(ii) $\frac{dy}{dx} = 2x - 4$

$$\Rightarrow y = \int (2x - 4) dx = x^2 - 4x + c$$

Point $(3, 6) \Rightarrow 6 = (3)^2 - 4(3) + c$

$$\Rightarrow 6 = 9 - 12 + c$$

$$\Rightarrow 6 = -3 + c \Rightarrow c = 9$$

Hence $y = x^2 - 4x + 9$

Exercise 4.2

Q1. (i) $\int e^{2x} dx = \frac{e^{2x}}{2} + c$

(ii) $\int 3e^x dx = 3e^x + c$

(iii) $\int 2e^{4x} dx = \frac{2 \cdot e^{4x}}{4} + c = \frac{e^{4x}}{2} + c$

(iv) $\int e^{-3x} dx = \frac{-e^{-3x}}{3} + c$

Q2. (i) $\int (e^{3x} + 4) dx = \frac{e^{3x}}{3} + 4x + c$

(ii) $\int 4e^{\frac{1}{2}x} dx = \frac{4 \cdot e^{\frac{1}{2}x}}{\frac{1}{2}} + c = 8e^{\frac{1}{2}x} + c$

(iii) $\int \left(e^{4x} + \frac{1}{e^{4x}} \right) dx = \int e^{4x} + e^{-4x} dx$
 $= \frac{e^{4x}}{4} + \frac{e^{-4x}}{-4} + c$
 $= \frac{e^{4x}}{4} - \frac{e^{-4x}}{4} + c$

Q3. $y = e^{x^2} \Rightarrow \frac{dy}{dx} = e^{x^2} \cdot \frac{d}{dx}(x^2) = 2x \cdot e^{x^2}$
Since $\frac{dy}{dx} = 2xe^{x^2} \Rightarrow \int 2xe^{x^2} dx = e^{x^2} + c$

Q4. (i) $\int \cos 3x dx = \frac{\sin 3x}{3} + c$

(ii) $\int \sin 4x dx = -\frac{\cos 4x}{4} + c$

(iii) $\int -\sin 5x dx = -\left(-\frac{\cos 5x}{5} \right) + c = \frac{\cos 5x}{5} + c$

(iv) $\int \cos kx dx = \frac{\sin kx}{k} + c$

Q5. (i) $\int 3 \cos 6x dx = 3 \cdot \frac{\sin 6x}{6} + c = \frac{\sin 6x}{2} + c$

(ii) $\int (\cos 2x - \sin 5x) dx = \frac{\sin 2x}{2} - \left(\frac{-\cos 5x}{5} \right) + c$
 $= \frac{\sin 2x}{2} + \frac{\cos 5x}{5} + c$

$$\begin{aligned}
 \text{(iii)} \quad \int 3 \cos(-9x) dx &= 3 \cdot \frac{\sin(-9x)}{-9} + c \\
 &= -\frac{\sin(-9x)}{3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Q6.} \quad \int 3(e^x - 4 \sin 3x + 2) dx \\
 &= 3 \left[e^x - 4 \left(\frac{-\cos 3x}{3} \right) + 2x \right] + c \\
 &= 3e^x + 4 \cos 3x + 6x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Q7. (i)} \quad \int (4e^{2x} + 4 \sin 3x) dx \\
 &= 4 \frac{e^{2x}}{2} + 4 \left(\frac{-\cos 3x}{3} \right) + c \\
 &= 2e^{2x} - \frac{4 \cos 3x}{3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int (3 \cos x - 2 \cos 4x) dx \\
 &= 3 \sin x - \frac{2 \sin 4x}{4} + c \\
 &= 3 \sin x - \frac{\sin 4x}{2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Q8.} \quad y = \cos 4x^2 &\Rightarrow \frac{dy}{dx} = -\sin 4x^2 \cdot \frac{d}{dx}(4x^2) \\
 &= -\sin 4x^2 \cdot 8x = -8x \sin 4x^2
 \end{aligned}$$

$$\text{Since } \frac{dy}{dx} = -8x \sin 4x^2 \Rightarrow \int -8x \sin 4x^2 dx = \cos 4x^2 + c$$

$$\begin{aligned}
 \text{Q9. (i)} \quad \int \frac{e^{2x} + 4}{e^x} dx &= \int \frac{e^{2x}}{e^x} + \frac{4}{e^x} dx = \int (e^x + 4e^{-x}) dx \\
 &= e^x + \frac{4e^{-x}}{-1} + c \\
 &= e^x - \frac{4}{e^x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int \frac{e^{x+2} + 3}{e^x} dx &= \int \frac{e^{x+2}}{e^x} + \frac{3}{e^x} dx \\
 &= \int (e^2 + 3e^{-x}) dx \\
 &= xe^2 + 3 \frac{e^{-x}}{-1} + c \\
 &= xe^2 - \frac{3}{e^x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \int \frac{1+3e^x}{e^{2x}} dx &= \int \frac{1}{e^{2x}} + \frac{3e^x}{e^{2x}} dx \\
 &= \int (e^{-2x} + 3e^{-x}) dx \\
 &= \frac{e^{-2x}}{-2} + 3 \frac{e^{-x}}{-1} + c \\
 &= -\frac{1}{2} e^{-2x} - 3e^{-x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Q10. (i)} \quad \int (e^x - e^{-x})^2 dx &= \int [(e^x)^2 - 2e^x \cdot e^{-x} + (e^{-x})^2] dx \\
 &= \int (e^{2x} - 2 + e^{-2x}) dx \\
 &= \frac{1}{2} e^{2x} - 2x + \frac{e^{-2x}}{-2} + c \\
 &= \frac{1}{2} e^{2x} - 2x - \frac{1}{2} e^{-2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int (3+e^x)(2+e^{-x}) dx &= \int (6+3e^{-x}+2e^x+e^x \cdot e^{-x}) dx \\
 &= \int (6+3e^{-x}+2e^x+1) dx \\
 &= \int (7+3e^{-x}+2e^x) dx \\
 &= 7x + \frac{3e^{-x}}{-1} + 2e^x + c \\
 &= 7x - \frac{3}{e^x} + 2e^x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Q11.} \quad y = 7^x &\Rightarrow \ell \text{ n } y = \ell \text{ n } 7^x \\
 &\Rightarrow \ell \text{ n } y = x \ell \text{ n } 7 \\
 &\Rightarrow x = \frac{\ell \text{ n } y}{\ell \text{ n } 7} = \frac{1}{\ell \text{ n } 7} \cdot \ell \text{ n } y
 \end{aligned}$$

$$\text{(i)} \quad \frac{dx}{dy} = \frac{1}{\ell \text{ n } 7} \cdot \frac{1}{y}$$

$$\text{(ii)} \quad \frac{dy}{dx} = \ln 7 \cdot y = \ell \text{ n } 7 \cdot 7^x \quad \text{OR} \quad 7^x \ln 7$$

$$\begin{aligned}
 \text{(iii)} \quad \text{Since } \frac{dy}{dx} &= 7^x \cdot \ell \text{ n } 7 \\
 \int 7^x dx &= \frac{7^x}{\ell \text{ n } 7} + c
 \end{aligned}$$

Q12. $\frac{dy}{dx} = ae^{-x} + 2$

When $x = 0 \Rightarrow \frac{dy}{dx} = ae^{-(0)} + 2 = 5$
 $\Rightarrow a \cdot 1 + 2 = 5 \Rightarrow a = 3$

Hence $\frac{dy}{dx} = 3e^{-x} + 2$

$\Rightarrow y = \int (3e^{-x} + 2) dx$

$= 3 \frac{e^{-x}}{-1} + 2x + c = \frac{-3}{e^x} + 2x + c$

$y = -3$ when $x = 0 \Rightarrow -3 = \frac{-3}{e^0} + 2(0) + c$

$\Rightarrow -3 = \frac{-3}{1} + 0 + c \Rightarrow c = 0$

$\Rightarrow y = \frac{-3}{e^x} + 2x$

Q13. Tangent at $(1, e^2)$ passes through the origin $(0, 0)$

\Rightarrow gradient $m = \frac{e^2 - 0}{1 - 0} = e^2$

(i) When $x = 1 \Rightarrow$ gradient $= \frac{dy}{dx} = e^{k(1)} = e^2 \Rightarrow k = 2$

(ii) $\frac{dy}{dx} = e^{2x} \Rightarrow y = \int e^{2x} dx = \frac{e^{2x}}{2} + c$

Point $(1, e^2) \Rightarrow e^2 = \frac{e^{2(1)}}{2} + c \Rightarrow c = e^2 - \frac{e^2}{2} = \frac{e^2}{2}$

$\Rightarrow y = \frac{1}{2}e^{2x} + \frac{1}{2}e^2$

Q14. (i) $f(x) = 2xe^x \Rightarrow$ product rule: $u = 2x$ and $v = e^x$

$\Rightarrow \frac{du}{dx} = 2 \Rightarrow \frac{dv}{dx} = e^x$

$\frac{dy}{dx} = f'(x) = u \frac{dv}{dx} + v \frac{du}{dx}$

$= 2x \cdot e^x + e^x \cdot 2 = 2x e^x + 2e^x$

(ii) $\int (2x e^x + 2e^x) dx = 2x e^x + c$

$\Rightarrow \int 2x e^x dx + \int 2e^x dx = 2x e^x + c$

$\Rightarrow \int 2x e^x dx + 2e^x = 2x e^x + c$

$\Rightarrow \int 2x e^x dx = 2x e^x - 2e^x + c$

Q15. $f(x) = x \sin x \Rightarrow$ product rule: $u = x$ and $v = \sin x$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = f'(x) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$= x \cdot \cos x + \sin x \cdot 1 = \sin x + x \cos x$$

$$\int (\sin x + x \cos x) dx = x \sin x + c$$

$$\Rightarrow \int \sin x dx + \int x \cos x dx = x \sin x + c$$

$$\Rightarrow -\cos x + \int x \cos x dx = x \sin x + c$$

$$\Rightarrow \int x \cos x dx = x \sin x + \cos x + c$$

Q16. (i) $f(x) = 4xe^{2x} \Rightarrow$ product rule: $u = 4x$ and $v = e^{2x}$

$$\Rightarrow \frac{du}{dx} = 4 \Rightarrow \frac{dv}{dx} = 2e^{2x}$$

$$\frac{dy}{dx} = f'(x) = u \frac{dv}{dx} + v \frac{du}{dx} = 4x \cdot 2e^{2x} + e^{2x} \cdot 4$$

$$= 8x e^{2x} + 4e^{2x}$$

(ii) $\int (8x e^{2x} + 4e^{2x}) dx = 4x e^{2x} + c$

$$\Rightarrow \int 8x \cdot e^{2x} dx + \int 4 e^{2x} dx = 4x e^{2x} + c$$

$$\Rightarrow \int 8x e^{2x} dx + 4 \frac{e^{2x}}{2} = 4x e^{2x} + c$$

$$\Rightarrow \int 8x e^{2x} dx = 4x e^{2x} - 2e^{2x} + c$$

Q17. $y = 2x \cdot e^{3x} + \cos x$

$$\frac{dy}{dx} = 2x \cdot e^{3x} \cdot 3 + e^{3x} \cdot 2 - \sin x$$

$$= 6x e^{3x} + 2e^{3x} - \sin x$$

Hence $\int (6x e^{3x} + 2e^{3x} - \sin x) dx = 2x e^{3x} + \cos x + c$

$$\Rightarrow \int 6x e^{3x} dx + \int 2e^{3x} dx - \int \sin x dx = 2x e^{3x} + \cos x + c$$

$$\Rightarrow \int 6x e^{3x} dx + \frac{2 \cdot e^{3x}}{3} + \cos x = 2x e^{3x} + \cos x + c$$

$$\Rightarrow \int 6x e^{3x} dx = 2x e^{3x} - \frac{2}{3} e^{3x} + c$$

Exercise 4.3

Q1. (i) $v = \frac{ds}{dt} = 5t + 4$

$$\Rightarrow s = \int (5t + 4) dt$$

$$= \frac{5t^2}{2} + 4t + c$$

$$s = 0 \text{ when } t = 0 \Rightarrow \frac{5}{2}(0)^2 + 4(0) + c = 0 \Rightarrow c = 0$$

$$\Rightarrow s = \frac{5t^2}{2} + 4t$$

(ii) $t = 4 \Rightarrow s = \frac{5(4)^2}{2} + 4(4) = 56 \text{ metres}$

Q2. $v = \frac{ds}{dt} = t^2 - 4t + 3$

(i) $a = \frac{d^2s}{dt^2} = 2t - 4$

$$t = 5 \Rightarrow a = 2(5) - 4 = 6 \text{ m/sec}^2$$

(ii) $s = \int (t^2 - 4t + 3) dt$

$$= \frac{t^3}{3} - \frac{4t^2}{2} + 3t + c = \frac{t^3}{3} - 2t^2 + 3t + c$$

$$s = 4 \text{ when } t = 3 \Rightarrow \frac{(3)^3}{3} - 2(3)^2 + 3(3) + c = 4$$

$$\Rightarrow 9 - 18 + 9 + c = 4 \Rightarrow c = 4$$

$$\Rightarrow s = \frac{t^3}{3} - 2t^2 + 3t + 4$$

(iii) $t = 1 \Rightarrow s = \frac{(1)^3}{3} - 2(1)^2 + 3(1) + 4 = 5\frac{1}{3} \text{ m}$

Q3. $a = 6t - 12$

(i) $v = \int (6t - 12) dt = \frac{6t^2}{2} - 12t + c$

$$= 3t^2 - 12t + c$$

$$v = 9 \text{ when } t = 0 \Rightarrow 3(0)^2 - 12(0) + c = 9$$

$$\Rightarrow c = 9$$

$$\Rightarrow v = 3t^2 - 12t + 9$$

$$\begin{aligned}
\text{(ii)} \quad s &= \int (3t^2 - 12t + 9) dt \\
&= \frac{3t^3}{3} - 12 \frac{t^2}{2} + 9t + c \\
&= t^3 - 6t^2 + 9t + c \\
s = 6 \text{ when } t = 0 &\Rightarrow (0)^3 - 6(0)^2 + 9(0) + c = 6 \\
&\Rightarrow c = 6 \\
\Rightarrow s &= t^3 - 6t^2 + 9t + 6 \\
\text{(iii)} \quad \text{Body at rest} &\Rightarrow v = 0 \\
&\Rightarrow 3t^2 - 12t + 9 = 0 \\
&\Rightarrow t^2 - 4t + 3 = 0 \\
&\Rightarrow (t-1)(t-3) = 0 \\
&\Rightarrow t = 1 \text{ OR } t = 3
\end{aligned}$$

Q4. $a = (2t - 3) \text{ cm/sec}^2$

$$\begin{aligned}
\text{(i)} \quad v &= \int (2t - 3) dt = t^2 - 3t + c \\
v = 3 \text{ when } t = 0 &\Rightarrow (0)^2 - 3(0) + c = 3 \\
&\Rightarrow c = 3 \\
\Rightarrow v &= t^2 - 3t + 3 \\
s &= \int (t^2 - 3t + 3) dt \\
&= \frac{t^3}{3} - 3 \frac{t^2}{2} + 3t + c \\
s = 2 \text{ when } t = 0 &\Rightarrow \frac{(0)^3}{3} - \frac{3(0)^2}{2} + 3(0) + c = 2 \Rightarrow c = 2 \\
\Rightarrow s &= \frac{t^3}{3} - \frac{3t^2}{2} + 3t + 2 \\
\text{(ii)} \quad t = 2 &\Rightarrow v = (2)^2 - 3(2) + 3 = 1 \text{ m/sec} \\
t = 2 &\Rightarrow s = \frac{(2)^3}{3} - 3 \frac{(2)^2}{2} + 3(2) + 2 = 4 \frac{2}{3} \text{ m}
\end{aligned}$$

Q5. $a = -10 \text{ m/sec}^2$

$$\begin{aligned}
\text{(i)} \quad v &= \int -10 dt = -10t + c \\
v = 25 \text{ when } t = 0 &\Rightarrow -10(0) + c = 25 \Rightarrow c = 25 \\
&\Rightarrow v = (-10t + 25) \text{ m/sec} \\
\text{(ii)} \quad s &= \int (-10t + 25) dt \\
&= -10 \frac{t^2}{2} + 25t + c = -5t^2 + 25t + c \\
s = 0 \text{ when } t = 0 &\Rightarrow -5(0)^2 + 25(0) + c = 0 \Rightarrow c = 0 \\
&\Rightarrow s = (-5t^2 + 25t) \text{ m}
\end{aligned}$$

- (iii) Maximum height occurs when $v = 0$
 $\Rightarrow -10t + 25 = 0$
 $\Rightarrow -10t = -25 \Rightarrow t = \frac{5}{2} \text{ sec}$
- (iv) $t = \frac{5}{2} \Rightarrow s = -5\left(\frac{5}{2}\right)^2 + 25\left(\frac{5}{2}\right) = \frac{125}{4} \text{ m}$
- (v) Return to the point of projection $\Rightarrow s = 0$
 $\Rightarrow -5t^2 + 25t = 0$
 $\Rightarrow t^2 - 5t = 0$
 $\Rightarrow t(t - 5) = 0$
 $t = 0 \quad \text{OR} \quad t = 5$
 $\Rightarrow \text{Answer: } t = 5 \text{ sec}$

Q6. $\frac{dN}{dt} = 4e^t + 10$

- (i) $N = \int (4e^t + 10) dt = 4e^t + 10t + c$
- (ii) $t = 0 \Rightarrow N = 4e^0 + 10(0) + c = 10$
 $4 + c = 10 \Rightarrow c = 6$
 $\Rightarrow N = 4e^t + 10t + 6$
 $t = 5 \Rightarrow N = 4e^5 + 10(5) + 6 = 4e^5 + 56$
 $= 649.65 = 650$

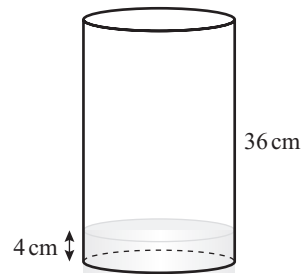
Q7. $v = 0.6t - 0.004t^2$

- (i) $s = \int (0.6t - 0.004t^2) dt$
 $= 0.6 \cdot \frac{t^2}{2} - 0.004 \cdot \frac{t^3}{3} + c$
 $s = 0 \text{ when } t = 0 \Rightarrow 0.3(0)^2 - 0.004 \frac{(0)^3}{3} + c = 0 \Rightarrow c = 0$
 $\Rightarrow s = 0.3t^2 - \frac{0.004t^3}{3}$
- (ii) $t = 2 \frac{1}{2} \text{ mins} = 150 \text{ secs}$
 $\Rightarrow s = 0.3(150)^2 - \frac{0.004(150)^3}{3} = 2250 \text{ m}$

Q8. $\frac{dh}{dt} = 2t - 3$

- (i) $h = \int (2t - 3) dt = t^2 - 3t + c$
 $h = 4 \text{ when } t = 0 \Rightarrow (0)^2 - 3(0) + c = 4 \Rightarrow c = 4$
 $\Rightarrow h = t^2 - 3t + 4$

(ii) Height of container = 36 cm
 $\Rightarrow 36 = t^2 - 3t + 4$
 $\Rightarrow t^2 - 3t - 32 = 0$
 $\Rightarrow t = \frac{3 \pm \sqrt{9 + 128}}{2}$
 $\Rightarrow t = 7.4 \quad \text{OR} \quad t = -4.4$
 $\Rightarrow t = 7.4$
Hence, $t = 7.4$ secs



Exercise 4.4

Q1. $\int_1^2 6x \, dx = \left[\frac{6x^2}{2} \right]_1^2 = [3x^2]_1^2$
 $= [3(2)^2] - [3(1)^2] = 12 - 3 = 9$

Q2. $\int_1^3 (3x^2 - 2x) \, dx = \left[\frac{3x^3}{3} - \frac{2x^2}{2} \right]_1^3 = [x^3 - x^2]_1^3$
 $= [(3)^3 - (3)^2] - [(1)^3 - (1)^2]$
 $= [27 - 9] - [1 - 1] = 18$

Q3. $\int_1^4 (3x^2 - 4) \, dx = \left[\frac{3x^3}{3} - 4x \right]_1^4 = [x^3 - 4x]_1^4$
 $= [(4)^3 - 4(4)] - [(1)^3 - 4(1)]$
 $= (64 - 16) - (1 - 4) = 51$

Q4. $\int_1^2 (x^3 + 2x) \, dx = \left[\frac{x^4}{4} + \frac{2x^2}{2} \right]_1^2 = \left[\frac{x^4}{4} + x^2 \right]_1^2$
 $= \left[\frac{(2)^4}{4} + (2)^2 \right] - \left[\frac{(1)^4}{4} + (1)^2 \right]$
 $= (4 + 4) - \left(\frac{1}{4} + 1 \right) = 6\frac{3}{4}$

Q5. $\int_1^3 (x^2 - x + 1) \, dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + x \right]_1^3$
 $= \left[\frac{(3)^3}{3} - \frac{(3)^2}{2} + 3 \right] - \left[\frac{(1)^3}{3} - \frac{(1)^2}{2} + 1 \right]$
 $= \left[9 - \frac{9}{2} + 3 \right] - \left[\frac{1}{3} - \frac{1}{2} + 1 \right]$
 $= 7\frac{1}{2} - \frac{5}{6} = 6\frac{2}{3}$

$$\begin{aligned}
 \text{Q6.} \quad \int_{-1}^2 (2x-5)dx &= \left[\frac{2x^2}{2} - 5x \right]_{-1}^2 = [x^2 - 5x]_{-1}^2 \\
 &= [(2)^2 - 5(2)] - [(-1)^2 - 5(-1)] \\
 &= (4-10) - (1+5) = -6-6 = -12
 \end{aligned}$$

$$\begin{aligned}
 \text{Q7.} \quad \int_0^1 x^2(3-x)dx &= \int_0^1 (3x^2 - x^3)dx = \left[\frac{3x^3}{3} - \frac{x^4}{4} \right]_0^1 \\
 &= \left[x^3 - \frac{x^4}{4} \right]_0^1 = \left[(1)^3 - \frac{(1)^4}{4} \right] - \left[(0)^3 - \frac{(0)^4}{4} \right] \\
 &= 1 - \frac{1}{4} - 0 = \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{Q8.} \quad \int_1^9 \sqrt{x} \, dx &= \int_1^9 x^{\frac{1}{2}} dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^9 = \left[\frac{2}{3} x^{\frac{3}{2}} \right]_1^9 \\
 &= \left[\frac{2}{3} (9)^{\frac{3}{2}} \right] - \left[\frac{2}{3} (1)^{\frac{3}{2}} \right] \\
 &= \frac{2}{3} (27) - \frac{2}{3} = 18 - \frac{2}{3} = 17\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Q9.} \quad \int_2^4 \frac{1}{x^2} \, dx &= \int_2^4 x^{-2} dx = \left[\frac{x^{-1}}{-1} \right]_2^4 = \left[-\frac{1}{x} \right]_2^4 \\
 &= \left(-\frac{1}{4} \right) - \left(-\frac{1}{2} \right) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{Q10.} \quad \int_4^9 \frac{dx}{\sqrt{x}} &= \int_4^9 x^{-\frac{1}{2}} dx = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_4^9 = [2\sqrt{x}]_4^9 \\
 &= [2\sqrt{9}] - [2\sqrt{4}] = 6 - 4 = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{Q11.} \quad \int_0^2 \frac{x^3 - 2x^2 + 4x}{x} dx &= \int_0^2 (x^2 - 2x + 4) dx \\
 &= \left[\frac{x^3}{3} - x^2 + 4x \right]_0^2 = \left[\frac{(2)^3}{3} - (2)^2 + 4(2) \right] - \left[\frac{(0)^3}{3} - (0)^2 + 4(0) \right] \\
 &= \left[\frac{8}{3} - 4 + 8 \right] - [0 - 0 + 0] = 6\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
\text{Q12.} \quad \int_1^4 (\sqrt{x} - 2)^2 dx &= \int_1^4 (x - 4\sqrt{x} + 4) dx = \int_1^4 (x - 4x^{\frac{1}{2}} + 4) dx \\
&= \left[\frac{x^2}{2} - 4 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 4x \right]_1^4 = \left[\frac{x^2}{2} - \frac{8}{3} x^{\frac{3}{2}} + 4x \right]_1^4 \\
&= \left[\frac{(4)^2}{2} - \frac{8}{3} (4)^{\frac{3}{2}} + 4(4) \right] - \left[\frac{(1)^2}{2} - \frac{8}{3} (1)^{\frac{3}{2}} + 4(1) \right] \\
&= \left(8 - \frac{64}{3} + 16 \right) - \left(\frac{1}{2} - \frac{8}{3} + 4 \right) \\
&= 2\frac{2}{3} - 1\frac{5}{6} = \frac{5}{6}
\end{aligned}$$

$$\begin{aligned}
\text{Q13.} \quad \int_{-2}^{-1} \frac{2}{x^3} dx &= \int_{-2}^{-1} 2x^{-3} dx = \left[\frac{2x^{-2}}{-2} \right]_{-2}^{-1} = \left[-\frac{1}{x^2} \right]_{-2}^{-1} \\
&= \left[-\frac{1}{-1^2} \right] - \left[-\frac{1}{-2^2} \right] \\
&= -1 + \frac{1}{4} = -\frac{3}{4}
\end{aligned}$$

$$\begin{aligned}
\text{Q14.} \quad \int_1^{16} \left(\frac{\sqrt{x} - 4}{\sqrt{x}} \right) dx &= \int_1^{16} \left(\frac{\sqrt{x}}{\sqrt{x}} - \frac{4}{\sqrt{x}} \right) dx = \int_1^{16} (1 - 4x^{-\frac{1}{2}}) dx \\
&= \left[x - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^{16} = \left[x - 8\sqrt{x} \right]_1^{16} \\
&= (16 - 8\sqrt{16}) - (1 - 8\sqrt{1}) = -16 + 7 = -9
\end{aligned}$$

$$\begin{aligned}
\text{Q15.} \quad \int_1^4 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx &= \int_1^4 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}} \right) dx = \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right) dx \\
&= \left[\frac{2x^{\frac{3}{2}}}{3} + 2\sqrt{x} \right]_1^4 = \left[\frac{2}{3} (4)^{\frac{3}{2}} + 2\sqrt{4} \right] - \left[\frac{2(1)^{\frac{3}{2}}}{3} + 2\sqrt{1} \right] \\
&= \left(\frac{16}{3} + 4 \right) - \left(\frac{2}{3} + 2 \right) = 9\frac{1}{3} - 2\frac{2}{3} = 6\frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\text{Q16.} \quad \int_1^2 (x-1)(x-2) dx &= \int_1^2 (x^2 - 3x + 2) dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x \right]_1^2 \\
&= \left[\frac{(2)^3}{3} - \frac{3(2)^2}{2} + 2(2) \right] - \left[\frac{(1)^3}{3} - \frac{3(1)^2}{2} + 2(1) \right] \\
&= \left(\frac{8}{3} - 6 + 4 \right) - \left(\frac{1}{3} - \frac{3}{2} + 2 \right) = \frac{2}{3} - \frac{5}{6} = -\frac{1}{6}
\end{aligned}$$

Q17.
$$\frac{x^2 - 16}{2x + 8} = \frac{(x-4)(x+4)}{2(x+4)} = \frac{x-4}{2}$$

$$\begin{aligned}\int_0^1 \frac{x^2 - 16}{2x + 8} dx &= \frac{1}{2} \int_0^1 (x-4) dx = \frac{1}{2} \left[\frac{x^2}{2} - 4x \right]_0^1 \\ &= \frac{1}{2} \left[\frac{(1)^2}{2} - 4(1) \right] - \frac{1}{2} \left[\frac{(0)^2}{2} - 4(0) \right] \\ &= \frac{1}{2} \left[\frac{1}{2} - 4 \right] - \frac{1}{2} [0 - 0] = -1\frac{3}{4}\end{aligned}$$

Q18.
$$\int_0^k (2x - 4) dx = -3$$

$$\begin{aligned}\Rightarrow [x^2 - 4x]_0^k &= -3 \\ \Rightarrow [k^2 - 4k] - [(0)^2 - 4(0)] &= -3 \\ \Rightarrow k^2 - 4k + 3 &= 0 \\ \Rightarrow (k-1)(k-3) &= 0 \quad \Rightarrow k=1 \quad \text{OR} \quad k=3\end{aligned}$$

Q19.
$$\int_0^k (x^2 - 3x) dx = 0$$

$$\begin{aligned}\Rightarrow \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^k &= 0 \\ \Rightarrow \left[\frac{k^3}{3} - \frac{3k^2}{2} \right] - \left[\frac{(0)^3}{3} - \frac{3(0)^2}{2} \right] &= 0 \\ \Rightarrow \frac{k^3}{3} - \frac{3k^2}{2} &= 0 \\ \Rightarrow 2k^3 - 9k^2 &= 0 \\ \Rightarrow k^2(2k - 9) &= 0 \\ \Rightarrow k^2 = 0 \quad \text{OR} \quad 2k - 9 &= 0 \\ \Rightarrow k = 0 \quad \text{OR} \quad k = \frac{9}{2} \\ \text{Since } k > 0 \Rightarrow k &= \frac{9}{2}\end{aligned}$$

Q20.
$$x^3 - 8 = (x-2)(x^2 + 2x + 4)$$

$$\begin{aligned}\int_0^2 \frac{x^3 - 8}{x - 2} dx &= \int_0^2 \frac{(x-2)(x^2 + 2x + 4)}{x - 2} dx \\ &= \int_0^2 (x^2 + 2x + 4) dx \\ &= \left[\frac{x^3}{3} + x^2 + 4x \right]_0^2 \\ &= \left[\frac{(2)^3}{3} + (2)^2 + 4(2) \right] - \left[\frac{(0)^3}{3} + (0)^2 + 4(0) \right] \\ &= \frac{8}{3} + 4 + 8 - 0 \\ &= 14\frac{2}{3}\end{aligned}$$

$$\begin{aligned}
\text{Q21.} \quad & \int_0^1 nx^2 dx = 1 \\
\Rightarrow & \left[\frac{nx^3}{3} \right]_0^1 = 1 \\
\Rightarrow & \left[\frac{n(1)^3}{3} \right] - \left[\frac{n(0)^3}{3} \right] = 1 \\
\Rightarrow & \frac{n}{3} - 0 = 1 \Rightarrow n = 3
\end{aligned}$$

$$\begin{aligned}
\text{Q22. (i)} \quad & \int_0^{\frac{\pi}{4}} \cos 2x dx = \left[\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{4}} = \left[\frac{\sin 2\left(\frac{\pi}{4}\right)}{2} \right] - \left[\frac{\sin 2(0)}{2} \right] \\
& = \left[\frac{\sin \frac{\pi}{2}}{2} \right] - \left[\frac{\sin 0}{2} \right] \\
& = \frac{1}{2} - 0 = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad & \int_0^{\frac{\pi}{6}} \sin 3x dx = \left[\frac{-\cos 3x}{3} \right]_0^{\frac{\pi}{6}} = \left[\frac{-\cos 3\left(\frac{\pi}{6}\right)}{3} \right] - \left[\frac{-\cos 3(0)}{3} \right] \\
& = -\frac{\cos \frac{\pi}{2}}{3} + \frac{\cos 0}{3} = 0 + \frac{1}{3} = \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad & \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 5 \sin x dx = \left[-5 \cos x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \left[-5 \cos \left(\frac{\pi}{2} \right) \right] - \left[-5 \cos \left(\frac{\pi}{3} \right) \right] \\
& = -5(0) + 5 \left(\frac{1}{2} \right) = \frac{5}{2}
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad & \int_0^{\frac{\pi}{2}} [2 \cos x + 1] dx = [2 \sin x + x]_0^{\frac{\pi}{2}} \\
& = \left[2 \sin \frac{\pi}{2} + \frac{\pi}{2} \right] - [2 \sin 0 + 0] \\
& = 2 + \frac{\pi}{2} - 0 = 2 + \frac{\pi}{2}
\end{aligned}$$

$$\text{Q23. (i)} \quad \int_0^2 e^{4x} dx = \left[\frac{e^{4x}}{4} \right]_0^2 = \frac{e^{4(2)}}{4} - \frac{e^{4(0)}}{4} = \frac{1}{4} [e^8 - 1]$$

$$\text{(ii)} \quad \int_{-1}^1 e^{x+3} dx = [e^{x+3}]_{-1}^1 = e^{1+3} - e^{-1+3} = e^4 - e^2$$

$$\begin{aligned} \text{(iii)} \quad \int_0^1 e^{\frac{x}{2}} dx &= \left[\frac{e^{\frac{x}{2}}}{\frac{1}{2}} \right]_0^1 = \left[2e^{\frac{x}{2}} \right]_0^1 = 2e^{\frac{1}{2}} - 2e^{\frac{0}{2}} \\ &= 2e^{\frac{1}{2}} - 2 \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad \int_0^1 (e^{-2x} + 1) dx &= \left[\frac{e^{-2x}}{-2} + x \right]_0^1 = \left[-\frac{1}{2}e^{-2x} + x \right]_0^1 \\ &= \left[-\frac{1}{2}e^{-2(1)} + 1 \right] - \left[-\frac{1}{2}e^{-2(0)} + 0 \right] \\ &= -\frac{1}{2}e^{-2} + 1 + \frac{1}{2} = \frac{3}{2} - \frac{1}{2e^2} \\ &= \frac{1}{2} \left[3 - \frac{1}{e^2} \right] \end{aligned}$$

$$\begin{aligned} \text{Q24. (i)} \quad \int_0^1 (2e^{\frac{x}{3}} + 2) dx &= \left[\frac{2e^{\frac{x}{3}}}{\frac{1}{3}} + 2x \right]_0^1 \\ &= \left[6e^{\frac{x}{3}} + 2x \right]_0^1 \\ &= \left[6e^{\frac{1}{3}} + 2(1) \right] - \left[6e^{\frac{0}{3}} + 2(0) \right] \\ &= 6e^{\frac{1}{3}} + 2 - 6 - 0 = 6e^{\frac{1}{3}} - 4 \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \int_{-2}^2 \frac{e^x + e^{-x}}{2} dx &= \frac{1}{2} \int_{-2}^2 (e^x + e^{-x}) dx \\ &= \frac{1}{2} \left[e^x + \frac{e^{-x}}{-1} \right]_{-2}^2 \\ &= \frac{1}{2} [e^x - e^{-x}]_{-2}^2 = \frac{1}{2} [e^2 - e^{-2}] - \frac{1}{2} [e^{-2} - e^{-(-2)}] \\ &= \frac{1}{2} e^2 - \frac{1}{2e^2} - \frac{1}{2e^2} + \frac{1}{2} e^2 \\ &= e^2 - \frac{1}{e^2} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \int_1^3 5^x dx &= \left[\frac{5^x}{\ell \ln 5} \right]_1^3 = \frac{5^3}{\ell \ln 5} - \frac{5^1}{\ell \ln 5} \\ &= \frac{125}{\ell \ln 5} - \frac{5}{\ell \ln 5} = \frac{120}{\ell \ln 5} \end{aligned}$$

$$\text{(iv)} \quad \int_0^e 7^x dx = \left[\frac{7^x}{\ell \ln 7} \right]_0^e = \frac{7^e}{\ell \ln 7} - \frac{7^0}{\ell \ln 7} = \frac{7^e}{\ell \ln 7} - \frac{1}{\ell \ln 7}$$

Q25. $f(x) = \frac{\cos x}{\sin x} \Rightarrow$ Quotient rule: $u = \cos x$ and $v = \sin x$

$$\Rightarrow \frac{du}{dx} = -\sin x \quad \Rightarrow \frac{dv}{dx} = \cos x$$

$$f'(x) = \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-1(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

Hence $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x} dx = -\left[\frac{\cos x}{\sin x} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$$= -\left[\frac{\cos \frac{\pi}{2}}{\sin \frac{\pi}{2}} \right] + \left[\frac{\cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} \right]$$

$$= -\left(\frac{0}{1} \right) + \frac{1}{\frac{1}{\sqrt{2}}} = 0 + 1 = 1$$

Q26. $f(x) = x \sin 3x \Rightarrow$ product rule: $u = x$ and $v = \sin 3x$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = \cos 3x \cdot 3$$

$$= 3 \cos 3x$$

$$f'(x) = \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} = x.(3 \cos 3x) + \sin 3x \cdot 1$$

$$= 3x \cos 3x + \sin 3x$$

Hence $\int (3x \cos 3x + \sin 3x) dx = x \sin 3x$

$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x \cos 3x + \int_0^{\frac{\pi}{6}} \sin 3x dx = [x \sin 3x]_0^{\frac{\pi}{6}}$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x \cos 3x + \left[\frac{-\cos 3x}{3} \right]_0^{\frac{\pi}{6}} = \frac{\pi}{6} \sin 3 \left(\frac{\pi}{6} \right) - 0 \cdot \sin 3(0)$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x \cos 3x + \left[\frac{-\cos 3 \left(\frac{\pi}{6} \right)}{3} \right] - \left[\frac{-\cos 3(0)}{3} \right] = \frac{\pi}{6} \sin \frac{\pi}{2}$$

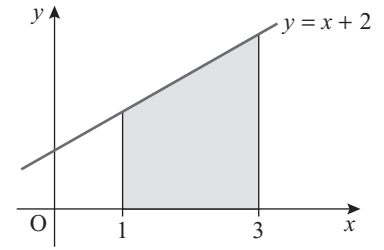
$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x \cos 3x + \left[\frac{-\cos \frac{\pi}{2}}{3} \right] + \frac{\cos 0}{3} = \frac{\pi}{6} \cdot 1$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x \cos 3x - \frac{0}{3} + \frac{1}{3} = \frac{\pi}{6}$$

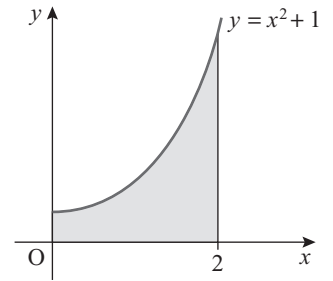
$$\Rightarrow \int_0^{\frac{\pi}{6}} 3x \cos 3x = \frac{\pi}{6} - \frac{1}{3}$$

Exercise 4.5

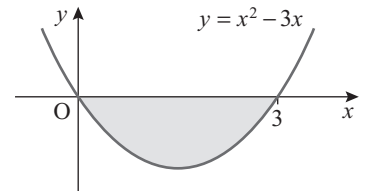
Q1.
$$\begin{aligned}\text{Area} &= \int_1^3 (x+2) dx = \left[\frac{x^2}{2} + 2x \right]_1^3 \\ &= \left[\frac{(3)^2}{2} + 2(3) \right] - \left[\frac{(1)^2}{2} + 2(1) \right] \\ &= \frac{9}{2} + 6 - \frac{1}{2} - 2 = 10\frac{1}{2} - 2\frac{1}{2} = 8 \text{ sq. units}\end{aligned}$$



Q2.
$$\begin{aligned}\text{Area} &= \int_0^2 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_0^2 \\ &= \left[\frac{(2)^3}{3} + 2 \right] - \left[\frac{(0)^3}{3} + 0 \right] \\ &= \frac{8}{3} + 2 - 0 = 4\frac{2}{3} \text{ sq. units}\end{aligned}$$

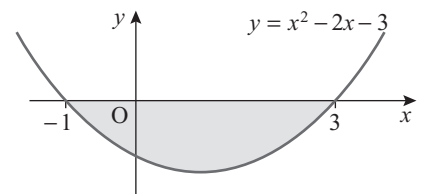


Q3.
$$\begin{aligned}\text{Area} &= \int_0^3 (x^2 - 3x) dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^3 \\ &= \left[\frac{(3)^3}{3} - \frac{3(3)^2}{2} \right] - \left[\frac{(0)^3}{3} - \frac{3(0)^2}{2} \right] \\ &= 9 - \frac{27}{2} - 0 = -4\frac{1}{2}\end{aligned}$$



Hence Area = $4\frac{1}{2}$ sq. units

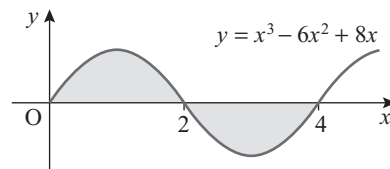
Q4.
$$\begin{aligned}\text{Area} &= \int_{-1}^3 (x^2 - 2x - 3) dx = \left[\frac{x^3}{3} - x^2 - 3x \right]_{-1}^3 \\ &= \left[\frac{(3)^3}{3} - (3)^2 - 3(3) \right] - \left[\frac{(-1)^3}{3} - (-1)^2 - 3(-1) \right] \\ &= (9 - 9 - 9) - \left(-\frac{1}{3} - 1 + 3 \right) \\ &= -9 - 1\frac{2}{3} = -10\frac{2}{3}\end{aligned}$$



Hence Area = $10\frac{2}{3}$ sq. units

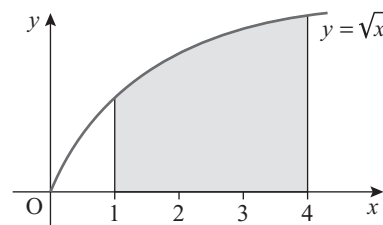
Q5.

$$\begin{aligned}
 \text{Area} &= \int_0^2 (x^3 - 6x^2 + 8x) dx + \int_2^4 (x^3 - 6x^2 + 8x) dx \\
 &= \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2} \right]_0^2 + \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2} \right]_2^4 \\
 &= \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 + \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_2^4 \\
 &= \left[\frac{(2)^4}{4} - 2(2)^3 + 4(2)^2 \right] - \left[\frac{0}{4} - 2(0) + 4(0) \right] + \left[\frac{(4)^4}{4} - 2(4)^3 + 4(4)^2 \right] - \left[\frac{(2)^4}{4} - 2(2)^3 + 4(2)^2 \right] \\
 &= [4 - 16 + 16] - [0] + [64 - 128 + 64] - [4 - 16 + 16] \\
 &= 4 + [-4] \Rightarrow \text{Area} = 4 + 4 = 8 \text{ sq.units}
 \end{aligned}$$



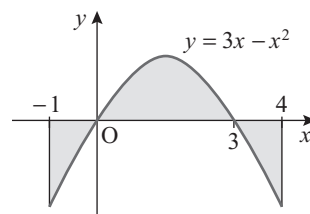
Q6.

$$\begin{aligned}
 \text{Area} &= \int_1^4 \sqrt{x} dx = \int_1^4 x^{\frac{1}{2}} dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 = \left[\frac{2x^{\frac{3}{2}}}{3} \right]_1^4 \\
 &= \left[\frac{2(4)^{\frac{3}{2}}}{3} \right] - \left[\frac{2(1)^{\frac{3}{2}}}{3} \right] = \frac{16}{3} - \frac{2}{3} = \frac{14}{3} = 4\frac{2}{3} \text{ sq.units}
 \end{aligned}$$



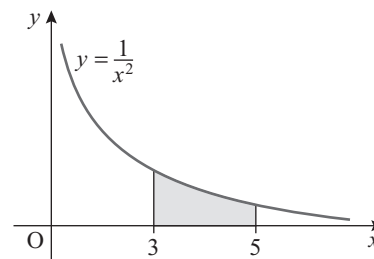
Q7.

$$\begin{aligned}
 \text{Area} &= \int_{-1}^0 (3x - x^2) dx + \int_0^3 (3x - x^2) dx + \int_3^4 (3x - x^2) dx \\
 &= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_{-1}^0 + \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 + \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_3^4 \\
 &= \left[\frac{3(0)^2}{2} - \frac{(0)^3}{3} \right] - \left[\frac{3(-1)^2}{2} - \frac{(-1)^3}{3} \right] + \left[\frac{3(3)^2}{2} - \frac{(3)^3}{3} \right] - \left[\frac{3(0)^2}{2} - \frac{(0)^3}{3} \right] \\
 &\quad + \left[\frac{3(4)^2}{2} - \frac{(4)^3}{3} \right] - \left[\frac{3(3)^2}{2} - \frac{(3)^3}{3} \right] \\
 &= 0 - \frac{11}{6} + 4\frac{1}{2} - 0 + 2\frac{2}{3} - 4\frac{1}{2} \\
 &= -\frac{11}{6} + 4\frac{1}{2} - \frac{11}{6} \\
 \Rightarrow \text{Area} &= \frac{11}{6} + 4\frac{1}{2} + \frac{11}{6} = 8\frac{1}{6} \text{ sq.units}
 \end{aligned}$$



Q8.

$$\begin{aligned}
 \text{Area} &= \int_3^5 \frac{1}{x^2} dx = \int_3^5 x^{-2} dx = \left[\frac{x^{-1}}{-1} \right]_3^5 \\
 &= \left[-\frac{1}{x} \right]_3^5 = \left[-\frac{1}{5} \right] - \left[-\frac{1}{3} \right] = \frac{2}{15} \text{ sq.units}
 \end{aligned}$$



Q9.

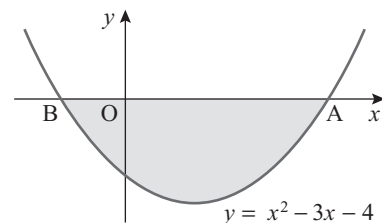
$$y = x^2 - 3x - 4$$

Points A and B are on x -axis $\Rightarrow y = 0$

$$\Rightarrow x^2 - 3x - 4 = 0$$

$$\Rightarrow (x-4)(x+1) = 0 \Rightarrow x = 4 \quad \text{OR} \quad x = -1$$

$$\Rightarrow A = (4, 0) \text{ and } B = (-1, 0)$$



$$\begin{aligned} \text{Area} &= \int_{-1}^4 (x^2 - 3x - 4) dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x \right]_{-1}^4 \\ &= \left[\frac{(4)^3}{3} - \frac{3(4)^2}{2} - 4(4) \right] - \left[\frac{(-1)^3}{3} - \frac{3(-1)^2}{2} - 4(-1) \right] \\ &= \left[\frac{64}{3} - 24 - 16 \right] - \left[-\frac{1}{3} - \frac{3}{2} + 4 \right] \\ &= -18\frac{2}{3} - 2\frac{1}{6} = -20\frac{5}{6} \\ \Rightarrow \text{Area} &= 20\frac{5}{6} \text{ sq. units} \end{aligned}$$

Q10.

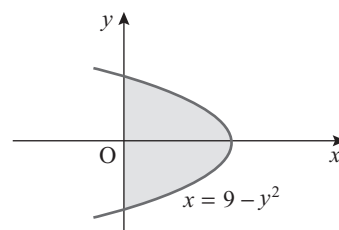
$$x = 9 - y^2$$

On y -axis, $x = 0 \Rightarrow 9 - y^2 = 0$

$$\Rightarrow (3+y)(3-y) = 0 \Rightarrow y = -3 \quad \text{OR} \quad y = 3$$

Points are $(0, 3)$ and $(0, -3)$

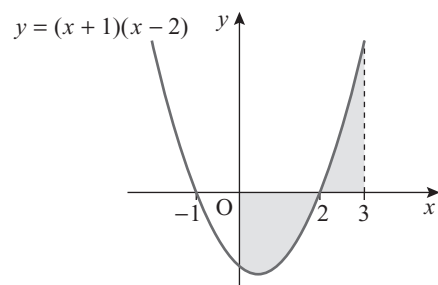
$$\begin{aligned} \text{Area} &= \int_a^b x dy = \int_{-3}^3 (9 - y^2) dy \\ &= \left[9y - \frac{y^3}{3} \right]_{-3}^3 \\ &= \left[9(3) - \frac{(3)^3}{3} \right] - \left[9(-3) - \frac{(-3)^3}{3} \right] \\ &= (27 - 9) - (-27 + 9) = 18 + 18 = 36 \text{ sq. units} \end{aligned}$$

**Q11.**

$$y = (x+1)(x-2) = x^2 - x - 2$$

$$\begin{aligned} \text{Area} &= \int_0^2 (x^2 - x - 2) dx + \int_2^3 (x^2 - x - 2) dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_0^2 + \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_2^3 \\ &= \left[\frac{(2)^3}{3} - \frac{(2)^2}{2} - 2(2) \right] - \left[\frac{(0)^3}{3} - \frac{(0)^2}{2} - 2(0) \right] \\ &\quad + \left[\frac{(3)^3}{3} - \frac{(3)^2}{2} - 2(3) \right] - \left[\frac{(2)^3}{3} - \frac{(2)^2}{2} - 2(2) \right] \\ &= \left(\frac{8}{3} - 2 - 4 \right) - 0 + \left(9 - \frac{9}{2} - 6 \right) - \left(\frac{8}{3} - 2 - 4 \right) \\ &= -3\frac{1}{3} + 1\frac{5}{6} \end{aligned}$$

$$\text{Hence Area} = 3\frac{1}{3} + 1\frac{5}{6} = \frac{31}{6} \text{ sq. units}$$



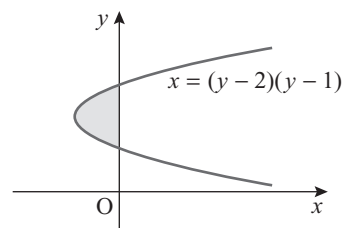
Q12.

$$x = (y-2)(y-1) \Rightarrow \text{on } y\text{-axis, } x = 0 \Rightarrow (y-2)(y-1) = 0 \\ \Rightarrow y = 2 \text{ or } y = 1$$

$$x = (y-2)(y-1) = y^2 - 3y + 2$$

$$\begin{aligned} \text{Area} &= \int_1^2 (y^2 - 3y + 2) dy = \left[\frac{y^3}{3} - \frac{3y^2}{2} + 2y \right]_1^2 \\ &= \left[\frac{(2)^3}{3} - \frac{3(2)^2}{2} + 2(2) \right] - \left[\frac{(1)^3}{3} - \frac{3(1)^2}{2} + 2(1) \right] \\ &= \left[\frac{8}{3} - 6 + 4 \right] - \left[\frac{1}{3} - \frac{3}{2} + 2 \right] \\ &= \frac{2}{3} - \frac{5}{6} = -\frac{1}{6} \end{aligned}$$

$$\text{Hence Area} = \frac{1}{6} \text{ sq. units}$$

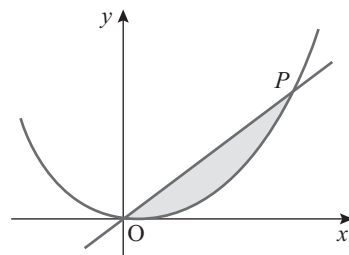
**Q13. (i)**

$$y = 2x \cap y = x^2$$

$$\Rightarrow x^2 = 2x \Rightarrow x^2 - 2x = 0$$

$$\Rightarrow x(x-2) = 0 \Rightarrow x = 0, x = 2$$

$$\text{At } P: x = 2 \Rightarrow y = 2(2) = 4 \Rightarrow P = (2, 4)$$

**(ii)**

$$\begin{aligned} \text{Area} &= \int_0^2 2x \, dx - \int_0^2 x^2 \, dx \\ &= \left[x^2 \right]_0^2 - \left[\frac{x^3}{3} \right]_0^2 \\ &= \left[(2)^2 \right] - \left[(0)^2 \right] - \left[\left(\frac{(2)^3}{3} \right) - \left(\frac{(0)^3}{3} \right) \right] \\ &= (4 - 0) - \left(\frac{8}{3} - 0 \right) = 4 - 2\frac{2}{3} = 1\frac{1}{3} \text{ sq. units} \end{aligned}$$

Q14.

$$y = -x + 8 \cap y = 5x - x^2$$

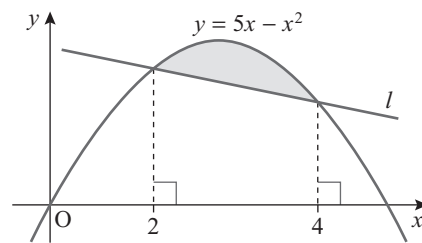
$$\Rightarrow -x + 8 = 5x - x^2$$

$$\Rightarrow x^2 - 6x + 8 = 0$$

$$\Rightarrow (x-2)(x-4) = 0 \Rightarrow x = 2 \text{ OR } x = 4$$

$$\text{Area} = \int_2^4 (5x - x^2) dx - \int_2^4 (-x + 8) dx$$

$$\begin{aligned} &= \left[\frac{5x^2}{2} - \frac{x^3}{3} \right]_2^4 - \left[\frac{-x^2}{2} + 8x \right]_2^4 \\ &= \left[\left[\frac{5(4)^2}{2} - \frac{(4)^3}{3} \right] - \left[\frac{5(2)^2}{2} - \frac{(2)^3}{3} \right] \right] - \left[\left[\frac{-(-4)^2}{2} + 8(4) \right] - \left[\frac{-(-2)^2}{2} + 8(2) \right] \right] \\ &= \left[40 - 21\frac{1}{3} \right] - \left[10 - \frac{8}{3} \right] - [(-8 + 32) - (-2 + 16)] \\ &= \left[18\frac{2}{3} - 7\frac{1}{3} \right] - [24 - 14] \\ &= 11\frac{1}{3} - 10 = 1\frac{1}{3} \text{ sq. units} \end{aligned}$$



Q15. (i) $x + y - 1 = 0 \Rightarrow y = -x + 1$

Line $y = -x + 1 \cap$ Curve $y = -x^2 - x + 2$

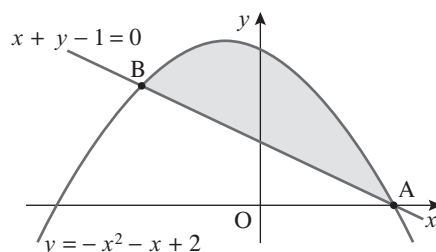
$$\Rightarrow -x + 1 = -x^2 - x + 2$$

$$\Rightarrow x^2 - 1 = 0$$

$$\Rightarrow (x+1)(x-1) = 0 \Rightarrow x = -1 \text{ OR } x = 1$$

At B, $x = -1 \Rightarrow y = -(-1) + 1 = 2 \Rightarrow B = (-1, 2)$

At A, $x = 1 \Rightarrow y = -1 + 1 = 0 \Rightarrow A = (1, 0)$



(ii) Shaded Area = Area under the curve – Area under the line

$$= \int_{-1}^1 (-x^2 - x + 2) dx - \int_{-1}^1 (-x + 1) dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-1}^1 - \left[-\frac{x^2}{2} + x \right]_{-1}^1$$

$$= \left[-\frac{(1)^3}{3} - \frac{(1)^2}{2} + 2(1) \right] - \left[-\frac{(-1)^3}{3} - \frac{(-1)^2}{2} + 2(-1) \right] - \left[\left[\frac{-(-1)^2}{2} + (-1) \right] - \left[\frac{-(1)^2}{2} + (1) \right] \right]$$

$$= \left[\left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{1}{3} - \frac{1}{2} - 2 \right) \right] - \left[\left(-\frac{1}{2} + 1 \right) - \left(-\frac{1}{2} - 1 \right) \right]$$

$$= \left(1\frac{1}{6} + 2\frac{1}{6} \right) - \left(\frac{1}{2} + 1\frac{1}{2} \right)$$

$$= 3\frac{1}{3} - 2 = 1\frac{1}{3} \text{ sq. units}$$

Q16.

$$y = 2x \cap y^2 = 8x \text{ OR } y = \sqrt{8x^{\frac{1}{2}}}$$

$$\Rightarrow y^2 = 4x^2 \cap y^2 = 8x$$

$$\Rightarrow 4x^2 = 8x \Rightarrow x^2 = 2x \Rightarrow x^2 - 2x = 0$$

$$\Rightarrow x(x - 2) = 0$$

$$\Rightarrow x = 0 \text{ OR } x = 2$$

When $x = 0 \Rightarrow y = 2(0) = 0 \Rightarrow \text{Point}(0, 0)$

When $x = 2 \Rightarrow y = 2(2) = 4 \Rightarrow \text{Point}(2, 4)$

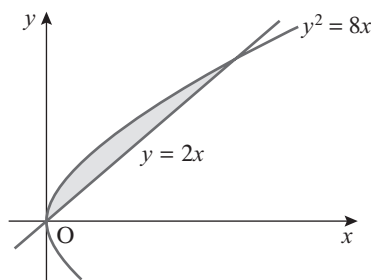
Shaded area = Area under the curve – area under the line

$$= \int_0^2 \sqrt{8} x^{\frac{1}{2}} dx - \int_0^2 2x dx$$

$$= \left[\sqrt{8} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^2 - \left[x^2 \right]_0^2$$

$$= \left[\frac{2\sqrt{8}}{3} (2)^{\frac{3}{2}} - 0 \right] - \left[[2^2] - 0 \right]$$

$$= \frac{2\sqrt{8}}{3} \cdot 2\sqrt{2} - 4 = \frac{16}{3} - 4 = 1\frac{1}{3} \text{ sq. units}$$



Q17. (i) $y^2 = 4x \cap x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

$$\Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow \frac{x^4}{16} = 4x$$

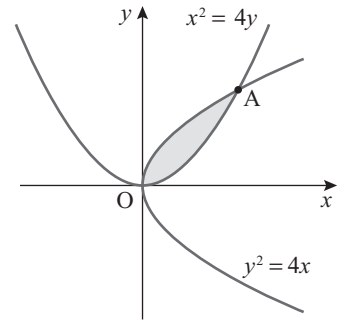
$$\Rightarrow x^4 = 64x$$

$$\Rightarrow x^4 - 64x = 0$$

$$\Rightarrow x(x-4)(x^2+4x+16) = 0$$

$$\Rightarrow x = 0, x = 4$$

When $x = 4 \Rightarrow y = \frac{(4)^2}{4} = 4 \Rightarrow A = (4, 4)$



(ii) $y^2 = 4x \Rightarrow y = \sqrt{4x} = 2\sqrt{x} = 2x^{\frac{1}{2}}$

Shaded Area = Area under curve $y = 2x^{\frac{1}{2}}$ - area under curve $y = \frac{x^2}{4}$

$$\text{Shaded area} = \int_0^4 2x^{\frac{1}{2}} dx - \int_0^4 \frac{x^2}{4} dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^4 - \left[\frac{1}{4} \cdot \frac{x^3}{3} \right]_0^4$$

$$= \left[\frac{4}{3}(x)^{\frac{3}{2}} \right]_0^4 - \left[\frac{x^3}{12} \right]_0^4$$

$$= \left[\frac{4}{3}(4)^{\frac{3}{2}} - 0 \right] - \left[\frac{(4)^3}{12} - 0 \right]$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} = 5\frac{1}{3} \text{ sq. units}$$

Q18. (i) $y = 2x - x^2 \Rightarrow$ on x -axis, $y = 0$

$$\Rightarrow 2x - x^2 = 0$$

$$\Rightarrow x(2-x) = 0 \Rightarrow x = 0 \quad \text{OR} \quad x = 2$$

$$\Rightarrow Q = (2, 0)$$

Line: $y = -2x \cap$ Curve: $y = 2x - x^2$

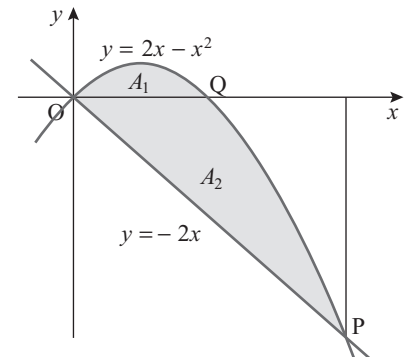
$$\Rightarrow -2x = 2x - x^2$$

$$\Rightarrow x^2 - 4x = 0$$

$$\Rightarrow x(x-4) = 0$$

$$\Rightarrow x = 0 \quad \text{OR} \quad x = 4$$

When $x = 4 \Rightarrow y = -2(4) = -8 \Rightarrow P = (4, -8)$



$$\begin{aligned}
 \text{(ii) Area } A_1 &= \int_0^2 (2x - x^2) dx \\
 &= \left[x^2 - \frac{x^3}{3} \right]_0^2 \\
 &= \left[(2)^2 - \frac{(2)^3}{3} \right] - \left[(0)^2 - \frac{(0)^3}{3} \right] \\
 &= 4 - \frac{8}{3} - 0 = 1\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Area } A_2 &= \int_0^4 (-2x) dx - \int_2^4 (2x - x^2) dx \\
 &= \left[-x^2 \right]_0^4 - \left[x^2 - \frac{x^3}{3} \right]_2^4 \\
 &= \left[-(4)^2 - (-0)^2 \right] - \left[\left[(4)^2 - \frac{(4)^3}{3} \right] - \left[(2)^2 - \frac{(2)^3}{3} \right] \right] \\
 &= -16 - \left[\left(16 - \frac{64}{3} \right) - \left(4 - \frac{8}{3} \right) \right] \\
 &= -16 - \left[\frac{-16}{3} - \frac{4}{3} \right] \\
 &= -16 + \frac{20}{3} = -\frac{28}{3}
 \end{aligned}$$

$$\text{Hence Area } A_2 = \frac{28}{3} = 9\frac{1}{3}$$

$$\Rightarrow \text{Area shaded region} = 1\frac{1}{3} + 9\frac{1}{3} = 10\frac{2}{3} \text{ sq. units}$$

Q19. (i) $y = x^2 + 2 \cap y = 6$

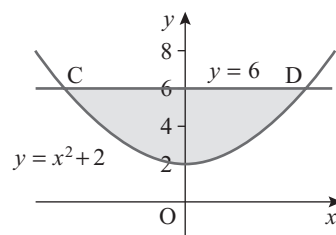
$$\Rightarrow x^2 + 2 = 6 \Rightarrow x^2 - 4 = 0$$

$$\Rightarrow (x+2)(x-2) = 0 \Rightarrow x = -2 \text{ OR } x = 2$$

Hence $C = (-2, 6)$ and $D = (2, 6)$

(ii) Shaded area = area under line: $y = 6$ - area under curve: $y = x^2 + 2$

$$\begin{aligned}
 &= \int_{-2}^2 6 dx - \int_{-2}^2 (x^2 + 2) dx \\
 &= [6x]_{-2}^2 - \left[\frac{x^3}{3} + 2x \right]_{-2}^2 \\
 &= [6(2) - 6(-2)] - \left[\left[\frac{(2)^3}{3} + 2(2) \right] - \left[\frac{(-2)^3}{3} + 2(-2) \right] \right] \\
 &= 12 + 12 - \left[\left(\frac{8}{3} + 4 \right) - \left(\frac{-8}{3} - 4 \right) \right] \\
 &= 24 - 13\frac{1}{3} = 10\frac{2}{3} \text{ sq. units}
 \end{aligned}$$

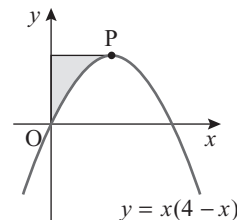


Q20. (i) $y = x(4 - x) = 4x - x^2$

$$\frac{dy}{dx} = 4 - 2x = 0 \text{ for maximum}$$

$$\Rightarrow 2x = 4 \Rightarrow x = 2$$

$$\Rightarrow y = 4(2) - (2)^2 = 4 \Rightarrow P = (2, 4)$$

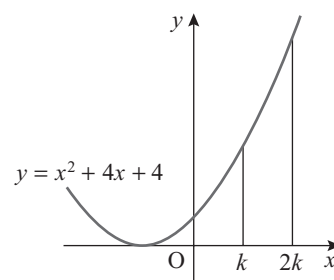


- (ii) Shaded area = area between line: $x = 2$ and y -axis
 – area between curve: $y = 4x^2 - x^2 [OP]$ and x -axis

$$\begin{aligned} \Rightarrow \text{Shaded area} &= \int_0^4 2dy - \int_0^2 (4x - x^2)dx \\ &= [2y]_0^4 - \left[2x^2 - \frac{x^3}{3} \right]_0^2 \\ &= [2(4) - 2(0)] - \left[\left[2(2)^2 - \frac{(2)^3}{3} \right] - \left[2(0)^2 - \frac{(0)^3}{3} \right] \right] \\ &= (8 - 0) - \left[\left(8 - \frac{8}{3} \right) - (0 - 0) \right] \\ &= 8 - 8 + \frac{8}{3} = 2\frac{2}{3} \text{ sq. units} \end{aligned}$$

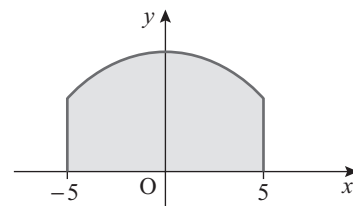
Q21.

$$\begin{aligned} \int_0^{2k} (x^2 + 4x + 4)dx &= 4 \int_0^k (x^2 + 4x + 4)dx \\ &= \left[\frac{x^3}{3} + 2x^2 + 4x \right]_0^{2k} = 4 \left[\frac{x^3}{3} + 2x^2 + 4x \right]_0^k \\ &= \left[\frac{(2k)^3}{3} + 2(2k)^2 + 4(2k) \right] - \left[\frac{(0)^3}{3} + 2(0)^2 + 4(0) \right] = 4 \left[\frac{k^3}{3} + 2k^2 + 4k \right] - 4(0) \\ &= \frac{8k^3}{3} + 8k^2 + 8k - 0 = \frac{4k^3}{3} + 8k^2 + 16k - 0 \\ &\Rightarrow 8k^3 + 24k = 4k^3 + 48k \\ &\Rightarrow 4k^3 - 24k = 0 \\ &\Rightarrow k(k^2 - 6) = 0 \Rightarrow k = 0 \text{ OR } k^2 = 6 \\ &\Rightarrow k = \sqrt{6} \end{aligned}$$



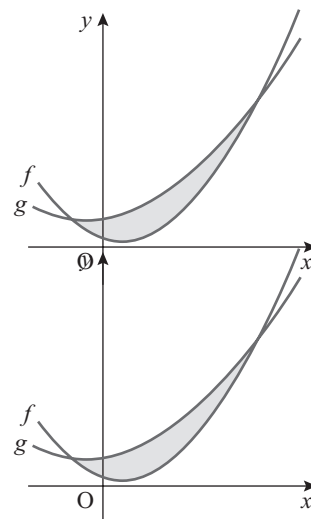
Q22. (i) Area = $\int_{-5}^5 (6 - 0.08x^2 - 0.0006x^4)dx$

$$\begin{aligned} &= \left[6x - 0.08 \frac{x^3}{3} - 0.0006 \frac{x^5}{5} \right]_{-5}^5 \\ &= \left[6(5) - 0.08 \frac{(5)^3}{3} - 0.0006 \frac{(5)^5}{5} \right] - \left[6(-5) - 0.08 \frac{(-5)^3}{3} - 0.0006 \frac{(-5)^5}{5} \right] \\ &= \left(30 - \frac{10}{3} - \frac{3}{8} \right) - \left(-30 + \frac{10}{3} + \frac{3}{8} \right) = 52.58 = 52.6 \text{ sq. units} \end{aligned}$$



- (ii) Volume of the tunnel = $52.6 \cdot 14 = 736.4 = 736 \text{ m}^3$

Q23. (i) $y = 2x^2 - 3x + 2 \cap y = x^2 + x + 7$
 $\Rightarrow 2x^2 - 3x + 2 = x^2 + x + 7$
 $\Rightarrow x^2 - 4x - 5 = 0$
 $\Rightarrow (x+1)(x-5) = 0 \Rightarrow x = -1 \text{ OR } x = 5$
 $x = -1 \Rightarrow g(-1) = (-1)^2 + (-1) + 7 = 1 - 1 + 7 = 7 \Rightarrow \text{point } (-1, 7)$
 $x = 5 \Rightarrow g(5) = (5)^2 + (5) + 7 = 25 + 5 + 7 = 37 \Rightarrow \text{point } (5, 37)$

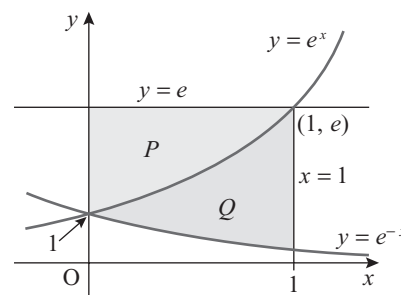


(ii) Shaded area = area under $g(x)$ - area under $f(x)$

$$\begin{aligned} &= \int_{-1}^5 (x^2 + x + 7) dx - \int_{-1}^5 (2x^2 - 3x + 2) dx \\ &= \int_{-1}^5 (x^2 + x + 7) dx = \left[\frac{x^3}{3} + \frac{x^2}{2} + 7x \right]_{-1}^5 \\ &= \left[\frac{(5)^3}{3} + \frac{(5)^2}{2} + 7(5) \right] - \left[\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 7(-1) \right] \\ &= \left(\frac{125}{3} + \frac{25}{2} + 35 \right) - \left(-\frac{1}{3} + \frac{1}{2} - 7 \right) = 96 \\ &\int_{-1}^5 (2x^2 - 3x + 2) dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x \right]_{-1}^5 \\ &= \left[\frac{2(5)^3}{3} - \frac{3(5)^2}{2} + 2(5) \right] - \left[\frac{2(-1)^3}{3} - \frac{3(-1)^2}{2} + 2(-1) \right] \\ &= \left(\frac{250}{3} - \frac{75}{2} + 10 \right) - \left(-\frac{2}{3} - \frac{3}{2} - 2 \right) = 60 \end{aligned}$$

$$\Rightarrow \text{Shaded Area} = 96 - 60 = 36 \text{ sq. units}$$

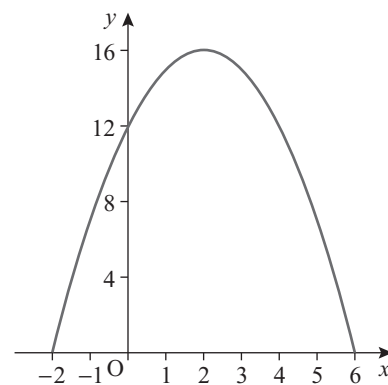
Q24. (i) Area $P = \int_0^1 e dx - \int_0^1 e^x dx$
 $= [ex]_0^1 - [e^x]_0^1$
 $= [e(1) - e(0)] - [e^1 - e^0]$
 $= (e - 0) - (e - 1) = e - e + 1 = 1 \text{ sq. units}$



(ii) Area $Q = \int_0^1 e^x dx - \int_0^1 e^{-x} dx$
 $= [e^x]_0^1 - \left[\frac{e^{-x}}{-1} \right]_0^1$
 $= [e^x]_0^1 - \left[\frac{-1}{e^x} \right]_0^1$
 $= [e^1 - e^0] - \left[-\frac{1}{e^1} + \frac{1}{e^0} \right]$
 $= e - 1 + \frac{1}{e} - 1 = \left(e + \frac{1}{e} - 2 \right) \text{ sq. units}$

Exercise 4.6

$$\begin{aligned}\text{Q1. (i) Average value} &= \frac{f(0) + f(1) + f(2) + f(3) + f(4) + f(5) + f(6)}{7} \\ &= \frac{12 + 15 + 16 + 15 + 12 + 7 + 0}{7} = \frac{77}{7} = 11\end{aligned}$$



$$\begin{aligned}\text{(ii) Average value} &= \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{6-0} \int_0^6 (-x^2 + 4x + 12) dx \\ &= \frac{1}{6} \left[\frac{-x^3}{3} + 2x^2 + 12x \right]_0^6 \\ &= \frac{1}{6} \left[\frac{-(6)^3}{3} + 2(6)^2 + 12(6) \right] - \frac{1}{6} \left[\frac{-(0)^3}{3} + 2(0)^2 + 12(0) \right] \\ &= \frac{1}{6} [-72 + 72 + 72] - 0 = 12\end{aligned}$$

(iii) Method (ii) gives the exact estimate

$$\begin{aligned}\text{Q2. (i) } f(x) &= 2x - 4 \\ f(2) &= 2(2) - 4 = 0 \\ f(3) &= 2(3) - 4 = 2 \\ f(4) &= 2(4) - 4 = 4 \\ f(5) &= 2(5) - 4 = 6 \\ \Rightarrow \text{average value} &= \frac{0 + 2 + 4 + 6}{4} = \frac{12}{4} = 3\end{aligned}$$

OR

$$\begin{aligned}\text{Average value} &= \frac{1}{5-2} \int_2^5 (2x-4) dx \\ &= \frac{1}{3} [x^2 - 4x]_2^5 \\ &= \frac{1}{3} [(5)^2 - 4(5)] - \frac{1}{3} [(2)^2 - 4(2)] \\ &= \frac{1}{3} [5] - \frac{1}{3} [-4] = \frac{5}{3} + \frac{4}{3} = 3\end{aligned}$$

$$\begin{aligned}\text{(ii) Average value} &= \frac{1}{2-0} \int_0^2 (x^2 - x) dx \\ &= \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^2 \\ &= \frac{1}{2} \left[\frac{(2)^3}{3} - \frac{(2)^2}{2} \right] - \frac{1}{2} \left[\frac{(0)^3}{3} - \frac{(0)^2}{2} \right] \\ &= \frac{1}{2} \left[\frac{8}{3} - 2 \right] - 0 = \frac{1}{3}\end{aligned}$$

$$\begin{aligned}
 \text{(iii) Average value} &= \frac{1}{2-0} \int_0^2 (2x - x^2) dx \\
 &= \frac{1}{2} \left[x^2 - \frac{x^3}{3} \right]_0^2 \\
 &= \frac{1}{2} \left[(2)^2 - \frac{(2)^3}{3} \right] - \frac{1}{2} \left[(0)^2 - \frac{(0)^3}{3} \right] = \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Q3. Average value} &= \frac{1}{4-0} \int_0^4 x^3 dx \\
 &= \frac{1}{4} \left[\frac{x^4}{4} \right]_0^4 \\
 &= \frac{1}{4} \left[\frac{(4)^4}{4} \right] - \frac{1}{4} \left[\frac{(0)^4}{4} \right] \\
 &= \frac{1}{4} (64) - 0 = 16
 \end{aligned}$$

$$\begin{aligned}
 \text{Q4. } f(x) &= x^2 + 4 \\
 \text{Average value} &= \frac{1}{3-(-2)} \int_{-2}^3 (x^2 + 4) dx \\
 &= \frac{1}{5} \left[\frac{x^3}{3} + 4x \right]_{-2}^3 \\
 &= \frac{1}{5} \left[\frac{(3)^3}{3} + 4(3) \right] - \frac{1}{5} \left[\frac{(-2)^3}{3} + 4(-2) \right] \\
 &= \frac{1}{5} (21) - \frac{1}{5} \left(\frac{-32}{3} \right) = 6 \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Q5. (i) } f(x) &= \sin x \\
 \text{Average value} &= \frac{1}{\frac{\pi}{2}-0} \int_0^{\frac{\pi}{2}} \sin x \, dx \\
 &= \frac{2}{\pi} \left[-\cos x \right]_0^{\frac{\pi}{2}} \\
 &= \frac{2}{\pi} \left[-\cos \frac{\pi}{2} \right] - \left[-\cos 0 \right] \\
 &= \frac{2}{\pi} (-0) + (1) = \frac{2}{\pi}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } f(x) &= \cos x \\
 \text{Average value} &= \frac{1}{2\pi-0} \int_0^{2\pi} \cos x \, dx \\
 &= \frac{1}{2\pi} \left[\sin x \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} (\sin 2\pi) - \frac{1}{2\pi} (\sin 0) = 0 - 0 = 0
 \end{aligned}$$

(iii) $f(x) = e^x$

$$\begin{aligned}\text{Average value} &= \frac{1}{3-0} \int_0^3 e^x dx \\ &= \frac{1}{3} [e^x]_0^3 \\ &= \frac{1}{3}(e^3) - \frac{1}{3}(e^0) = \frac{1}{3}(e^3 - 1)\end{aligned}$$

(iv) $f(x) = e^{4x}$

$$\begin{aligned}\text{Average value} &= \frac{1}{2-0} \int_0^2 e^{4x} dx \\ &= \frac{1}{2} \left[\frac{e^{4x}}{4} \right]_0^2 \\ &= \frac{1}{2} \left(\frac{e^{4(2)}}{4} \right) - \frac{1}{2} \left[\frac{e^{4(0)}}{4} \right] \\ &= \frac{e^8}{8} - \frac{1}{8}\end{aligned}$$

Q6. $f(x) = x + 1$

$$\begin{aligned}\text{Average value} &= \frac{1}{k-2} \int_2^k (x+1) dx = 8 \\ &= \frac{1}{k-2} \left[\frac{x^2}{2} + x \right]_2^k = 8 \\ &= \frac{1}{k-2} \left[\frac{k^2}{2} + k \right] - \frac{1}{k-2} \left[\frac{(2)^2}{2} + 2 \right] = 8 \\ &\Rightarrow \frac{k^2}{2} + k - 4 = 8(k-2) = 8k - 16 \\ &\Rightarrow k^2 + 2k - 8 = 16k - 32 \\ &\Rightarrow k^2 - 14k + 24 = 0 \\ &\Rightarrow (k-2)(k-12) = 0 \\ &\Rightarrow k = 2 \text{ OR } k = 12 \Rightarrow k = 12\end{aligned}$$

Q7. $f(x) = x^3$

$$\begin{aligned}\text{Average value} &= \frac{1}{k-0} \int_0^k x^3 dx = 16 \\ &\Rightarrow \frac{1}{k} \left[\frac{x^4}{4} \right]_0^k = 16 \\ &\Rightarrow \frac{1}{k} \left[\frac{k^4}{4} \right] - \frac{1}{k} \left[\frac{(0)^4}{4} \right] = 16 \\ &\Rightarrow \frac{k^3}{4} = 16 \\ &\Rightarrow k^3 = 64 \Rightarrow k = \sqrt[3]{64} = 4\end{aligned}$$

Q8. (i) $f(x) = \frac{1}{x^2} = x^{-2}$

$$\begin{aligned}\text{Average value} &= \frac{1}{5-1} \int_1^5 x^{-2} dx \\ &= \frac{1}{4} \left[\frac{x^{-1}}{-1} \right]_1^5 \\ &= \frac{1}{4} \left[-\frac{1}{x} \right]_1^5 \\ &= \frac{1}{4} \left[-\frac{1}{5} \right] - \frac{1}{4} \left[-\frac{1}{1} \right] \\ &= -\frac{1}{20} + \frac{1}{4} = \frac{1}{5}\end{aligned}$$

(ii) $f(x) = 5 \left(\cos \frac{x}{2} \right)$

$$\begin{aligned}\text{Average value} &= \frac{1}{2\pi-0} \int_0^{2\pi} 5 \left(\cos \frac{x}{2} \right) dx \\ &= \frac{1}{2\pi} \left[\frac{5 \sin \frac{x}{2}}{\frac{1}{2}} \right]_0^{2\pi} = \frac{1}{2\pi} \left[10 \sin \frac{x}{2} \right]_0^{2\pi} \\ &= \frac{1}{2\pi} \left[10 \sin \frac{2\pi}{2} \right] - \frac{1}{2\pi} \left[10 \sin \frac{0}{2} \right] \\ &= \frac{1}{2\pi} [10(0)] - \frac{1}{2\pi} [10(0)] \\ &= \frac{1}{2\pi} (0) - \frac{1}{2\pi} (0) = 0\end{aligned}$$

Q9. $V = \pi \frac{h^3}{12}$

$$\begin{aligned}\text{Average volume} &= \frac{1}{8-2} \int_2^8 \pi \frac{h^3}{12} dh \\ &= \frac{1}{6} \left[\frac{\pi}{12} \frac{h^4}{4} \right]_2^8 \\ &= \frac{\pi}{288} [h^4]_2^8 \\ &= \frac{\pi}{288} (8)^4 - \frac{\pi}{288} (2)^4 \\ &= \frac{\pi}{288} [4096 - 16] = \frac{85\pi}{6} \text{ cm}^3\end{aligned}$$

Q10. $v = 9.8t$

$$\begin{aligned}\text{Average velocity} &= \frac{1}{3-0} \int_0^3 (9.8t) dt \\ &= \frac{1}{3} \left[9.8 \frac{t^2}{2} \right]_0^3 \\ &= \frac{1}{3} \left[9.8 \frac{(3)^2}{2} \right] - \frac{1}{3} \left[9.8 \frac{(0)^2}{2} \right] \\ &= \frac{1}{3} \left(\frac{441}{10} \right) - 0 = \frac{147}{10} \text{ m / sec}\end{aligned}$$

Q11. (i) $v = 3t^2 - 4$

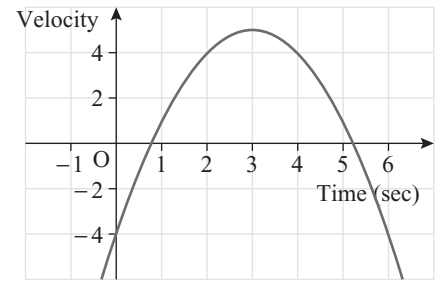
$$\begin{aligned}\text{Average velocity} &= \frac{1}{3-1} \int_1^3 (3t^2 - 4) dt \\ &= \frac{1}{2} \left[\frac{3t^3}{3} - 4t \right]_1^3 \\ &= \frac{1}{2} [t^3 - 4t]_1^3 \\ &= \frac{1}{2} [(3)^3 - 4(3)] - \frac{1}{2} [(1)^3 - 4(1)] \\ &= \frac{1}{2}(15) - \frac{1}{2}(-3) = 9 \text{ m / sec}\end{aligned}$$

(ii) $v = 3t^2 - 4 \Rightarrow a = \frac{dv}{dt} = 6t$

$$\begin{aligned}\text{Average acceleration} &= \frac{1}{3-1} \int_1^3 6t dt \\ &= \frac{1}{2} \left[\frac{6t^2}{2} \right]_1^3 \\ &= \frac{1}{2} [3t^2]_1^3 \\ &= \frac{1}{2} [3(3)^2] - \frac{1}{2} [3(1)^2] \\ &= \frac{27}{2} - \frac{3}{2} = 12 \text{ m / sec}^2\end{aligned}$$

Q12. (i) $v = 5 - (t - 3)^2 = 5 - (t^2 - 6t + 9) = -t^2 + 6t - 4$

$$\begin{aligned}\text{Average velocity} &= \frac{1}{6-0} \int_0^6 (-t^2 + 6t - 4) dt \\ &= \frac{1}{6} \left[\frac{-t^3}{3} + \frac{6t^2}{2} - 4t \right]_0^6 \\ &= \frac{1}{6} \left[\frac{-t^3}{3} + 3t^2 - 4t \right]_0^6 \\ &= \frac{1}{6} \left[\frac{-(6)^3}{3} + 3(6)^2 - 4(6) \right] - \frac{1}{6} \left[\frac{-(0)^3}{3} + 3(0)^2 - 4(0) \right] \\ &= \frac{1}{6} [-72 + 108 - 24] - \frac{1}{6}(0) = 2 \text{ m/sec}\end{aligned}$$



(ii) $\text{velocity} = 2 \Rightarrow -t^2 + 6t - 4 = 2$
 $\Rightarrow t^2 - 6t + 6 = 0$
 $\Rightarrow t = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(6)}}{2(1)}$
 $= \frac{6 \pm \sqrt{36 - 24}}{2} = \frac{6 \pm \sqrt{12}}{2}$
 $= \frac{6 \pm 2\sqrt{3}}{2} = (3 \pm \sqrt{3}) \text{ sec}$

Q13. $T = 30x$

$$\begin{aligned}\text{Average tension} &= \frac{1}{0.2 - 0.1} \int_{0.1}^{0.2} 30x \, dx \\ &= \frac{1}{0.1} \left[30 \frac{x^2}{2} \right]_{0.1}^{0.2} \\ &= 10 \left[15x^2 \right]_{0.1}^{0.2} \\ &= 10 \left[15(0.2)^2 \right] - 10 \left[15(0.1)^2 \right] \\ &= 6 - 1.5 = 4.5 \text{ newtons}\end{aligned}$$

Q14. (i) $y = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}} \Rightarrow \frac{1}{y} = \sqrt{x}$

$$\begin{aligned}\text{Average value} &= \frac{1}{4-1} \int_1^4 x^{\frac{1}{2}} dx \\ &= \frac{1}{3} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 \\ &= \frac{2}{9} \left[x^{\frac{3}{2}} \right]_1^4 \\ &= \frac{2}{9} (4)^{\frac{3}{2}} - \frac{2}{9} (1)^{\frac{3}{2}} \\ &= \frac{16}{9} - \frac{2}{9} = \frac{14}{9}\end{aligned}$$

$$\begin{aligned}
 \text{(ii) Area} &= \int_1^4 x^{-\frac{1}{2}} dx = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 \\
 &= \left[2\sqrt{x} \right]_1^4 = 2(\sqrt{4}) - 2(\sqrt{1}) = 2 \text{ sq. units}
 \end{aligned}$$

Q15. $pv^{\frac{3}{4}} = 30 \Rightarrow p = \frac{30}{p^{\frac{3}{4}}} = 30p^{-\frac{3}{4}}$

$$\begin{aligned}
 \text{Average pressure} &= \frac{1}{16-1} \int_1^{16} 30p^{-\frac{3}{4}} dp \\
 &= \frac{30}{15} \left[\frac{p^{\frac{1}{4}}}{\frac{1}{4}} \right]_1^{16} = 2.4 \left[p^{\frac{1}{4}} \right]_1^{16} \\
 &= 8(16)^{\frac{1}{4}} - 8(1)^{\frac{1}{4}} \\
 &= 16 - 8 = 8
 \end{aligned}$$

Q16. $v = 40 - 10t$

$$\begin{aligned}
 \text{Average velocity} &= \frac{1}{3-1} \int_1^3 (40 - 10t) dt \\
 &= \frac{1}{2} \left[40t - 5t^2 \right]_1^3 \\
 &= \frac{1}{2} [40(3) - 5(3)^2] - \frac{1}{2} [40(1) - 5(1)^2] \\
 &= \frac{1}{2} [120 - 45] - \frac{1}{2} [40 - 5] \\
 &= \frac{75}{2} - \frac{35}{2} = \frac{40}{2} = 20 \text{ m/sec}
 \end{aligned}$$

Revision Exercise (Core)

Q1. (i) $\int (2x + 5) dx = x^2 + 5x + c$

(ii) $\int (3x^2 - 2x + 4) dx = x^3 - x^2 + 4x + c$

(iii) $\int \left(x^2 + \frac{1}{x^2} \right) dx = \int (x^2 + x^{-2}) dx$

$$= \frac{x^3}{3} + \frac{x^{-1}}{-1} + c = \frac{x^3}{3} - \frac{1}{x} + c$$

Q2. (i) $\int \sin 3x dx = -\frac{\cos 3x}{3} + c$

(ii) $\int \cos 5x dx = \frac{\sin 5x}{5} + c$

$$\begin{aligned}
 \text{(iii)} \quad & \int (2 \sin x + 3 \cos 2x) dx \\
 &= -2 \cos x + 3 \frac{\sin 2x}{2} + c \\
 &= -2 \cos x + \frac{3}{2} \sin 2x + c
 \end{aligned}$$

$$\text{Q3. (i)} \quad \int e^{5x} dx = \frac{e^{5x}}{5} + c$$

$$\text{(ii)} \quad \int (e^{2x} + e^{-x}) dx = \frac{e^{2x}}{2} + \frac{e^{-x}}{-1} + c = \frac{e^{2x}}{2} - e^{-x} + c$$

$$\text{(iii)} \quad \int (4 + e^{3x}) dx = 4x + \frac{e^{3x}}{3} + c$$

$$\begin{aligned}
 \text{Q4.} \quad & \frac{dy}{dx} = x^2 - 3x + 2 \\
 & y = \int (x^2 - 3x + 2) dx \\
 &= \frac{x^3}{3} - 3 \frac{x^2}{2} + 2x + c = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Q5. (i)} \quad & \int \frac{x^3 - 2}{x^2} dx = \int \left(\frac{x^3}{x^2} - \frac{2}{x^2} \right) dx \\
 &= \int (x - 2x^{-2}) dx \\
 &= \frac{x^2}{2} - \frac{2x^{-1}}{-1} + c = \frac{x^2}{2} + \frac{2}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \int (\sqrt{x} - 3) dx = \int (x^{\frac{1}{2}} - 3) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3x + c = \frac{2}{3} x^{\frac{3}{2}} - 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad & \int (\sqrt{x} + 3)^2 dx = \int (x + 6\sqrt{x} + 9) dx \\
 &= \int (x + 6x^{\frac{1}{2}} + 9) dx \\
 &= \frac{x^2}{2} + 6 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 9x + c \\
 &= \frac{x^2}{2} + 4x^{\frac{3}{2}} + 9x + c
 \end{aligned}$$

$$\begin{aligned}
\text{Q6. (i)} \quad & \int_0^3 (2x^2 - 4x + 1) dx \\
&= \left[2 \frac{x^3}{3} - 2x^2 + x \right]_0^3 \\
&= \left[\frac{2}{3}(3)^3 - 2(3)^2 + 3 \right] - \left[\frac{2}{3}(0)^3 - 2(0)^2 + 0 \right] \\
&= 18 - 18 + 3 - 0 = 3
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad & \int_0^{\frac{\pi}{4}} \cos 2x \, dx = \left[\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{4}} \\
&= \frac{\sin 2\left(\frac{\pi}{4}\right)}{2} - \frac{\sin 2(0)}{2} \\
&= \frac{\sin \frac{\pi}{2}}{2} - \sin 0 = \frac{1}{2} - 0 = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad & \int_0^{\frac{\pi}{3}} (\cos 3\theta + \sin 3\theta) d\theta \\
&= \left[\frac{\sin 3\theta}{3} - \frac{\cos 3\theta}{3} \right]_0^{\frac{\pi}{3}} \\
&= \left[\frac{\sin 3(\frac{\pi}{3})}{3} - \frac{\cos 3(\frac{\pi}{3})}{3} \right] - \left[\frac{\sin 3(0)}{3} - \frac{\cos 3(0)}{3} \right] \\
&= \left[\frac{\sin \pi}{3} - \frac{\cos \pi}{3} \right] - \left[\frac{\sin 0}{3} - \frac{\cos 0}{3} \right] \\
&= \left[\frac{0}{3} - \frac{(-1)}{3} \right] - \left[\frac{0}{3} - \frac{1}{3} \right] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\text{Q7.} \quad & \int_0^1 \left(x^{\frac{1}{2}} + x^{\frac{3}{2}} \right) dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^1 \\
&= \left[\frac{2}{3} x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}} \right]_0^1 \\
&= \left[\frac{2}{3} (1)^{\frac{3}{2}} + \frac{2}{5} (1)^{\frac{5}{2}} \right] - \left[\frac{2}{3} (0)^{\frac{3}{2}} + \frac{2}{5} (0)^{\frac{5}{2}} \right] \\
&= \frac{2}{3} + \frac{2}{5} - 0 - 0 = \frac{16}{15}
\end{aligned}$$

$$\begin{aligned}
\text{Q8. (i)} \quad & \int_0^3 (e^{2x} + 1) dx = \left[\frac{e^{2x}}{2} + x \right]_0^3 = \left[\frac{e^{2(3)}}{2} + 3 \right] - \left[\frac{e^{2(0)}}{2} + 0 \right] \\
&= \frac{e^6}{2} + 3 - \frac{1}{2} - 0 = \frac{1}{2} e^6 + 2\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int_0^2 2e^{-2x} dx &= \left[\frac{2e^{-2x}}{-2} \right]_0^2 = \left[\frac{-1}{e^{2x}} \right]_0^2 \\
 &= \left[-\frac{1}{e^{2(2)}} \right] - \left[\frac{-1}{e^{2(0)}} \right] = \frac{-1}{e^4} + 1
 \end{aligned}$$

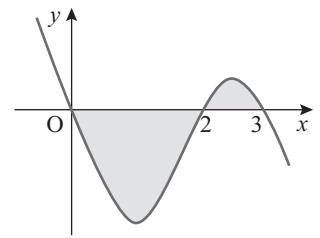
$$\begin{aligned}
 \text{(iii)} \quad \int_1^2 \left(e^{2x} + \frac{4}{x^2} \right) dx &= \int_1^2 (e^{2x} + 4x^{-2}) dx \\
 &= \left[\frac{e^{2x}}{2} + \frac{4x^{-1}}{-1} \right]_1^2 = \left[\frac{e^{2x}}{2} - \frac{4}{x} \right]_1^2 \\
 &= \left[\frac{e^{2(2)}}{2} - \frac{4}{2} \right] - \left[\frac{e^{2(1)}}{2} - \frac{4}{1} \right] \\
 &= \frac{1}{2}e^4 - 2 - \frac{1}{2}e^2 + 4 \\
 &= \frac{1}{2}e^4 - \frac{1}{2}e^2 + 2
 \end{aligned}$$

Q9.

$$y = x(2-x)(x-3) = -x^3 + 5x^2 - 6x$$

$$\text{On } x\text{-axis} \Rightarrow y = 0 \Rightarrow x(2-x)(x-3) = 0$$

$$\Rightarrow x = 0, x = 2 \text{ or } x = 3$$



$$\begin{aligned}
 \text{Shaded Area} &= \int_0^2 (-x^3 + 5x^2 - 6x) dx + \int_2^3 (-x^3 + 5x^2 - 6x) dx \\
 &= \left[-\frac{x^4}{4} + 5\frac{x^3}{3} - 3x^2 \right]_0^2 + \left[-\frac{x^4}{4} + \frac{5x^3}{3} - 3x^2 \right]_2^3 \\
 &= \left[\left[-\frac{(2)^4}{4} + \frac{5(2)^3}{3} - 3(2)^2 \right] - \left[-\frac{(0)^4}{4} + \frac{5(0)^3}{3} - 3(0)^2 \right] \right] \\
 &\quad + \left[\left[-\frac{(3)^4}{4} + \frac{5(3)^3}{3} - 3(3)^2 \right] - \left[-\frac{(2)^4}{4} + \frac{5(2)^3}{3} - 3(2)^2 \right] \right] \\
 &= \left[\left(-4 + \frac{40}{3} - 12 \right) - (0 + 0 + 0) \right] + \left[\left(-\frac{81}{4} + 45 - 27 \right) - \left(-4 + \frac{40}{3} - 12 \right) \right] \\
 &= \left[-2\frac{2}{3} - 0 \right] + \left[-2\frac{1}{4} + 2\frac{2}{3} \right]
 \end{aligned}$$

$$\text{Area} = 2\frac{2}{3} + \frac{5}{12} = 3\frac{1}{12} \text{ sq. units}$$

Q10.

$$\frac{dy}{dx} = 15x^2 - 12x$$

$$\Rightarrow y = \int (15x^2 - 12x) dx$$

$$= 15\frac{x^3}{3} - 12\frac{x^2}{2} + c = 5x^3 - 6x^2 + c$$

$$\text{Point } (1,3) \Rightarrow 5(1)^3 - 6(1)^2 + c = 3$$

$$\Rightarrow 5 - 6 + c = 3 \Rightarrow c = 4$$

$$\text{Hence } f(x) = y = 5x^3 - 6x^2 + 4$$

Q11. $f(x) = 2x^2 - x$

$$\begin{aligned}\text{Average value} &= \frac{1}{4-0} \int_0^4 (2x^2 - x) dx \\ &= \frac{1}{4} \left[2 \frac{x^3}{3} - \frac{x^2}{2} \right]_0^4 \\ &= \frac{1}{4} \left[2 \frac{(4)^3}{3} - \frac{(4)^2}{2} \right] - \frac{1}{4} \left[2 \frac{(0)^3}{3} - \frac{(0)^2}{2} \right] \\ &= \frac{1}{4} \left(\frac{128}{3} - 8 \right) - \frac{1}{4} (0) = 8 \frac{2}{3}\end{aligned}$$

Q12. $\frac{dy}{dx} = e^{2x} - x$

$$y = \int (e^{2x} - x) dx = \frac{e^{2x}}{2} - \frac{x^2}{2} + c$$

$$y = 5 \text{ when } x = 0 \Rightarrow \frac{e^{2(0)}}{2} - \frac{(0)^2}{2} + c = 5$$

$$\Rightarrow \frac{1}{2} - 0 + c = 5 \Rightarrow c = 4 \frac{1}{2}$$

$$\text{Hence } y = \frac{1}{2} e^{2x} - \frac{x^2}{2} + 4 \frac{1}{2}$$

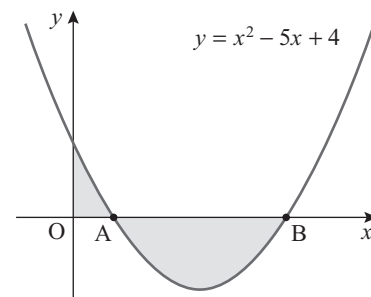
Q13. (i) $y = x^2 - 5x + 4$

$$\text{On } x \text{ axis, } y = 0 \Rightarrow x^2 - 5x + 4 = 0$$

$$\Rightarrow (x-1)(x-4) = 0$$

$$\Rightarrow x = 1 \quad \text{OR} \quad x = 4$$

$$\Rightarrow A = (1, 0) \quad \text{and} \quad B = (4, 0)$$



(ii) Shaded Area = $\int_0^1 (x^2 - 5x + 4) dx + \int_1^4 (x^2 - 5x + 4) dx$

$$\begin{aligned}&= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x \right]_0^1 + \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x \right]_1^4 \\ &= \left[\left[\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 4(1) \right] - \left[\frac{(0)^3}{3} - \frac{5(0)^2}{2} + 4(0) \right] \right] \\ &\quad + \left[\left[\frac{(4)^3}{3} - \frac{5(4)^2}{2} + 4(4) \right] - \left[\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 4(1) \right] \right] \\ &= \left[\left(\frac{1}{3} - \frac{5}{2} + 4 \right) - (0 - 0 + 0) \right] + \left[\left(\frac{64}{3} - 40 + 16 \right) - \left(\frac{1}{3} - \frac{5}{2} + 4 \right) \right] \\ &= 1 \frac{5}{6} + \left(-4 \frac{1}{2} \right)\end{aligned}$$

$$\text{Area} = 1 \frac{5}{6} + 4 \frac{1}{2} = 6 \frac{1}{3} \text{ sq. units}$$

Q14. (i) $v = 6t + 12t^2$

$$\begin{aligned}\text{Average velocity} &= \frac{1}{2-0} \int_0^2 (6t + 12t^2) dt \\ &= \frac{1}{2} \left[3t^2 + 4t^3 \right]_0^2 \\ &= \frac{1}{2} \left[3(2)^2 + 4(2)^3 \right] - \frac{1}{2} \left[3(0)^2 + 4(0)^3 \right] \\ &= \frac{1}{2} [12 + 32] - \frac{1}{2} (0) = 22 \text{ m/sec}\end{aligned}$$

(ii) $v = 6t + 12t^2 \Rightarrow a = \frac{dv}{dt} = 6 + 24t$

$$\begin{aligned}\text{Average acceleration} &= \frac{1}{5-1} \int_1^5 (6 + 24t) dt \\ &= \frac{1}{4} \left[6t + 12t^2 \right]_1^5 \\ &= \frac{1}{4} \left[6(5) + 12(5)^2 \right] - \frac{1}{4} \left[6(1) + 12(1)^2 \right] \\ &= \frac{1}{4} (330) - \frac{1}{4} (18) = 78 \text{ m/sec}^2\end{aligned}$$

Q15. $f(x) = x \cdot \sin 2x \Rightarrow$ Product Rule: $u = x$ and $v = \sin 2x$

$$\Rightarrow \frac{du}{dx} = 1 \quad \Rightarrow \frac{dv}{dx} = 2 \cos 2x$$

$$\begin{aligned}\frac{dy}{dx} = f'(x) &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= x \cdot 2 \cos 2x + \sin 2x \cdot 1 = 2x \cos 2x + \sin 2x\end{aligned}$$

$$\begin{aligned}\int (2x \cos 2x + \sin 2x) dx &= x \sin 2x + c \\ &= \int 2x \cos 2x dx + \int \sin 2x = x \sin 2x + c \\ \Rightarrow \int 2x \cos 2x dx - \frac{\cos 2x}{2} &= x \sin 2x + c \\ \Rightarrow \int 2x \cos 2x dx &= x \sin 2x + \frac{\cos 2x}{2} + c\end{aligned}$$

Revision Exercise 4 (Advanced)

Q1. (i) $\int_1^3 (9x^2 - 4x) dx = \left[\frac{9x^3}{3} - \frac{4x^2}{2} \right]_1^3 = \left[3x^3 - 2x^2 \right]_1^3$

$$\begin{aligned}&= \left[3(3)^3 - 2(3)^2 \right] - \left[3(1)^3 - 2(1)^2 \right] \\ &= (81 - 18) - (3 - 2) = 63 - 1 = 62\end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \int_0^a (9x^2 - 4x) dx = 0 \\
 & \Rightarrow [3x^3 - 2x^2]_0^a = 0 \\
 & \Rightarrow (3a^3 - 2a^2) - [3(0)^3 - 2(0)^2] = 0 \\
 & \Rightarrow 3a^3 - 2a^2 = 0 \Rightarrow a^2(3a - 2) = 0 \\
 & \Rightarrow a = 0 \quad \text{OR} \quad a = \frac{2}{3} \\
 & \text{Since } a > 0 \Rightarrow a = \frac{2}{3}
 \end{aligned}$$

Q2.

$$f(x) = (x+3)(2x-5) = 2x^2 + x - 15$$

$$\begin{aligned}
 \text{Average value} &= \frac{1}{5-1} \int_1^5 (2x^2 + x - 15) dx \\
 &= \frac{1}{4} \left[\frac{2x^3}{3} + \frac{x^2}{2} - 15x \right]_1^5 \\
 &= \frac{1}{4} \left[2 \frac{(5)^3}{3} + \frac{(5)^2}{2} - 15(5) \right] - \frac{1}{4} \left[2 \frac{(1)^3}{3} + \frac{(1)^2}{2} - 15(1) \right] \\
 &= \frac{1}{4} \left(20 \frac{5}{6} \right) - \frac{1}{4} \left(-13 \frac{5}{6} \right) = 8 \frac{2}{3}
 \end{aligned}$$

Q3. (i)

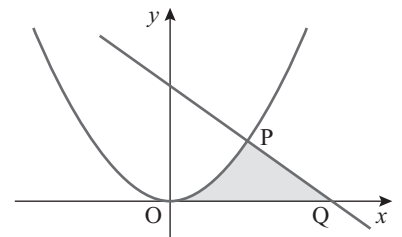
$$\begin{aligned}
 y = x^2 \cap y = -2x + 15 \\
 \Rightarrow x^2 = -2x + 15 \Rightarrow x^2 + 2x - 15 = 0 \\
 \Rightarrow (x+5)(x-3) = 0 \\
 \Rightarrow x = -5 \quad \text{OR} \quad x = 3
 \end{aligned}$$

$$\text{When } x = 3 \Rightarrow y = (3)^2 = 9 \Rightarrow P = (3, 9)$$

$$\text{LINE : } 2x + y = 15$$

$$\text{On } x\text{-axis, } y = 0 \Rightarrow 2x + 0 = 15 \Rightarrow x = 7 \frac{1}{2}$$

$$\Rightarrow Q = \left(7 \frac{1}{2}, 0 \right)$$



(ii) Shaded Area = Area under curve OP + Area under line PQ

$$\begin{aligned}
 &= \int_0^3 x^2 dx + \int_3^{7\frac{1}{2}} (-2x + 15) dx \\
 &= \left[\frac{x^3}{3} \right]_0^3 + \left[-x^2 + 15x \right]_3^{7\frac{1}{2}} \\
 &= \left[\frac{(3)^3}{3} - \frac{(0)^3}{3} \right] + \left[-\left(7 \frac{1}{2} \right)^2 + 15 \left(7 \frac{1}{2} \right) \right] - \left[-(3)^2 + 15(3) \right] \\
 &= 9 - 0 + \left[-56 \frac{1}{4} + 112 \frac{1}{2} \right] - [-9 + 45] \\
 &= 9 + 56 \frac{1}{4} - 36 = 29.25 \text{ sq. units}
 \end{aligned}$$

Q4.

$$V = \frac{1}{3}\pi(30h^2 - h^3)$$

$$\begin{aligned}\text{Average value} &= \frac{1}{4-0} \int_0^4 \frac{1}{3}\pi(30h^2 - h^3)dh \\ &= \frac{\pi}{12} \left[30 \frac{h^3}{3} - \frac{h^4}{4} \right]_0^4 = \frac{\pi}{12} \left[10h^3 - \frac{h^4}{4} \right]_0^4 \\ &= \frac{\pi}{12} \left[10(4)^3 - \frac{(4)^4}{4} \right] - \frac{\pi}{12} \left[10(0)^3 - \frac{(0)^4}{4} \right] \\ &= \frac{\pi}{12} [640 - 64] = 48\pi \text{ cm}^3\end{aligned}$$

Q5.

$$y = x^3 - 3x^2 + 5$$

$$\frac{dy}{dx} = 3x^2 - 6x = 0 \Rightarrow x^2 - 2x = 0$$

$$\Rightarrow x(x-2) = 0 \Rightarrow x = 0, x = 2$$

$$x = 0 \Rightarrow y = (0)^3 - 3(0)^2 + 5 = 5 \Rightarrow A = (0, 5)$$

$$\text{Line } \ell : y = 5 \text{ meets curve } \Rightarrow x^3 - 3x^2 + 5 = 5$$

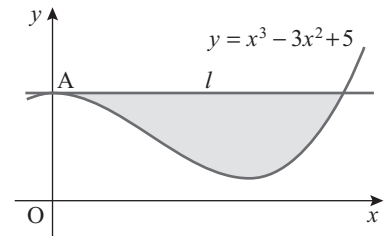
$$\Rightarrow x^3 - 3x^2 = 0$$

$$\Rightarrow x^2(x-3) = 0 \Rightarrow x = 0, x = 3$$

$$2^{\text{nd}} \text{ point} = (3, 5)$$

$$\text{Shared Area} = \text{Area under line } \ell - \text{Area under curve}$$

$$\begin{aligned}&= \int_0^3 5dx - \int_0^3 (x^3 - 3x^2 + 5)dx \\ &= [5x]_0^3 - \left[\frac{x^4}{4} - x^3 + 5x \right]_0^3 \\ &= (15 - 0) - \left[\left[\frac{(3)^4}{4} - (3)^3 + 5(3) \right] - \left[\frac{(0)^4}{4} - (0)^3 + 5(0) \right] \right] \\ &= 15 - 8\frac{1}{4} = 6\frac{3}{4} = \frac{27}{4} \text{ sq. units}\end{aligned}$$



Q6.

$$\frac{dy}{dx} = ae^{-x} + 1$$

$$\frac{dy}{dx} = 3 \text{ when } x = 0 \Rightarrow ae^{-0} + 1 = 3$$

$$\Rightarrow a \cdot 1 = 2 \Rightarrow a = 2$$

$$\frac{dy}{dx} = 2e^{-x} + 1$$

$$y = \int (2e^{-x} + 1)dx$$

$$= 2 \frac{e^{-x}}{-1} + x + c = \frac{-2}{e^x} + x + c$$

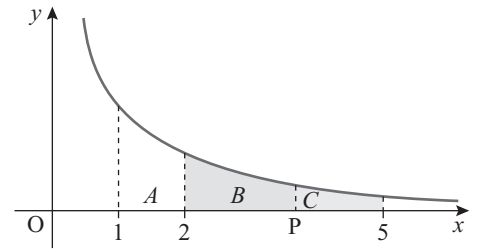
$$y = 5 \text{ when } x = 0 \Rightarrow \frac{-2}{e^0} + 0 + c = 5 \Rightarrow c = 7$$

$$\Rightarrow y = \frac{-2}{e^x} + x + 7$$

$$\text{When } x = 2 \Rightarrow y = \frac{-2}{e^2} + 2 + 7 = 9 - 2e^{-2}$$

Q7. (i) $y = \frac{10}{x^2} = 10x^{-2}$

$$\begin{aligned}\text{Area A} &= \int_1^2 10x^{-2} dx = \left[10 \frac{x^{-1}}{-1} \right]_1^2 = \left[\frac{-10}{x} \right]_1^2 \\ &= \left(-\frac{10}{2} \right) - \left(-\frac{10}{1} \right) = -5 + 10 = 5 \text{ sq. units}\end{aligned}$$

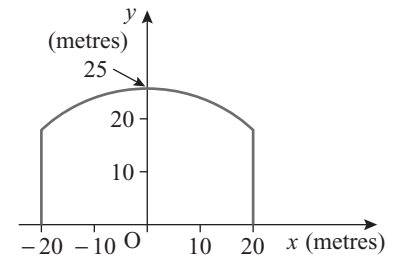


(ii) Area B = Area C

$$\begin{aligned}\Rightarrow \int_2^P 10x^{-2} dx &= \int_P^5 10x^{-2} dx \\ &= \left[\frac{-10}{x} \right]_2^P = \left[\frac{-10}{x} \right]_P^5 \\ &= \left(-\frac{10}{P} \right) - \left(-\frac{10}{2} \right) = \left(-\frac{10}{5} \right) - \left(-\frac{10}{P} \right) \\ &= -\frac{10}{P} + 5 = -2 + \frac{10}{P} \\ \Rightarrow \frac{-20}{P} &= -7 \Rightarrow 7P = 20 \Rightarrow P = \frac{20}{7}\end{aligned}$$

Q8. $\text{Area} = \int_{-20}^{20} (25 - 0.02x^2) dx$

$$\begin{aligned}&= \left[25x - 0.02 \frac{x^3}{3} \right]_{-20}^{20} \\ &= \left[25(20) - 0.02 \frac{(20)^3}{3} \right] - \left[25(-20) - 0.02 \frac{(-20)^3}{3} \right] \\ &= \left(500 - \frac{160}{3} \right) - \left(-500 + \frac{160}{3} \right) = 893 \frac{1}{3} \text{ m}^2\end{aligned}$$



$$\text{Volume} = 893 \frac{1}{3} \times 80 = 71,466 \frac{2}{3} \text{ m}^3$$

Q9. (i) $f(x) = x^2 \cdot \ln 3x \Rightarrow$ product rule: $u = x^2$ and $v = \ln 3x$

$$\Rightarrow \frac{du}{dx} = 2x \quad \Rightarrow \frac{dv}{dx} = \frac{1}{3x} \cdot 3 = \frac{1}{x}$$

$$\frac{dy}{dx} = f'(x) = u \frac{dv}{dx} + v \frac{du}{dx} = x^2 \cdot \frac{1}{x} + \ln 3x \cdot 2x = 2x \ln 3x + x$$

(ii) $\int (2x \ln 3x + x) dx = x^2 \ln 3x$

$$\Rightarrow \int 2x \ln 3x + \int x dx = x^2 \ln 3x + c$$

$$\Rightarrow \int 2x \ln 3x + \frac{x^2}{2} = x^2 \ln 3x + c$$

$$\Rightarrow \int 2x \ln 3x dx = x^2 \ln 3x - \frac{x^2}{2} + c$$

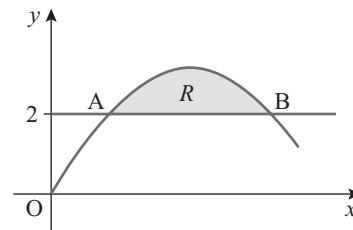
Q10. (i) $y = 2 \cap y = -2x^2 + 5x$

$$\Rightarrow 2 = -2x^2 + 5x$$

$$\Rightarrow 2x^2 - 5x + 2 = 0$$

$$\Rightarrow (2x-1)(x-2) = 0 \Rightarrow x = \frac{1}{2} \text{ OR } x = 2$$

Hence $A = \left(\frac{1}{2}, 2\right)$ and $B = (2, 2)$



(ii) Shaded area $R = \int_{\frac{1}{2}}^2 (-2x^2 + 5x) dx - \int_{\frac{1}{2}}^2 2 dx$

$$= \left[\frac{-2x^3}{3} + \frac{5x^2}{2} \right]_{\frac{1}{2}}^2 - [2x]_{\frac{1}{2}}^2$$

$$= \left[\left(\frac{-2(2)^3}{3} + \frac{5(2)^2}{2} \right) - \left(\frac{-2\left(\frac{1}{2}\right)^3}{3} + \frac{5\left(\frac{1}{2}\right)^2}{2} \right) \right] - \left[2(2) - 2\left(\frac{1}{2}\right) \right]$$

$$= \left(\frac{-16}{3} + 10 \right) - \left(-\frac{1}{12} + \frac{5}{8} \right) - 3 = \frac{9}{8} \text{ sq. units}$$

Q11. $k = 5v^2$

Average kinetic energy $= \frac{1}{7-1} \int_1^7 5v^2 dv$

$$= \frac{1}{6} \left[\frac{5v^3}{3} \right]_1^7 = \frac{1}{6} \left[5 \frac{(7)^3}{3} \right] - \frac{1}{6} \left[\frac{5(1)^3}{3} \right]$$

$$= \frac{1}{6} \left(\frac{1715}{3} \right) - \frac{1}{6} \left(\frac{5}{3} \right) = 95 \text{ joules}$$

Q12. (i) $a = 6t + 10$

$$v = \int (6t + 10) dt = 3t^2 + 10t$$

When $t = 5 \Rightarrow v = 3(5)^2 + 10(5) = 125 \text{ m/sec}$

(ii) $s = \int (3t^2 + 10t) dt$

$$= \frac{3t^3}{3} + \frac{10t^2}{2} + c = t^3 + 5t^2 + c$$

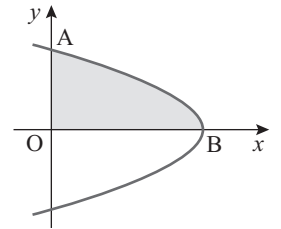
$s = 3$ when $t = 0 \Rightarrow (0)^3 + 5(0)^2 + c = 3 \Rightarrow c = 3$

$$\Rightarrow s = t^3 + 5t^2 + 3$$

(iii) $t = 3 \Rightarrow s = (3)^3 + 5(3)^2 + 3 = 75 \text{ m}$

$$\begin{aligned}
 \text{(iv) Average speed} &= \frac{1}{4-1} \int_1^4 (3t^2 + 10t) dt \\
 &= \frac{1}{3} \left[\frac{3t^3}{3} + \frac{10t^2}{2} \right]_1^4 = \frac{1}{3} [t^3 + 5t^2]_1^4 \\
 &= \frac{1}{3} [(4)^3 + 5(4)^2] - \frac{1}{3} [(1)^3 + 5(1)^2] \\
 &= \frac{1}{3} (64 + 80) - \frac{1}{3} (1 + 5) = 46 \text{ m/sec.}
 \end{aligned}$$

Q13. (i) $y^2 = 9(1-x) \Rightarrow y = \sqrt{9(1-x)} = 3\sqrt{1-x}$
 At A, $x = 0 \Rightarrow y = 3\sqrt{1-0} = 3 \Rightarrow A = (0, 3)$
 At B, $y = 0 \Rightarrow 3\sqrt{1-x} = 0 \Rightarrow x = 1 \Rightarrow B = (1, 0)$



(ii) $\int_0^3 \left(1 - \frac{y^2}{9}\right) dy = \left[y - \frac{1}{9} \left(\frac{y^3}{3} \right) \right]_0^3 = \left[y - \frac{y^3}{27} \right]_0^3$
 $= \left[3 - \frac{(3)^3}{27} \right] - \left[0 - \frac{(0)^3}{27} \right] = 3 - 1 = 2 \text{ sq. units}$

Q14. (i) $f'(x) = k(x-a)(x-b)$

(a) $a = 2, b = 4$

(b) $f'(x) = k(x-2)(x-4)$

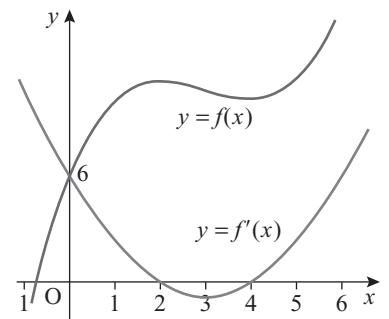
Point (0, 6) $\Rightarrow f'(0) = k(0-2)(0-4) = 6 \Rightarrow 8k = 6 \Rightarrow k = \frac{3}{4}$

(ii) $f'(x) = \frac{3}{4}(x-2)(x-4) = \frac{3}{4}(x^2 - 6x + 8)$

$$\begin{aligned}
 y = f(x) &= \frac{3}{4} \int (x^2 - 6x + 8) dx \\
 &= \frac{3}{4} \left[\frac{x^3}{3} - 3x^2 + 8x + c \right] = \frac{x^3}{4} - \frac{9}{4}x^2 + 6x + \frac{3}{4}c
 \end{aligned}$$

Point (0, 6) $\Rightarrow \frac{3}{4} \left[\frac{(0)^3}{3} - 3(0)^2 + 8(0) + c \right] = 6$
 $\Rightarrow 0 - 0 + 0 + c = 6 \cdot \frac{4}{3} = 8 \Rightarrow c = 8$

Hence $y = \frac{1}{4}x^3 - \frac{9}{4}x^2 + 6x + 6$



Q15.

$$\frac{dy}{dx} = ax + \frac{b}{x^2} = ax + bx^{-2}$$

$$\text{Turning point } (1, 0) \Rightarrow \frac{dy}{dx} = a(1) + \frac{b}{(1)^2} = 0 \Rightarrow a + b = 0$$

$$y = \int (ax + bx^{-2}) dx = a \frac{x^2}{2} + b \frac{x^{-1}}{-1} + c$$

$$= a \frac{x^2}{2} - \frac{b}{x} + c$$

$$\text{Point } (-1, -4) \Rightarrow a \frac{(-1)^2}{2} - \frac{b}{(-1)} + c = -4$$

$$\Rightarrow \frac{a}{2} + b + c = -4$$

$$\text{Point } (1, 0) \Rightarrow a \frac{(1)^2}{2} - \frac{b}{1} + c = 0$$

$$\Rightarrow \frac{a}{2} - b + c = 0$$

$$\text{and} \quad -\frac{a}{2} - b - c = 4$$

$$\text{add } \Rightarrow \quad \frac{-2b}{-2b} = 4 \quad \Rightarrow b = -2$$

$$\text{and} \quad a - 2 = 0 \Rightarrow a = 2$$

$$\frac{a}{2} - b + c = 0 \Rightarrow \frac{2}{2} - (-2) + c = 0$$

$$\Rightarrow 1 + 2 + c = 0 \Rightarrow c = -3$$

$$\text{Hence } y = \frac{2x^2}{2} - \frac{-2}{x} - 3 = x^2 + \frac{2}{x} - 3$$

Revision Exercise 4 (Extended-Response)

$$\text{Q1. (a)} \quad y = x - \frac{1}{x^2} = x - x^{-2}$$

$$\text{At } A, x = 2 \Rightarrow y = 2 - \frac{1}{2^2} = \frac{7}{4} \Rightarrow A = \left(2, \frac{7}{4}\right)$$

$$\frac{dy}{dx} = 1 - (-2x^{-3}) = 1 + \frac{2}{x^3}$$

$$\text{When } x = 2 \Rightarrow \text{slope} = \frac{dy}{dx} = 1 + \frac{2}{(2)^3} = 1 + \frac{1}{4} = \frac{5}{4}$$

$$\Rightarrow \text{Equation of Tangent: } y - y_1 = m(x - x_1)$$

$$\Rightarrow y - \frac{7}{4} = \frac{5}{4}(x - 2)$$

$$\Rightarrow 4y - 7 = 5x - 10$$

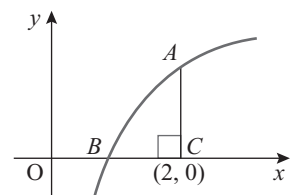
$$\Rightarrow 5x - 4y - 3 = 0$$

$$\text{(b)} \quad \text{On } x\text{-axis} \Rightarrow y = 0 \Rightarrow 5x - 4(0) - 3 = 0$$

$$\Rightarrow 5x = 3$$

$$\Rightarrow x = \frac{3}{5}$$

$$\Rightarrow \text{point } T = \left(\frac{3}{5}, 0\right)$$



$$(c) \quad y = x - \frac{1}{x^2} = \frac{x^3 - 1}{x^2}$$

$$\text{On } x\text{-axis, } y = 0 \Rightarrow \frac{x^3 - 1}{x^2} = 0 \Rightarrow x^3 - 1 = 0$$

$$\Rightarrow x^3 = 1 \Rightarrow x = \sqrt[3]{1} = 1$$

$$\Rightarrow B = (1, 0)$$

(d) Required Area = Area under line TA – area under curve BA

$$\text{Area under line TA} \Rightarrow 5x - 4y - 3 = 0$$

$$4y = 5x - 3$$

$$\Rightarrow y = \frac{1}{4}(5x - 3)$$

$$\text{Area} = \frac{1}{4} \int_{\frac{3}{5}}^2 (5x - 3) dx$$

$$= \frac{1}{4} \left[\frac{5x^2}{2} - 3x \right]_{\frac{3}{5}}^2$$

$$= \frac{1}{4} \left[\frac{5(2)^2}{2} - 3(2) \right] - \frac{1}{4} \left[5 \left(\frac{3}{5} \right)^2 - 3 \left(\frac{3}{5} \right) \right]$$

$$= \frac{1}{4} [10 - 6] - \frac{1}{4} \left[\frac{9}{10} - \frac{9}{5} \right]$$

$$= 1 - \frac{1}{4} \left(-\frac{9}{10} \right) = 1 \frac{9}{40}$$

$$\text{Area under curve BA} = \int_1^2 (x - x^{-2}) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^{-1}}{-1} \right]_1^2 = \left[\frac{x^2}{2} + \frac{1}{x} \right]_1^2$$

$$= \left[\frac{2^2}{2} + \frac{1}{2} \right] - \left[\frac{(1)^2}{2} + \frac{1}{1} \right]$$

$$= 2 \frac{1}{2} - 1 \frac{1}{2} = 1$$

$$\text{Hence required area} = 1 \frac{9}{40} - 1 = \frac{9}{40} \text{ sq. units}$$

$$(e) \quad \begin{array}{ccc} \text{Triangle ATC:} & C(2,0) & A\left(2, \frac{7}{4}\right) & T\left(\frac{3}{5}, 0\right) \\ & \downarrow & \downarrow & \\ & (0,0) & \left(0, \frac{7}{4}\right) & \left(-\frac{7}{5}, 0\right) \\ & & x_1 \ y_1 & x_2 \ y_2 \end{array}$$

$$\text{Area Triangle ATC} = \frac{1}{2} |x_1 y_2 - x_2 y_1|$$

$$= \frac{1}{2} \left| (0)(0) - \left(-\frac{7}{5}\right) \left(\frac{7}{4}\right) \right| = \frac{49}{40}$$

$$\Rightarrow \text{Ratio} = \frac{9}{40} : \frac{49}{40} = 9 : 49$$

Q2. (a)

$$y = 2x + \frac{8}{x^2} - 5 = 2x + 8x^{-2} - 5$$

$$\text{At P, } x = 1 \Rightarrow y = 2(1) + \frac{8}{(1)^2} - 5 = 2 + 8 - 5 = 5 \Rightarrow P = (1, 5)$$

$$\text{At Q, } x = 4 \Rightarrow y = 2(4) + \frac{8}{(4)^2} - 5 = 8 + \frac{1}{2} - 5 = 3\frac{1}{2} \Rightarrow Q = \left(4, 3\frac{1}{2}\right)$$

$$\text{Slope PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3\frac{1}{2} - 5}{4 - 1} = \frac{-1\frac{1}{2}}{3} = -\frac{1}{2}$$

$$\text{Equation PQ: } y - y_1 = m(x - x_1)$$

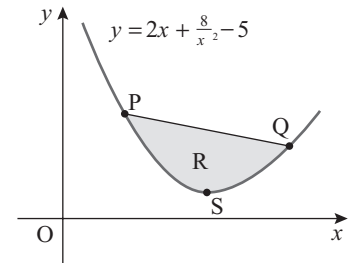
$$\Rightarrow y - 5 = \frac{-1}{2}(x - 1)$$

$$\Rightarrow 2y - 10 = -x + 1$$

$$\Rightarrow x + 2y - 11 = 0$$

$$2y = -x + 11$$

$$\Rightarrow y = \frac{1}{2}(-x + 11)$$



Shaded Area R = Area under line PQ – Area under curve PQ

$$\text{Area under line PQ} = \frac{1}{2} \int_1^4 (-x + 11) dx$$

$$= \frac{1}{2} \left[\frac{-x^2}{2} + 11x \right]_1^4$$

$$= \frac{1}{2} \left[\frac{-(4)^2}{2} + 11(4) \right] - \frac{1}{2} \left[\frac{-(1)^2}{2} + 11(1) \right]$$

$$= \frac{1}{2} [-8 + 44] - \frac{1}{2} \left[-\frac{1}{2} + 11 \right]$$

$$= \frac{1}{2} [36] - \frac{1}{2} \left[10\frac{1}{2} \right]$$

$$= 18 - 5\frac{1}{4}$$

$$= 12\frac{3}{4}$$

$$\text{Area under curve PQ} = \int_1^4 (2x + 8x^{-2} - 5) dx$$

$$= \left[\frac{2x^2}{2} + 8 \frac{x^{-1}}{-1} - 5x \right]_1^4$$

$$= \left[x^2 - \frac{8}{x} - 5x \right]_1^4$$

$$= \left[(4)^2 - \frac{8}{4} - 5(4) \right] - \left[(1)^2 - \frac{8}{1} - 5(1) \right]$$

$$= [16 - 2 - 20] - [1 - 8 - 5]$$

$$= -6 + 12 = 6$$

$$\text{Hence required area} = 12\frac{3}{4} - 6 = 6\frac{3}{4} \text{ sq. units}$$

(b) $y = 2x + 8x^{-2} - 5$

$$\Rightarrow \frac{dy}{dx} = 2 - 16x^{-3} = 2 - \frac{16}{x^3}$$

$$\text{When } x > 2 \Rightarrow 2 - \frac{16}{x^3} > 0$$

Since $\frac{dy}{dx} > 0$, y is increasing for $x > 2$.

(c) $\frac{dy}{dx} = 2 - \frac{16}{x^3}$

$$\text{Turning point} \Rightarrow \frac{dy}{dx} = 0 \Rightarrow 2 - \frac{16}{x^3} = 0$$

$$\Rightarrow 2x^3 - 16 = 0$$

$$\Rightarrow x^3 - 8 = 0$$

$$\Rightarrow x^3 = 8$$

$$\Rightarrow x = \sqrt[3]{8} = 2$$

$$x = 2 \Rightarrow y = 2(2) + \frac{8}{(2)^2} - 5$$

$$= 4 + \frac{8}{4} - 5 = 1$$

\Rightarrow Turning point $S = (2, 1)$

(d) Required area = area under line $y = 5$ – Area under line $y = 1$
between $x = 0$ and $x = 1$ + area under curve PS – area under
line $y = 1$ between $x = 1$ and $x = 2$

$$\text{Area} = \int_0^1 5dx - \int_0^1 1dx$$

$$= [5x]_0^1 - [x]_0^1$$

$$= [5(1) - 5(0)] - [1 - 0] = 5 - 1 = 4$$

$$\text{Area} = \int_1^2 (2x + 8x^{-2} - 5)dx - \int_1^2 1dx$$

$$= \left[x^2 - \frac{8}{x} - 5x \right]_1^2 - [x]_1^2$$

$$= \left[\left[(2)^2 - \frac{8}{2} - 5(2) \right] - \left[(1)^2 - \frac{8}{1} - 5(1) \right] \right] - [(2) - (1)]$$

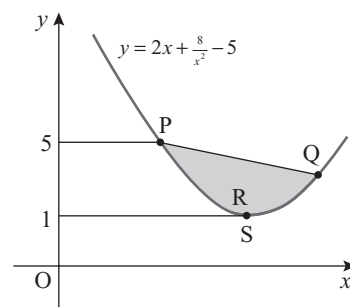
$$= [(4 - 4 - 10) - (1 - 8 - 5)] - 1$$

$$= [-10 + 12] - 1$$

$$= 2 - 1 = 1$$

Hence required area = 4 + 1

= 5 sq.units



Q3. (a)

$$y = \cos x \cap y = \sin 2x$$

$$\Rightarrow \sin 2x = \cos x$$

$$\Rightarrow 2 \sin x \cos x - \cos x = 0$$

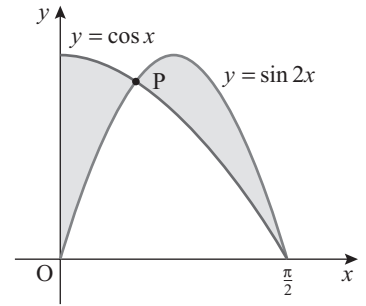
$$\Rightarrow \cos x(2 \sin x - 1) = 0$$

$$\Rightarrow \cos x = 0 \quad \text{OR} \quad 2 \sin x = 1$$

$$\Rightarrow x = \cos^{-1} 0 \quad \Rightarrow \sin x = \frac{1}{2}$$

$$\Rightarrow x = \frac{\pi}{2} \text{ (already on graph)} \quad \Rightarrow x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\text{Answer} = \frac{\pi}{6}$$



(b) Required area = area under curve $y = \cos x$ – area

under curve $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{6}$

+ area under curve $y = \sin 2x$ – area under

curve $y = \cos x$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$

$$\text{Area} = \int_0^{\frac{\pi}{6}} \cos x \, dx - \int_0^{\frac{\pi}{6}} \sin 2x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{6}} - \left[-\frac{\cos 2x}{2} \right]_0^{\frac{\pi}{6}}$$

$$= \left[\sin \frac{\pi}{6} - \sin 0 \right] - \left[\left(-\frac{\cos \frac{2\pi}{6}}{2} \right) - \left(-\frac{\cos 2(0)}{2} \right) \right]$$

$$= \left[\frac{1}{2} - 0 \right] - \left[\frac{-\frac{1}{2}}{2} - \left(\frac{-1}{2} \right) \right]$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x \, dx - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x \, dx$$

$$= \left[-\frac{\cos 2x}{2} \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[\left(-\frac{\cos 2(\frac{\pi}{2})}{2} \right) - \left(-\frac{\cos \frac{2\pi}{6}}{2} \right) \right] - \left[\sin \left(\frac{\pi}{2} \right) - \sin \left(\frac{\pi}{6} \right) \right]$$

$$= \left[-\left(\frac{-1}{2} \right) - \left(\frac{-\frac{1}{2}}{2} \right) \right] - \left(1 - \frac{1}{2} \right)$$

$$= \left(\frac{1}{2} + \frac{1}{4} \right) - \left(\frac{1}{2} \right) = \frac{1}{4}$$

Hence required area = $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ square unit

Q4. (a) $\frac{dV}{dt} = 120 + 26t - t^2$

Initial rate, $t = 0 \Rightarrow \frac{dV}{dt} = 120 + 26(0) - (0)^2 = 120 \text{ } \ell/\text{min}$

Twice the initial rate $= 2(120) = 240 \text{ } \ell/\text{min}$

$$\Rightarrow 120 + 26t - t^2 = 240$$

$$\Rightarrow t^2 - 26t + 120 = 0$$

$$\Rightarrow (t - 6)(t - 20) = 0$$

$$\Rightarrow t = 6 \text{ mins OR } t = 20 \text{ mins}$$

(b) $\frac{dV}{dt} = 120 + 26t - t^2$

$$\Rightarrow V = \int (120 + 26t - t^2) dt$$

$$= 120t + 26 \frac{t^2}{2} - \frac{t^3}{3} + c$$

$$= 120t + 13t^2 - \frac{t^3}{3} + c$$

$V = 0$ when $t = 0 \Rightarrow 120(0) + 13(0)^2 - \frac{(0)^3}{3} + c = 0$

$$\Rightarrow c = 0$$

$$\Rightarrow V = 120t + 13t^2 - \frac{t^3}{3}$$

(c) $t = 30 \Rightarrow V = 120(30) + 13(30)^2 - \frac{(30)^3}{3}$

$$= 3600 + 11,700 - 9000 = 6300 \text{ litres}$$

Initially, tank = 1500 litres

$$\Rightarrow \text{Total water} = 1500 + 6300 = 7800 \text{ litres}$$

Tank = 7000 litres

$$\Rightarrow \text{water lost} = 7800 - 7000 = 800 \text{ litres}$$

Q5. (a) $y = x^2$

Let point $M = (k, 0) \Rightarrow P = (k, k^2)$

Area rectangle OMPN $= |OM| \cdot |MP| = k \cdot k^2 = k^3$

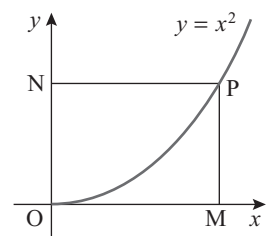
Area under curve $y = x^2 \Rightarrow \int_0^k x^2 dx$

$$= \left[\frac{x^3}{3} \right]_0^k$$

$$= \frac{k^3}{3} - \frac{(0)^3}{3} = \frac{k^3}{3}$$

Hence other region ONP $= k^3 - \frac{k^3}{3} = \frac{2k^3}{3}$

$$\Rightarrow \text{Ratio of areas} = \frac{2k^3}{3} : \frac{k^3}{3} = 2 : 1$$



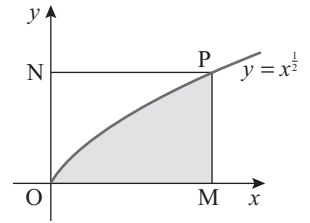
(b) Let point $M = (k, 0) \Rightarrow P = (k, k^{\frac{1}{2}})$

$$\text{Area rectangle OMPN} = |\text{OM}| \cdot |\text{MP}| = k \cdot k^{\frac{1}{2}} = k^{\frac{3}{2}}$$

$$\text{Area under curve } y = x^{\frac{1}{2}} \Rightarrow \int_0^k x^{\frac{1}{2}} dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^k = \left[\frac{2}{3} k^{\frac{3}{2}} \right] - \left[\frac{2}{3} (0)^{\frac{3}{2}} \right] = \frac{2}{3} k^{\frac{3}{2}}$$

$$\Rightarrow \text{Shaded area} = \frac{2}{3} \left[k^{\frac{3}{2}} \right] = \frac{2}{3} \text{ area rectangle OMPN}$$



(c) Let point $M = (k, 0) \Rightarrow P = (k, k^n)$

$$\begin{aligned} \text{Area rectangle OMPN} &= |\text{OM}| \cdot |\text{MP}| \\ &= k \cdot k^n = k^{n+1} \end{aligned}$$

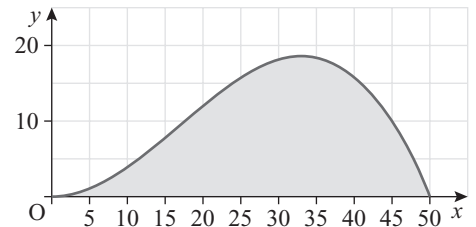
Area under the curve $y = x^n$:

$$\begin{aligned} \text{Area} &= \int_0^k x^n dx \\ &= \left[\frac{x^{n+1}}{n+1} \right]_0^k \\ &= \left[\frac{k^{n+1}}{n+1} \right] - \left[\frac{(0)^{n+1}}{n+1} \right] \\ &= \frac{1}{n+1} \left[k^{n+1} \right] \\ &= \frac{1}{n+1} \text{ area rectangle OMPN} \end{aligned}$$

Q6. $y = \frac{x^2}{1000} (50 - x)$

(a) (i) $x = 10\text{m} \Rightarrow \text{height } (y) = \frac{(10)^2}{1000} (50 - 10) = \frac{1}{10} (40) = 4\text{m}$

(ii) $x = 40\text{m} \Rightarrow \text{height}(y) = \frac{(40)^2}{1000} (50 - 40) = \frac{1600}{1000} (10) = 16\text{m}$



(b) $y = \frac{x^2}{1000} (50 - x) = \frac{50}{1000} x^2 - \frac{x^3}{1000} = \frac{1}{20} x^2 - \frac{1}{1000} x^3$

$$\text{slope} = \frac{dy}{dx} = \frac{1}{20} (2x) - \frac{1}{1000} (3x^2) = \frac{1}{10} x - \frac{3}{1000} x^2$$

(i) $x = 10\text{m} \Rightarrow \text{slope} = \frac{1}{10} (10) - \frac{3}{1000} (10)^2 = 1 - 0.3 = 0.7$

(ii) $x = 40\text{m} \Rightarrow \text{slope} = \frac{1}{10} (40) - \frac{3}{1000} (40)^2 = 4 - 4.8 = -0.8$

(c) (i) Height is a maximum $\Rightarrow \frac{dy}{dx} = 0$

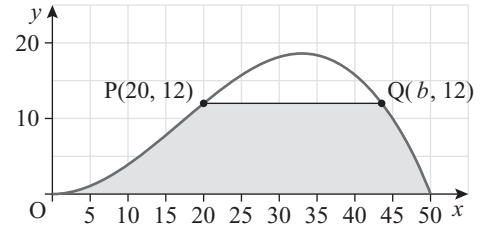
$$\Rightarrow \frac{1}{10}x - \frac{3}{1000}x^2 = 0$$

$$\Rightarrow 100x - 3x^2 = 0$$

$$\Rightarrow x(100 - 3x) = 0$$

$$\Rightarrow x = 0 \text{ (Not valid) OR } 100 = 3x$$

$$\Rightarrow x = \frac{100}{3}$$



(ii) When $x = \frac{100}{3} \Rightarrow \text{height } (y) = \frac{\left(\frac{100}{3}\right)^2}{1000} \left(50 - \frac{100}{3}\right)$

$$= \frac{10,000}{9000} \left(\frac{50}{3}\right)$$

$$= \frac{10}{9} \left(\frac{50}{3}\right) = \frac{500}{27} \text{ m}$$

(d) Area = $\int_0^{50} \left(\frac{1}{20}x^2 - \frac{1}{1000}x^3 \right) dx$

$$= \left[\frac{1}{20} \left(\frac{x^3}{3} \right) - \frac{1}{1000} \left(\frac{x^4}{4} \right) \right]_0^{50}$$

$$= \left[\frac{1}{60}(x^3) - \frac{1}{4000}(x^4) \right]_0^{50}$$

$$= \left[\frac{1}{60}(50)^3 - \frac{1}{4000}(50)^4 \right] - \left[\frac{1}{60}(0)^3 - \frac{1}{4000}(0)^4 \right]$$

$$= \frac{12500}{6} - \frac{6250}{4} = \frac{3125}{6} \text{ m}^2$$

(e) (i) $Q(b,12)$ has $y = 12$

$$\Rightarrow \frac{1}{20}x^2 - \frac{1}{1000}x^3 = 12$$

$$\Rightarrow 50x^2 - x^3 = 12000$$

$$\Rightarrow x^3 - 50x^2 + 12000 = 0$$

$P(20,12)$ has $x = 20 \Rightarrow$ factor $(x - 20)$

$$\Rightarrow \begin{array}{r} x^2 - 30x - 600 \\ x - 20 \overline{) x^3 - 50x^2 + 12000} \end{array}$$

$$\begin{array}{r} \text{(subtract)} \quad x^3 - 20x^2 \\ \hline 0 - 30x^2 + 12000 \end{array}$$

$$\begin{array}{r} \text{(subtract)} \quad -30x^2 + 600x \\ \hline 0 \quad -600x + 12000 \end{array}$$

$$\begin{array}{r} \text{(subtract)} \quad -600x + 12000 \\ \hline 0 \end{array}$$

$$\begin{aligned} x^2 - 30x - 600 = 0 &\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &\Rightarrow x = \frac{30 \pm \sqrt{(-30)^2 - 4(1)(-600)}}{2(1)} \\ &= \frac{30 \pm \sqrt{3300}}{2} = 15 \pm 5\sqrt{33} \\ &\Rightarrow Q = (15 + 5\sqrt{33}, 12) \end{aligned}$$

(ii) $a = 20$ and $b = 15 + 5\sqrt{33}$

$$\begin{aligned} R &= \int_{20}^{15+5\sqrt{33}} 12dx \\ &= [12x]_{20}^{15+5\sqrt{33}} \\ &= 12(15 + 5\sqrt{33}) - 12(20) \\ &= 180 + 60\sqrt{33} - 240 \\ &= 60\sqrt{33} - 60 \end{aligned}$$

$$\text{Area top of the mound} = \int_{20}^{15+5\sqrt{33}} \frac{x^2}{1000} (50 - x) dx - R$$

Therefore, $a = 20$, $b = 15 + 5\sqrt{33}$ and $R = 60\sqrt{33} - 60$

Q7. (a)

$$f(x) = 1 + e^x$$

$$f(-x) = 1 + e^{-x} = 1 + \frac{1}{e^x}$$

$$\begin{aligned} f(x) \times f(-x) &= (1 + e^x) \left(1 + \frac{1}{e^x} \right) \\ &= 1 \left(1 + \frac{1}{e^x} \right) + e^x \left(1 + \frac{1}{e^x} \right) \\ &= 1 + \frac{1}{e^x} + e^x + \frac{e^x}{e^x} \\ &= 1 + \frac{1}{e^x} + e^x + 1 = 2 + e^x + \frac{1}{e^x} \end{aligned}$$

$$\begin{aligned} f(x) + f(-x) &= (1 + e^x) + \left(1 + \frac{1}{e^x} \right) \\ &= 1 + e^x + 1 + \frac{1}{e^x} \\ &= 2 + e^x + \frac{1}{e^x} \end{aligned}$$

$$\text{Hence } f(x) \times f(-x) = f(x) + f(-x)$$

(b)

$$\begin{aligned} \frac{dy}{dx} &= \frac{3 - e^{2x}}{e^x} \\ &= \frac{3}{e^x} - \frac{e^{2x}}{e^x} \\ &= 3e^{-x} - e^x \end{aligned}$$

$$\begin{aligned} y &= \int (3e^{-x} - e^x) dx \\ &= 3 \frac{e^{-x}}{-1} - e^x + c \\ &= \frac{-3}{e^x} - e^x + c \end{aligned}$$

$$\begin{aligned} y = 4 \text{ when } x = 0 &\Rightarrow \frac{-3}{e^0} - e^0 + c = 4 \\ &\Rightarrow \frac{-3}{1} - 1 + c = 4 \\ &\Rightarrow -3 - 1 + c = 4 \Rightarrow c = 8 \\ &\Rightarrow y = \frac{-3}{e^x} - e^x + 8 \end{aligned}$$

(c) Shaded area = $\int_0^2 e^{2x} dx - \int_0^2 e^{-x} dx$

$$= \left[\frac{e^{2x}}{2} \right]_0^2 - \left[\frac{e^{-x}}{-1} \right]_0^2$$

$$= \left[\frac{e^{2x}}{2} + \frac{1}{e^x} \right]_0^2$$

$$= \left[\frac{e^{2(2)}}{2} + \frac{1}{e^2} \right] - \left[\frac{e^{2(0)}}{2} + \frac{1}{e^0} \right]$$

$$= \left(\frac{e^4}{2} + \frac{1}{e^2} \right) - \left(\frac{1}{2} + \frac{1}{1} \right)$$

$$= \frac{e^4}{2} + \frac{1}{e^2} - \frac{3}{2}$$

