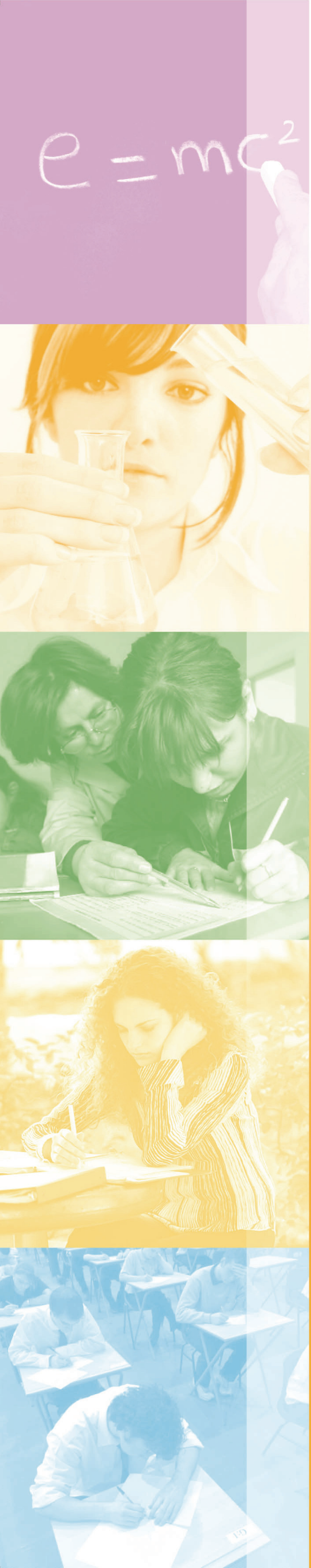

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Pre-Leaving Certificate Examination, 2017

Mathematics

Higher Level

Marking Scheme

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Pre-Leaving Certificate Examination, 2017

Mathematics

Higher Level – Paper 1
Marking Scheme (300 marks)

Structure of the Marking Scheme

Students' responses are marked according to different scales, depending on the types of response anticipated.
Scales labelled A divide students' responses into two categories (correct and incorrect).
Scales labelled B divide responses into three categories (correct, partially correct, and incorrect), and so on.
These scales and the marks that they generate are summarised in the following table:

Scale label	A	B	C	D
No. of categories	2	3	4	5
5 mark scale		**0, 2, 5**	**0, 2, 4, 5**	**0, 2, 3, 4, 5**
10 mark scale			**0, 4, 7, 10**	**0, 4, 6, 8, 10**
15 mark scale				**0, 6, 10, 13, 15**

A general descriptor of each point on each scale is given below. More specific directions in relation to interpreting the scales in the context of each question are given in the scheme, where necessary.

Marking scales – level descriptors

A-scales (two categories)
- incorrect response (no credit)
- correct response (full credit)

B-scales (three categories)
- response of no substantial merit (no credit)
- partially correct response (partial credit)
- correct response (full credit)

C-scales (four categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

D-scales (five categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response about half-right (middle partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

In certain cases, typically involving ❶ incorrect rounding, ❷ omission of units, ❸ a misreading that does not oversimplify the work <u>or</u> ❹ an arithmetical error that does not oversimplify the work, a mark that is one mark below the full-credit mark may also be awarded. Such cases are flagged with an asterisk.
Thus, for example, scale **10C*** indicates that 9 marks may be awarded.

- The * for units to be applied only if the student's answer is fully correct.
- The * to be applied once only **<u>within each section (a), (b), (c), *etc.*</u>** of all questions.
- The * penalty is not applied to currency solutions.

Unless otherwise specified, accept correct answer with <u>or</u> without work shown.

Accept students' work in one part of a question for use in subsequent parts of the question, unless this oversimplifies the work involved.

Summary of Marks – 2017 LC Maths (Higher Level, Paper 1)

Q.1	**(a)**		5C (0, 2, 4, 5)
	(b)		10D* (0, 4, 6, 8, 10)
	(c)		10D (0, 4, 6, 8, 10)
			25

Q.2	**(a)**		10D (0, 4, 6, 8, 10)
	(b)	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5C (0, 2, 4, 5)
			25

Q.3	**(a)**		15D (0, 6, 10, 13, 15)
	(b)	**(i)**	5C (0, 2, 4, 5)
		(ii)	5D (0, 2, 3, 4, 5)
			25

Q.4	**(a)**	**(i)**	5C (0, 2, 4, 5)
		(ii)	5B (0, 2, 5)
	(b)	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5C (0, 2, 4, 5)
			25

Q.5	**(a)**	**(i)**	10D* (0, 4, 6, 8, 10)
		(ii)	5C* (0, 2, 4, 5)
	(b)		10D (0, 4, 6, 8, 10)
			25

Q.6	**(a)**	**(i)**	5C* (0, 2, 4, 5)
		(ii)	10D* (0, 4, 6, 8, 10)
	(b)		10D* (0, 4, 6, 8, 10)
			25

Q.7	**(a)**		5C (0, 2, 4, 5)
	(b)		10D (0, 4, 6, 8, 10)
	(c)	**(i)**	5C (0, 2, 4, 5)
		(ii)	5C (0, 2, 4, 5)
		(iii)	5C (0, 2, 4, 5)
	(d)		10D (0, 4, 6, 8, 10)
	(e)	**(i)**	5C (0, 2, 4, 5)
		(ii)	5C (0, 2, 4, 5)
			50

Q.8	**(a)**	**(i)**	5B (0, 2, 5)
		(ii)	5C (0, 2, 4, 5)
	(b)	**(i)**	15D* (0, 4, 6, 8, 10)
		(ii)	10D* (0, 4, 6, 8, 10)
	(c)	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5D* (0, 2, 3, 4, 5)
			50

Q.9	**(a)**	**(i)**	10D* (0, 4, 6, 8, 10)
		(ii)	10D* (0, 4, 6, 8, 10)
	(b)	**(i)**	5C (0, 2, 4, 5)
		(ii)	5C* (0, 2, 4, 5)
		(iii)	10D* (0, 4, 6, 8, 10)
	(c)		10D* (0, 4, 6, 8, 10)
			50

Assumptions about these marking schemes on the basis of past SEC marking schemes should be avoided. While the underlying assessment principles remain the same, the exact details of the marking of a particular type of question may vary from a similar question asked by the SEC in previous years in accordance with the contribution of that question to the overall examination in the current year. In setting these marking schemes, we have strived to determine how best to ensure the fair and accurate assessment of students' work and to ensure consistency in the standard of assessment from year to year. Therefore, aspects of the structure, detail and application of the marking schemes for these examinations are subject to change from past SEC marking schemes and from one year to the next without notice.

General Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer all questions.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Pre-Leaving Certificate Examination, 2017

Mathematics

Higher Level – Paper 1
Marking Scheme (300 marks)

Section A	Concepts and Skills	150 marks

Answer **all six** questions from this section.

Question 1 **(25 marks)**

1(a) Simplify fully.

$$\frac{x^2-9}{2x^2-11x+15} \div \frac{x^2+3x}{4x^3-10x^2}$$ **(5C)**

$$\begin{aligned}\frac{x^2-9}{2x^2-11x+15} \div \frac{x^2+3x}{4x^3-10x^2} &= \frac{x^2-9}{2x^2-11x+15} \times \frac{4x^3-10x^2}{x^2+3x}\\ &= \frac{(x-3)(x+3)}{(2x-5)(x-3)} \times \frac{2x^2(2x-5)}{x(x+3)}\\ &= \frac{2x^2}{x}\\ &= 2x\end{aligned}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* inverts correctly second fraction and changes division to multiplication.
	–	Some correct factorising, *e.g.* $x^2-9=(x-3)(x+3)$.
High partial credit: (4 marks)	–	Both fractions fully factorised correctly (with second fraction inverted and division changed to multiplication), *i.e.* $\frac{(x-3)(x+3)}{(2x-5)(x-3)} \times \frac{2x^2(2x-5)}{x(x+3)}$ or equivalent, but fails to simplify to simplest form.

Question 1 (cont'd.)

1(b) Find the range of values of x for which

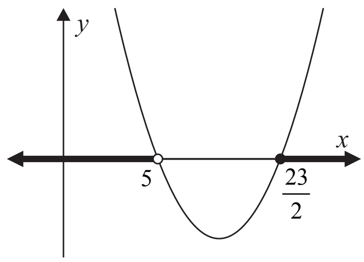
$\frac{3x-2}{x-5} \le 5$, where $x \in \mathbb{R}$ and $x \ne 5$. **(10D*)**

$$\frac{3x-2}{x-5} \le 5$$
$$\Rightarrow \frac{3x-2}{x-5} \times (x-5)^2 \le 5 \times (x-5)^2$$
$$\Rightarrow (3x-2)(x-5) \le 5(x-5)^2$$
$$\Rightarrow 3x^2 - 17x + 10 \le 5(x^2 - 10x + 25)$$
$$\Rightarrow 3x^2 - 17x + 10 \le 5x^2 - 50x + 125$$
$$\Rightarrow 2x^2 - 33x + 115 \ge 0$$
$$\Rightarrow (2x-23)(x-5) \ge 0$$

Consider:

$$(2x-23)(x-5) = 0$$
$$\Rightarrow 2x - 23 = 0$$
$$\Rightarrow x = \frac{23}{2}$$

and
$$x - 5 = 0$$
$$\Rightarrow x = 5$$



$$2x^2 - 33x + 115 \ge 0$$
$$\Rightarrow x \ge \frac{23}{2}$$
$$\Rightarrow x < 5 \quad \text{as } x \ne 5$$

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant correct step, *e.g.* multiplies both sides by $(x-5)^2$.
	–	Finds particular values of x for which the inequality is true.
	–	Some correct use of quadratic formula.
Mid partial credit: (6 marks)	–	Solves the relevant quadratic equation to find the roots, $x = 5$ and $x = \frac{23}{2}$.
High partial credit: (8marks)	–	Wrong shape to graph, but otherwise correct.
	–	Deduces incorrectly using correct values of x.
	–	Deduces correctly for one case only, *i.e.* $x < 5$ or $x \ge \frac{23}{2}$.
	–	Solution set shown on graph only.

* If solution is given as $x \le 5$ and $x \ge \frac{23}{2}$, award 9 marks.

Question 1 (cont'd.)

1(c) Prove that the equation $px^2 - (2p + 1)x + 2 = 0$ has real roots for all values of $p \in \mathbb{R}$ and hence, or otherwise, write down the roots of the equation in terms of p. **(10D)**

$$px^2 - (2p+1)x + 2 = 0$$

Real roots:

$$\Rightarrow \quad b^2 - 4ac \geq 0$$

Consider:

$$\begin{aligned} b^2 - 4ac &= [-(2p+1)]^2 - 4(p)(2) \\ &= 4p^2 + 4p + 1 - 8p \\ &= 4p^2 - 4p + 1 \\ &= (2p-1)^2 \\ &\geq 0 \text{ for all } p \in \mathbb{R} \end{aligned}$$

$\Rightarrow \quad px^2 - (2p+1)x + 2 = 0$ has real roots for all values of $p \in \mathbb{R}$

Roots:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2p + 1 \pm \sqrt{(2p-1)^2}}{2p} = \frac{2p + 1 \pm (2p-1)}{2p}$$

$$\Rightarrow \quad x = \frac{2p + 1 + (2p-1)}{2p} = \frac{4p}{2p} = 2$$

$$\text{and} \quad x = \frac{2p + 1 - (2p-1)}{2p} = \frac{2}{2p} = \frac{1}{p}$$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down $b^2 - 4ac \geq 0$
	–	Some correct substitution into correct '$-b$' formula <u>and stops</u> <u>or</u> continues incorrectly.
Mid partial credit: (6 marks)	–	Proves that roots are real for $p \in \mathbb{R}$ <u>and stops</u>.
	–	Finds both roots (not proving roots are real) <u>and stops</u>.
	–	Finds $b^2 - 4ac$ correctly, but fails to finish and fully correct substitution in quadratic formula.
High partial credit: (8 marks)	–	Proves roots are real for $p \in \mathbb{R}$ and correct substitution in quadratic formula, but not fully simplified.

Question 2 **(25 marks)**

2(a) Given that $4z - 3\bar{z} = \dfrac{1-18i}{2-i}$, express z in the form $a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. **(10D)**

$$
\begin{aligned}
& \text{Let } z && = a + bi \\
\Rightarrow\ & \bar{z} && = a - bi \\
\Rightarrow\ & 4(a+bi) - 3(a-bi) && = \frac{1-18i}{2-i} \times \frac{2+i}{2+i} \\
\Rightarrow\ & 4a + 4bi - 3a + 3bi && = \frac{2 + i - 36i - 18i^2}{5} \\
& && = \frac{20 - 35i}{5} \\
\Rightarrow\ & a + 7bi && = 4 - 7i \\
\Rightarrow\ & a && = 4 \\
\text{and}\ & 7b && = -7 \\
\Rightarrow\ & b && = -1 \\
\Rightarrow\ & z && = 4 - i
\end{aligned}
$$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down $\bar{z} = a - bi$ <u>or</u> multiplies $\frac{1-18i}{2-i}$ by $\frac{2+i}{2+i}$.
Medium partial credit: (6 marks)	–	Simplifies fully $4z - 3\bar{z}$ to $a + 7bi$ <u>or</u> $\frac{1-18i}{2-i}$ to $\frac{20-35i}{5}$ <u>and stops</u> <u>or</u> continues incorrectly.
High partial credit: (8 marks)	–	Simplifies both sides fully, *i.e.* $a + 7bi = \frac{20-35i}{5}$, but only one value ($a$ <u>or</u> b) correct.

Question 2 (cont'd.)

2(b) The complex number w has modulus $3\frac{3}{8}$ and argument $\frac{2\pi}{3}$.

(i) Use De Moivre's Theorem to find, in polar form, the three complex cube roots of w. (That is, find the three values of v for which $v^3 = w$.) **(10D)**

$$w = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow \quad v^3 = 3\frac{3}{8}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$$

$$= \frac{27}{8}[\cos(\frac{2\pi}{3} + 2n\pi) + i\sin(\frac{2\pi}{3} + 2n\pi)]$$

$$\Rightarrow \quad v = \left(\frac{27}{8}\right)^{\frac{1}{3}}[\cos(\frac{2\pi}{3} + 2n\pi) + i\sin(\frac{2\pi}{3} + 2n\pi)]^{\frac{1}{3}}$$

$$= \frac{3}{2}[\cos(\frac{2\pi}{9} + \frac{2n\pi}{3}) + i\sin(\frac{2\pi}{9} + \frac{2n\pi}{3})]$$

For $n = 0$

$$v_1 = \frac{3}{2}[\cos(\frac{2\pi}{9}) + i\sin(\frac{2\pi}{9})]$$

For $n = 1$

$$v_2 = \frac{3}{2}[\cos(\frac{8\pi}{9}) + i\sin(\frac{8\pi}{9})]$$

For $n = 2$

$$v_3 = \frac{3}{2}[\cos(\frac{14\pi}{9}) + i\sin(\frac{14\pi}{9})]$$

Scale 10D (0, 4, 6, 8, 10)

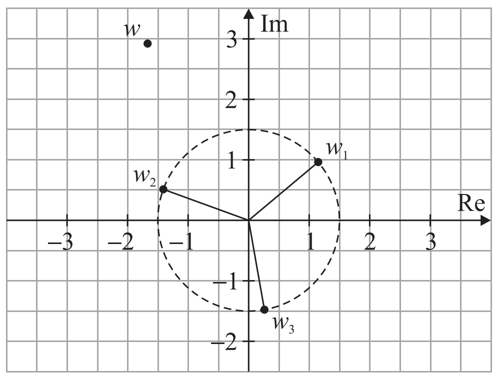
Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down $w = r(\cos\theta + i\sin\theta)$ with $r = 3\frac{3}{8}$, $\theta = \frac{2\pi}{3}$ <u>or</u> $v^3 = 3\frac{3}{8}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ <u>and stops</u>.
Mid partial credit: (6 marks)	–	Correct substitution with manipulation, *i.e.* $v = \left(\frac{27}{8}\right)^{\frac{1}{3}}[\cos(\frac{2\pi}{3} + 2n\pi) + i\sin(\frac{2\pi}{3} + 2n\pi)]^{\frac{1}{3}}$ <u>and stops</u> <u>or</u> continues incorrectly.
High partial credit: (8 marks)	–	Finds correct general term for v, but fails to substitute $n = 1, 2, 3$ into expression. *i.e.* $v = \frac{3}{2}[\cos(\frac{2\pi}{9} + \frac{2n\pi}{3}) + i\sin(\frac{2\pi}{9} + \frac{2n\pi}{3})]$ <u>and stops</u>.
	–	Finds $v_1 = \frac{3}{2}[\cos(\frac{2\pi}{9}) + i\sin(\frac{2\pi}{9})]$, but fails to find <u>or</u> finds incorrect v_2 and v_2.

Question 2 (cont'd.)

2(b) (cont'd.)

(ii) w is marked on the Argand diagram below.
On the same diagram, show your answers to part **(i)** and hence, write down the equation of the curve on which all three roots lie. **(5C)**

❶ Argand diagram



❷ Equation of curve

Curve: – circle with centre (0, 0), radius $\frac{3}{2}$

$\Rightarrow \quad x^2 + y^2 = \frac{9}{4}$

or $\quad 4x^2 + 4y^2 = 9$

** Accept students' answers from part (b)(i) if not oversimplified.
** Accept students' answers using other methods to find the roots of v^3.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Plots correctly one root from part (i).
High partial credit: (4 marks)	– –	Plots correctly all three roots from part (i). Plots correctly one root <u>and</u> writes down correct equation of the curve.

Question 3 **(25 marks)**

3(a) One root of the equation $4x^3 - 8x^2 + kx + 2 = 0$ is $\frac{1}{2}$.

Find the value of $k \in \mathbb{R}$ and hence the other roots of the equation. **(15D)**

❶ Find the value of k

$$\begin{aligned} f(\tfrac{1}{2}) &= 4(\tfrac{1}{2})^3 - 8(\tfrac{1}{2})^2 + k(\tfrac{1}{2}) + 2 \\ &= \frac{4}{8} - \frac{8}{4} + \frac{k}{2} + 2 \\ &= \frac{1}{2} - 2 + \frac{k}{2} + 2 \\ &= \frac{1}{2} + \frac{k}{2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \frac{1}{2} + \frac{k}{2} &= 0 \\ \Rightarrow \quad k + 1 &= 0 \\ \Rightarrow \quad k &= -1 \end{aligned}$$

❷ <u>Other roots of equation</u>

$f(x) = 4x^3 - 8x^2 - x + 2$

$x = \frac{1}{2}$ is a root of the equation

$\Rightarrow$ $(2x - 1)$ is a factor of the equation

<u>Consider</u>

$$\begin{array}{r|l} & 2x^2 - 3x - 2 \\ 2x - 1 & 4x^3 - 8x^2 - x + 2 \\ & \underline{-4x^3 + 2x^2} \\ & -6x^2 - x + 2 \\ & \underline{+6x^2 - 3x} \\ & -4x + 2 \\ & \underline{+4x - 2} \\ & 0 \end{array}$$

$$\begin{aligned} \Rightarrow \quad (2x - 1)(2x^2 - 3x - 2) &= 0 \\ \Rightarrow \quad (2x - 1)(2x + 1)(x - 2) &= 0 \\ \Rightarrow \quad 2x + 1 &= 0 \\ \Rightarrow \quad x &= -\frac{1}{2} \\ \Rightarrow \quad x - 2 &= 0 \\ \Rightarrow \quad x &= 2 \end{aligned}$$

Scale 15D (0, 6, 10, 13, 15)

Low partial credit: (6 marks)	–	Any relevant correct step, *e.g.* writes down $f(\frac{1}{2}) = 4(\frac{1}{2})^3 - 8(\frac{1}{2})^2 + k(\frac{1}{2}) + 2$ <u>and stops</u>.
	–	Writes down $2x - 1$ is a factor of equation and attempts to divide.
Mid partial credit: (10 marks)	–	Finds correct value for k and some correct division in dividing $2x - 1$ into equation.
High partial credit: (13 marks)	–	Finds $2x^2 - 3x - 2$ correctly using division, but fails to find <u>or</u> finds incorrect roots.

Question 3 (cont'd.)

3(b) (i) Express $\log_9 xy$ in terms of $\log_3 x$ and $\log_3 y$. **(5C)**

$$\begin{aligned}\log_9 xy &= \log_9 x + \log_9 y \\ &= \frac{\log_3 x}{\log_3 9} + \frac{\log_3 y}{\log_3 9} \\ &= \frac{\log_3 x}{2} + \frac{\log_3 y}{2}\end{aligned}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down $\log_9 xy = \log_9 x + \log_9 y$ <u>or</u> $= \frac{\log_3 xy}{\log_3 9}$ <u>and stops</u>.
High partial credit: (4 marks)	–	Give final answer as $\frac{\log_3 x}{\log_3 9} + \frac{\log_3 y}{\log_3 9}$, $\frac{\log_3 xy}{2}$ <u>or</u> $\frac{\log_3 x + \log_3 y}{9}$.

Question 3 (cont'd.)

3(b) (cont'd.)

(ii) Hence, or otherwise, solve the simultaneous equations for x and y:

$$\log_9 xy = \frac{5}{2}$$
$$\log_3 x . \log_3 y = -6.$$

Express your answers in their simplest form. **(5D)**

❶
$$\log_9 xy = \frac{5}{2}$$
$$\Rightarrow \frac{\log_3 x}{2} + \frac{\log_3 y}{2} = \frac{5}{2}$$
$$\Rightarrow \log_3 x + \log_3 y = 5$$
$$\Rightarrow \log_3 x = 5 - \log_3 y$$

❷
$$\log_3 x . \log_3 y = -6$$
$$\Rightarrow (5 - \log_3 y).\log_3 y = -6$$
$$\Rightarrow 5\log_3 y - (\log_3 y)^2 = -6$$
$$\Rightarrow (\log_3 y)^2 - 5\log_3 y - 6 = 0$$
$$\Rightarrow (\log_3 y + 1)(\log_3 y - 6) = 0$$

①
$$\log_3 y + 1 = 0$$
$$\Rightarrow \log_3 y = -1$$
$$\Rightarrow y = 3^{-1} \ \underline{\text{or}} \ \frac{1}{3} \ \underline{\text{or}} \ 0\cdot333333...$$

$$\log_3 x = 5 - \log_3 y = 5 - (-1) = 6$$
$$\Rightarrow x = 3^6 = 729$$

<u>and</u>

②
$$\log_3 y - 6 = 0$$
$$\Rightarrow \log_3 y = 6$$
$$\Rightarrow y = 3^6 = 729$$

$$\log_3 x = 5 - \log_3 y = 5 - 6 = -1$$
$$\Rightarrow x = 3^{-1} \ \underline{\text{or}} \ \frac{1}{3} \ \underline{\text{or}} \ 0\cdot333333...$$

** Accept students' answers from part (b)(i) if not oversimplified.

Scale 5D (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* $\log_3 x$ <u>or</u> $\log_3 y$ isolated, *i.e.* $\log_3 x = 5 - \log_3 y$.
Mid partial credit: (3 marks)	–	Substitutes $\log_3 x = 5 - \log_3 y$ correctly into second equation <u>and</u> forms correct quadratic equation.
High partial credit: (4 marks)	– –	Correct values for $\log_3 x$ and $\log_3 y$ *i.e.* $\log_3 x = -1$ and $\log_3 x = 6$, $\log_3 y = 6$ and $\log_3 y = -1$, but fails to finish <u>or</u> finishes incorrectly. Finds one correct solution and finishes correctly.

Question 4 **(25 marks)**

4(a) A curve is defined by the equation $(x - 3)^2 + y^2 = 25$.

(i) Find $\frac{dy}{dx}$ in terms of x. **(5C)**

❶ Isolate y and differentiate:

$$(x-3)^2 + y^2 = 25$$
$$\Rightarrow y^2 = 25 - (x-3)^2$$
$$\Rightarrow y = \pm\sqrt{25-(x-3)^2}$$
$$\Rightarrow \frac{dy}{dx} = \pm\frac{1}{2}[25-(x-3)^2]^{-\frac{1}{2}}[0-2(x-3)]$$
$$= \pm\frac{3-x}{\sqrt{25-(x-3)^2}}$$

❷ Using implicit differentiation:

$$(x-3)^2 + y^2 = 25$$
$$\Rightarrow 2(x-3)(1) + 2y\frac{dy}{dx} = 0$$
$$\Rightarrow 2y\frac{dy}{dx} = -2(x-3)$$
$$\Rightarrow \frac{dy}{dx} = \frac{-2x+6}{2y}$$
$$= \frac{3-x}{y}$$
$$(x-3)^2 + y^2 = 25$$
$$\Rightarrow y^2 = 25-(x-3)^2$$
$$\Rightarrow y = \pm\sqrt{25-(x-3)^2}$$
$$\Rightarrow \frac{dy}{dx} = \pm\frac{3-x}{\sqrt{25-(x-3)^2}}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* attempts to isolates y (correct transpositions) (method ❶) <u>and stops</u>.
	–	Differentiates any term correctly (method ❷), *e.g.* $\frac{d}{dx}(x-3)^2 = 2(x-3)(1)$ $\frac{d}{dx}(y)^2 = 2y\frac{dy}{dx}$.
High partial credit: (4 marks)	–	Isolates y correctly [ans. $\pm\sqrt{25-(x-3)^2}$] and some correct differentiation (method ❶), *e.g.* $\pm\frac{1}{2}[25-(x-3)^2]^{-\frac{1}{2}}$ <u>and stops</u> <u>or</u> continues incorrectly.
	–	Differentiates all term correctly (method ❷), *i.e.* $2(x-3)(1) + 2y\frac{dy}{dx} = 0$ <u>and</u> isolates $\frac{dy}{dx} = \frac{3-x}{y}$, but fails to give answer in terms of x only.

Question 4 (cont'd.)

4(a) (cont'd.)

(ii) Hence, find the equation of the tangent to the curve at the point (6, 4). **(5B)**

❶ Slope of tangent @ (6, 4)

$$\text{Slope, } m = \frac{dy}{dx}$$
$$= \pm\frac{3-x}{\sqrt{25-(x-3)^2}}$$
$$\Rightarrow m\ @\ (6, 4) = \pm\frac{3-6}{\sqrt{25-(6-3)^2}}$$
$$= \pm\frac{-3}{\sqrt{16}}$$
$$= -\frac{3}{4} \quad \text{... slope} < 0 \text{ for } 3 < x < 8 \text{ in first quadrant}$$

Equation of tangent

Point (6, 4), $m = -\frac{3}{4}$

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 4 = -\frac{3}{4}(x-6)$$
$$\Rightarrow 4(y-4) = -3(x-6)$$
$$\Rightarrow 4y - 16 = -3x + 18$$
$$\Rightarrow 3x + 4y - 34 = 0$$

or

❷ Slope of tangent @ (6, 4)

$$\text{Slope, } m = \frac{dy}{dx}$$
$$= \frac{3-x}{y}$$
$$\Rightarrow m\ @\ (6, 4) = \frac{3-6}{4}$$
$$= -\frac{3}{4}$$

Equation of tangent

Point (6, 4), $m = -\frac{3}{4}$

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 4 = -\frac{3}{4}(x-6)$$
$$\Rightarrow 4(y-4) = -3(x-6)$$
$$\Rightarrow 4y - 16 = -3x + 18$$
$$\Rightarrow 3x + 4y - 34 = 0$$

** Accept students' answers from part (a)(i) if not oversimplified.

Scale 5B (0, 2, 5)

Partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down formula for the equation of a line with x_1 and/or y_1 substituted.
	–	Finds correct slope at (6, 4) and stops.

Question 4 (cont'd.)

4(b) (i) Show that the curve $y = \frac{2}{x-3}$, where $x \neq 3$ and $x \in \mathbb{R}$, has no turning points and no points of inflection. **(10D)**

❶ Turning points

$$\Rightarrow \frac{dy}{dx} = 0$$

$$y = \frac{2}{x-3}$$

$$= 2(x-3)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = -2(x-3)^{-2}(1)$$

$$= \frac{-2}{(x-3)^2}$$

$$\neq 0 \quad \text{... as } -2 \neq 0$$

$\Rightarrow y = \frac{2}{x-3}$ has no turning points

❷ Points of inflection

$$\Rightarrow \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{dy}{dx} = -2(x-3)^{-2}(1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4(x-3)^{-3}(1)$$

$$= \frac{4}{(x-3)^3}$$

$$\neq 0 \quad \text{... as } 4 \neq 0$$

$\Rightarrow y = \frac{2}{x-3}$ has no points of inflection

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down $\frac{dy}{dx} = 0$ at a turning point or $\frac{d^2y}{dx^2} = 0$ at a point of inflection and stops.
	–	Finds $\frac{dy}{dx} = -2(x-3)^{-2}(1)$, but no conclusion given.
Mid partial credit: (6 marks)	–	Finds $\frac{dy}{dx}$ correctly and concludes not equal to zero and stops.
	–	Finds $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ correctly, but neither equated to zero (*i.e.* no deductions).
High partial credit: (8 marks)	–	Shows correctly that there are no turning points and finds $\frac{d^2y}{dx^2}$ or vice versa, but fails to finish.
	–	Finds both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ correctly and equated to 0, but does not show why this means there are no turning points or points of inflection.

Question 4 (cont'd.)

4(b) (cont'd.)

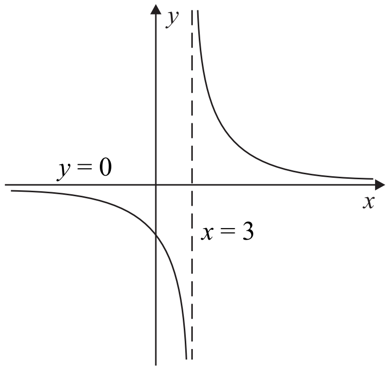
(ii) Comment on the shape of the curve for all $x \in \mathbb{R}$. **(5C)**

❶ $$\frac{dy}{dx} = \frac{-2}{(x-3)^2}$$

< 0 for all $x \in \mathbb{R}, x \neq 3$

$\Rightarrow$ curve is decreasing for all values of x

❷ Curve has two asymptotes at $x = 3$ and $y = 0$



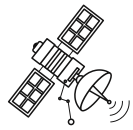
** Accept students' answers from part (b)(i) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Mentions that the curve is in two sections, has a break, is not continuous or has an asymptote at $x = 3$ <u>or</u> at y = 0.
High partial credit: (4 marks)	– –	States that the curve is decreasing for all values of x. States that the curve has two asymptotes at $x = 3$ and $y = 0$.

Question 5 **(25 marks)**

(a) The power supply to a space satellite is provided by means of a generator that converts heat released by the decay of a radioisotope into electricity. The power output, in watts, may be calculated using the function



$$w(t) = Ae^{bt},$$

where t is the time, in days, from when the satellite is launched into space. The initial power output at the launch of the satellite is 60 watts.

(i) Given that after 14 days the power output falls to 56 watts, calculate the value of b, correct to three decimal places. **(10D*)**

$$\begin{aligned}
& w(t) &&= Ae^{bt} \\
& w(0) &&= Ae^{b(0)} \\
& &&= 60 \\
\Rightarrow\ & Ae^{b(0)} &&= 60 \\
\Rightarrow\ & Ae^{0} &&= 60 \\
\Rightarrow\ & A(1) &&= 60 \\
\Rightarrow\ & A &&= 60 \\
\Rightarrow\ & w(t) &&= 60e^{bt} \\
& w(14) &&= 60e^{b(14)} \\
& &&= 56 \\
\Rightarrow\ & 60e^{b(14)} &&= 56 \\
\Rightarrow\ & e^{14b} &&= \frac{56}{60} \\
\Rightarrow\ & \ln e^{14b} &&= \ln\frac{56}{60} \\
\Rightarrow\ & 14b &&= -0{\cdot}068992... \\
\Rightarrow\ & b &&= -0{\cdot}004928... \\
& &&\cong -0{\cdot}005
\end{aligned}$$

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* substitutes correctly into function for $t = 0$, $w = 60$ [ans. $Ae^{0} = 60$] <u>or</u> $t = 14$, $w = 56$ [ans. $Ae^{14b} = 56$].
	–	Finds correct value of A [ans. 60].
Mid partial credit: (6 marks)	–	Finds $e^{14b} = \frac{56}{60}$ <u>or</u> $\frac{14}{15}$ <u>and stops</u>.
High partial credit: (8 marks)	–	Finds $e^{14b} = \frac{56}{60}$ and uses $\log_e$ correctly to simplify b term, but fails to find correct value of b.
	–	Finds $14b = \ln\frac{56}{60}$, but fails to find correct value of b.

* Deduct 1 mark off correct answer only if not rounded <u>or</u> incorrectly rounded - apply only once to each section (a), (b), (c), *etc*. of question.

Question 5 (cont'd.)

5(a) (cont'd.)

(ii) The satellite cannot function properly when the power output falls below 5 watts. After how many days will the satellite fail to function properly? **(5C*)**

$$
\begin{aligned}
w(t) &= 60e^{-0\cdot 005t} \\
&= 5 \\
\Rightarrow \quad 60e^{-0\cdot 005t} &= 5 \\
\Rightarrow \quad e^{-0\cdot 005t} &= \frac{5}{60} \\
\Rightarrow \quad \ln e^{-0\cdot 005t} &= \ln \frac{1}{12} \\
\Rightarrow \quad -0\cdot 005t &= -\ln 12 \\
\Rightarrow \quad t &= \frac{\ln 12}{0\cdot 005} \\
&= 496\cdot 981329...
\end{aligned}
$$

⇒ satellite will fail to function properly after 497 days

** Accept students' answers from part (b)(i) if not oversimplified.

Scale 5C* (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* substitutes correctly into function $w(t) = Ae^{bt}$ for $w = 5$ using A and b values from part (i).
High partial credit: (4 marks)	–	Finds $e^{-0\cdot 005t} = \frac{5}{60}$ <u>or</u> $\frac{1}{12}$ [accept student's values from (i)] <u>and</u> uses $\log_e$ correctly, *e.g.* $-0\cdot 005t = -\ln 12$, but fails to find correct value of t.

* Deduct 1 mark off correct answer only if '496 days' given as final answer.
* Deduct 1 mark off correct answer only if not rounded <u>or</u> incorrectly rounded - apply only once to each section (a), (b), (c), *etc*. of question.

Question 5 (cont'd.)

5(b) Find the value of the constant k for which $w(t+k) = \frac{1}{2}w(t)$, for all $t \geq 0$.

Give your answer in the form $p\ln q$, where $p, q \in \mathbb{N}$. **(10D)**

$$
\begin{aligned}
& w(t+k) &&= \frac{1}{2}w(t) \\
\Rightarrow\ & 60e^{b(t+k)} &&= \frac{1}{2}[60e^{bt}] \\
\Rightarrow\ & 60e^{bt+bk} &&= 30e^{bt} \\
\Rightarrow\ & e^{bt}.e^{bk} &&= \frac{30}{60}e^{bt} \\
& &&= \frac{1}{2}e^{bt} \\
\Rightarrow\ & e^{bk} &&= \frac{1}{2} \\
\Rightarrow\ & \ln e^{bk} &&= \ln\frac{1}{2} \\
\Rightarrow\ & bk &&= -\ln 2 \\
\Rightarrow\ & -0{\cdot}005k &&= -\ln 2 \\
\Rightarrow\ & k &&= \frac{\ln 2}{0{\cdot}005} \\
& &&= 200\ln 2 \ \underline{\text{or}}\ 100\ln 4 \ \underline{\text{or}}\ 50\ln 16 \ \underline{\text{or}}\ 25\ln 256
\end{aligned}
$$

** Accept students' answers from part (a) if not oversimplified.

Scale 10D (0, 3, 5, 8, 10)

Low partial credit: (3 marks)	–	Any relevant first step, *e.g.* writes down $w(t+k) = 60e^{b(t+k)}$ <u>or equivalent</u> [accept students' values for A and b from part (i)].
Mid partial credit: (5 marks)	–	Finds $e^{bk} = \frac{1}{2}$, $e^{-0\cdot005k} = \frac{1}{2}$ [accept students' values for b].
High partial credit: (8 marks)	–	Finds correct value of k, but not in the required form, *e.g.* $\frac{-\ln\frac{1}{2}}{0{\cdot}005}$, $-20\ln\frac{1}{2}$ <u>or</u> $k = 138{\cdot}629436...$

Question 6 **(25 marks)**

6(a) Fiona arranged to pay €120 at the end of each week for 25 years into a pension fund that earns an annual equivalent rate (AER) of 3·9%.

(i) Show that the rate of interest, compounded weekly, which corresponds to an AER of 3·9% is 0·0736%, correct to four decimal places. [1 year = 52 weeks] **(5C*)**

r = annual percentage rate (APR)
i = weekly percentage rate

$$F = P(1+r)$$
$$= P(1+i)^t$$
$$\Rightarrow 1(1+r) = 1(1+i)^t$$
$$\Rightarrow 1(1+0{\cdot}039) = 1(1+i)^{52}$$
$$\Rightarrow 1{\cdot}039 = (1+i)^{52}$$
$$\Rightarrow 1+i = (1{\cdot}039)^{\frac{1}{52}}$$
$$\Rightarrow i = 1{\cdot}0007360... - 1$$
$$= 0{\cdot}000736015...$$
$$\Rightarrow r = 0{\cdot}0736015\%$$
$$\cong 0{\cdot}0736\%$$

Scale 5C* (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula $F = P(1+i)^t$ and stops.
	–	Some correct substitution into correct formula (not stated) and stops or continues.
	–	Correct substitution into incorrect formula and stops or continues.
High partial credit: (4 marks)	–	Fully correct substitution into formula, *i.e.* $1(1+0{\cdot}039) = 1(1+i)^{52}$ or equivalent, but fails to find or finds incorrect rate.
	–	Final answer not given as a percentage, *i.e.* $r = 0{\cdot}000736015...$

* Deduct 1 mark off correct answer only if not rounded or incorrectly rounded - apply only once to each section (a), (b), (c), *etc*. of question.

6(a) (cont'd.)

(ii) Calculate, correct to the nearest euro, the total value of Fiona's pension fund when she retires. **(10D*)**

$$\#\text{payments} = 25 \times 52 = 1{,}300$$

$$F = P(1+i)^t = 120(1 + 0{\cdot}000736)^t$$

Week	Paid (€)	Value of payment on retirement (wk. 1,300)
1	120	$120(1{\cdot}000736)^{1,299}$
2	120	$120(1{\cdot}000736)^{1,298}$
3	120	$120(1{\cdot}000736)^{1,297}$
...	...	...
1,298	120	$120(1{\cdot}000736)^{2}$
1,299	120	$120(1{\cdot}000736)^{1}$
1,300	120	120

⇒ Geometric series with $n = 1{,}300$, $a = 120$ and $r = 1{\cdot}000736$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow S_{1,300} = \frac{120(1-1{\cdot}000736^{1,300})}{1-1{\cdot}000736} = 261{,}266{\cdot}798874... \cong €261{,}267$$

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* reference to $25 \times 52 = 1{,}300$ payments or value of first or subsequent payments at retirement $= 120(1{\cdot}000736)^n$, where $1 < n \le 1{,}300$.
Mid partial credit: (6 marks)	–	Recognises value of retirement fund as a sum of a GP with some correct substitution into S_n formula.
High partial credit: (8 marks)	–	Fully correct substitution into S_n formula, but fails to find or finds incorrect value of fund on retirement.

* Deduct 1 mark off correct answer only if not rounded or incorrectly rounded - apply only once to each section (a), (b), (c), *etc.* of question.

Question 6 (cont'd.)

6(b) On retirement, Fiona invests the total value of her pension fund in a scheme that earns an AER of 4·2%. Fiona will receive a fixed amount of money at the end of each month for twenty years, at which time the value of her investment will be zero. Calculate, correct to the nearest euro, the amount of each monthly payment **(10D*)**

Interest rate

r = annual percentage rate (APR)

i = monthly percentage rate

$F = P(1+r)$

$= P(1+i)^t$

$\Rightarrow 1(1+r) = 1(1+i)^t$

$\Rightarrow 1(1+0{\cdot}042) = 1(1+i)^{12}$

$\Rightarrow 1{\cdot}042 = (1+i)^{12}$

$\Rightarrow 1+i = (1{\cdot}042)^{\frac{1}{12}}$

$\Rightarrow i = 1{\cdot}042^{\frac{1}{12}} - 1$

$= 1{\cdot}003434... - 1$

$= 0{\cdot}003434...$

$\Rightarrow r = 0{\cdot}3434\%$

❶ Sum of geometric series

Let X = fixed monthly payment for 20 years

payments = 12×20

$= 240$

$F = P(1+i)^t$

$\Rightarrow P = \dfrac{F}{(1+i)^t}$

$= \dfrac{X}{(1+0{\cdot}003434)^t}$

$= \dfrac{X}{1{\cdot}003434^{\,t}}$

Month	Present value of future payment (P)	Future payment (F)
1	$\dfrac{X}{1{\cdot}003434^1}$	X
2	$\dfrac{X}{1{\cdot}003434^2}$	X
...	...	...
240	$\dfrac{X}{1{\cdot}003434^{240}}$	X

$\Rightarrow$ Geometric series with $n = 240$, $a = \dfrac{X}{1{\cdot}003434}$ and $r = \dfrac{1}{1{\cdot}003434}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow S_{240} = \frac{\frac{X}{1{\cdot}003434}\left(1-\frac{1}{1{\cdot}003434^{240}}\right)}{1-\frac{1}{1{\cdot}003434}}$$

$$= \frac{X(0{\cdot}558897...)}{0{\cdot}003422...}$$

$$= 163{\cdot}294980...X$$

$\Rightarrow 163{\cdot}294980...X = 261{,}267$

$\Rightarrow X = 1{,}599.969571...$

$\cong$ €1,600

Question 6 (cont'd.)

6(b) (cont'd.)

❷ Amortisation

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1}$$

$t = 12 \times 20$

$= 240$

$i = 0{\cdot}003434$

$P = 261{,}267$

$X =$ fixed monthly payment

$$\Rightarrow A = \frac{261{,}267(0{\cdot}003434)(1 + 0{\cdot}003434)^{240}}{(1{\cdot}003434)^{240} - 1}$$

$$= \frac{261{,}267(0{\cdot}003434)(1{\cdot}003434)^{240}}{(1{\cdot}003434)^{240} - 1}$$

$= 1599{\cdot}906538...$

$\cong$ €1,600

** Accept students' answers from part (a)(ii) if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* calculates correct monthly rate [ans. 0·003434379..., (rounded or not)] or number of payments [ans. 20 × 12 = 240].
Mid partial credit: (6 marks)	– –	Recognises sum of future payments as a sum of a GP with some correct substitution in S_n formula. Writes down correct relevant formula for amortisation with some correct substitution into formula.
High partial credit: (8 marks)	–	Fully correct substitution into S_n or amortisation formula, but fails to finish or finishes incorrectly.

* Deduct 1 mark off correct answer only if not rounded or incorrectly rounded - apply only once to each section (a), (b), (c), *etc.* of question.

Section B — Contexts and Applications — 150 marks

Answer **all three** questions from this section.

Question 7 (50 marks)

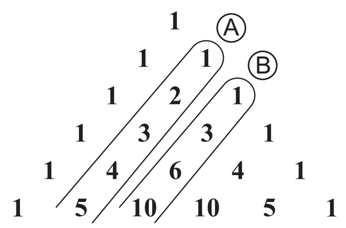
The diagram below shows the beginning of Pascal's triangle.

1	.... row 0
1 1	.... row 1
1 2 1	.... row 2
1 3 3 1	.... row 3

The rows of Pascal's triangle are conventionally enumerated, starting with row $r = 0$ at the top (row 0). The entries in each row are numbered from left to right, beginning with $k = 0$ (e.g. in row 3, $k_0 = 1$, $k_1 = 3$, $k_2 = 3$ and $k_4 = 1$).

The triangle may be constructed as follows: In row 0 (the topmost row), the entry is 1. Each entry in successive rows is found by adding the number above and to the left with the number above and to the right, treating blank entries as 0.

There are several patterns found within Pascal's triangle. Consider the two sequences, A and B, shown below.



7(a) Find an expression for T_n, the nth term, and S_n, the sum of the first n terms, of sequence A. **(5C)**

❶ T_n, the nth term

Sequence A: 1, 2, 3, 4, 5, ...

$\Rightarrow$ arithmetic series

$T_n = a + (n-1)d$

$a = 1$

$d = 1$

$\Rightarrow T_n = 1 + (n-1)1$

$= 1 + n - 1$

$= n$

❶ S_n, the sum of the first n terms

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[2(1) + (n-1)(1)]$$

$$= \frac{n}{2}[2 + n - 1]$$

$$= \frac{n}{2}[n + 1]$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down sequence A is linear (arithmetic) pattern with $a = 1$ and $d = 1$.
	–	Finds $T_n = n$ (by inspection or calculation) and stops.
High partial credit: (4 marks)	–	Finds correct T_n and writes down formula for S_n with a and d correctly identified, but not fully substituted / simplified.

Question 7 (cont'd.)

7(b) Find an expression for T_n, the nth term of sequence B. **(10D)**

❶ Sequence B: = 1, 3, 6, 10, ...

Term	1st Diff.	2nd Diff.
1		
	2	
3		**1**
	3	
6		**1**
	4	
10		

⇒ first differences are not constant, but the second differences are constant
⇒ terms form a quadratic sequence

$T_n = an^2 + bn + c$

$2a = 1$

$\Rightarrow a = \frac{1}{2}$

$\Rightarrow T_n = \frac{1}{2}n^2 + bn + c$

$T_1 = 1$

$\Rightarrow \frac{1}{2}(1)^2 + b(1) + c = 1$

$\Rightarrow \frac{1}{2} + b + c = 1$

$\Rightarrow b + c = \frac{1}{2}$ ①

$T_2 = 3$

$\Rightarrow \frac{1}{2}(2)^2 + b(2) + c = 3$

$\Rightarrow 2 + 2b + c = 3$

$\Rightarrow 2b + c = 1$ ②

$\Rightarrow -b - c = -\frac{1}{2}$ ① (×–1)

$2b + c = 1$ ②

$\Rightarrow b = \frac{1}{2}$

and $c = 0$

$\Rightarrow T_n = \frac{1}{2}n^2 + \frac{1}{2}n$

$= \frac{n}{2}(n + 1)$

or ❷ Sequence B: 1, 3, 6, 10, ...

$= \binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}, \ldots$

$\Rightarrow T_n = \binom{n+1}{2}$

$= \frac{(n+1)n}{2}$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down sequence B is quadratic pattern as second difference is constant.
	–	Finds $T_n = \frac{1}{2}n^2 + bn + c$ and stops.
Mid partial credit: (6 marks)	–	Writes down $T_n = \frac{1}{2}n^2 + bn + c$ and finds correct value of a and stops.
High partial credit: (8 marks)	–	Forms two correct equations in b and c, but fails to finish or finish incorrectly.

7(c) (i) Verify that the third entry of row 6 of Pascal's triangle is found by adding T_5 of sequence A and T_4 of sequence B. **(5C)**

Row 6: 1, 6, **15**, 20, 15, 6, 1

Sequence A:

$$T_n(A) = n$$
$$\Rightarrow T_5(A) = 5$$

Sequence B:

$$T_n(B) = \frac{n}{2}(n+1)$$
$$\Rightarrow T_4(B) = \frac{4(5)}{2} = 10$$
$$\Rightarrow T_5(A) + T_4(B) = 5 + 10 = 15$$
$$\Rightarrow \text{3rd entry of row 6} = T_5(A) + T_4(B)$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down at least first three terms of row 6 from Pascal's triangle and stops.
	–	Finds correct values of $T_5(A)$ and/or $T_4(B)$ and stops.
High partial credit: (4 marks)	–	Finds correct values of $T_5(A)$ and $T_4(B)$ and third entry of row 6, but no conclusion given.

(ii) Find an expression, in r, for the third entry of the rth row and hence, verify your answer to part **(i)** above. **(5C)**

❶ Expression, in r, for third entry of row r

Row r:

$$\text{Third entry} = T_{r-1}(A) + T_{r-2}(B)$$
$$T_{r-1}(A) = r - 1$$
$$T_{r-2}(B) = \frac{(r-2)(r-2+1)}{2} = \frac{(r-2)(r-1)}{2}$$
$$\Rightarrow \text{Third entry} = r - 1 + \frac{(r-2)(r-1)}{2} = \frac{2(r-1) + (r-2)(r-1)}{2} = \frac{2r - 2 + r^2 - 3r + 2}{2} = \frac{r^2 - r}{2} = \frac{r(r-1)}{2}$$

❷ Verify answer to part (i)

For $r = 6$

$$\text{Third entry} = \frac{6(6-1)}{2} = \frac{30}{2} = 15$$

Question 7 (cont'd.)

7(c) (ii) (cont'd.)

** Accept students' answers from part (c)(i) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down 'Third entry = $T_{r-1}(A) + T_{r-2}(B)$' and stops. Finds $T_{r-1}(A)$ or $T_{r-2}(B)$ correctly and stops.
High partial credit: (4 marks)	–	Finds $T_{r-1}(A)$ and $T_{r-2}(B)$ correctly and finds third entry in terms of r, but fails to verify answer to part (i).

(iii) An entry in Pascal's triangle is denoted $\binom{r}{k}$ and can be determined using the formula:

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k},$$

where r is the row number (top row = 0) and k is the entry number in row r (first entry = 0).
Using the above formula, verify your expression, in r, for the third entry in the rth row. **(5C)**

$$\begin{aligned}
\text{Third entry in row } r &= \binom{r}{2} \\
&= \binom{r-1}{1} + \binom{r-1}{2} \\
&= \frac{(r-1)!}{1!\,(r-1-1)!} + \frac{(r-1)!}{2!\,(r-1-2)!} \\
&= \frac{(r-1)!}{(r-2)!} + \frac{(r-1)!}{2!\,(r-3)!} \\
&= r-1 + \frac{(r-2)(r-1)}{(2)(1)} \\
&= \frac{2(r-1) + (r-2)(r-1)}{2} \\
&= \frac{2r-2+r^2-3r+2}{2} \\
&= \frac{r^2-r}{2} \\
&= \frac{r(r-1)}{2}
\end{aligned}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* substitutes 2 for k, *i.e.* $\binom{r}{2} = \binom{r-1}{1} + \binom{r-1}{2}$ and stops.
	–	Finds $\binom{r-1}{1}$ and/or $\binom{r-1}{2}$ correctly and stops.
High partial credit: (4 marks)	–	Finds $\binom{r}{2} = \binom{r-1}{1} + \binom{r-1}{2}$ and expands both $\binom{r-1}{1}$ and $\binom{r-1}{2}$ but fails to finish correctly.

Question 7 (cont'd.)

7(d) Prove by induction that S_n, the sum of the first n terms of sequence B, is $\dfrac{n(n+1)(n+2)}{6}$ for all $n \in \mathbb{N}$. **(10D)**

❶ P(n):

$$1 + 3 + 6 + 10 + \ldots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

❷ P(1):

Test hypothesis for $n = 1$

$$\frac{1(1+1)}{2} = \frac{1(1+1)(1+2)}{6}$$

$$\frac{1(2)}{2} = \frac{1(2)(3)}{6}$$

$$1 = \frac{6}{6} = 1$$

$\Rightarrow$ True for $n = 1$

❸ P(k):

Assume hypothesis for $n = k$ is true

$$\Rightarrow \quad 1 + 3 + 6 + 10 + \ldots + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$$

❹ P($k + 1$):

Test hypothesis for $n = k + 1$

To Prove:

$$1 + 3 + 6 + 10 + \ldots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)(k+3)}{6}$$

Proof:

$$1 + 3 + 6 + 10 + \ldots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2}$$

$$= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6}$$

$$= \frac{(k+1)(k+2)(k+3)}{6}$$

$\Rightarrow$ True for $n = k + 1$

So, P($k + 1$) is true whenever P(k) is true.

Since P(1) is true, then by induction P(n) is true for any positive integer n ($n \in \mathbb{N}$).

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correctly P(1) step and stops.
Mid partial credit: (6 marks)	–	Writes down correctly P(1) and P(k) or P(k + 1) steps.
High partial credit: (8 marks)	–	Writes down correctly P(1) step and P(k) and uses P(k) to prove P(k + 1) step, but fails to finish or finish incorrectly.
	–	Writes down all steps correctly, but no conclusion given.

Question 7 (cont'd.)

7(e) The coefficients of a binomial expansion can be found using Pascal's triangle.

(i) Using Pascal's triangle, or otherwise, expand $(a + b)^4 + (a - b)^4$ and simplify. **(5C)**

Row 4: 1, 4, 6, 4, 1

$$\Rightarrow \quad (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$\Rightarrow \quad \underline{(a-b)^4 = a^4 - 4a^3b + 6a^2b^3 - 4ab^3 + b^4}$$
$$\Rightarrow \quad (a+b)^4 + (a-b)^4 = 2a^4 + 12a^2b^2 + 2b^4$$
$$= 2(a^4 + 6a^2b^2 + b^4)$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* expands $(a + b)^4$ <u>or</u> $(a - b)^4$ correctly <u>and stops</u>.
High partial credit: (4 marks)	–	Expands both $(a + b)^4$ <u>and</u> $(a - b)^4$ correctly, but fails to find their sum <u>or</u> not fully simplified.

(ii) Hence, express $(x + \sqrt{x^2+1})^4 + (x - \sqrt{x^2+1})^4$ as a polynomial in terms of x. **(5C)**

$$(a+b)^4 + (a-b)^4 = 2(a^4 + 6a^2b^2 + b^4)$$

Let $a = x$ and $b = \sqrt{x^2+1}$

$$\Rightarrow \quad (x + \sqrt{x^2+1})^4 + (x - \sqrt{x^2+1})^4$$
$$= 2[x^4 + 6(x)^2(\sqrt{x^2+1})^2 + (\sqrt{x^2+1})^4]$$
$$= 2[x^4 + 6x^2(x^2 + 1) + (x^2 + 1)^2]$$
$$= 2[x^4 + 6x^4 + 6x^2 + x^4 + 2x^2 + 1]$$
$$= 2[8x^4 + 8x^2 + 1]$$
$$= 16x^4 + 16x^2 + 2$$

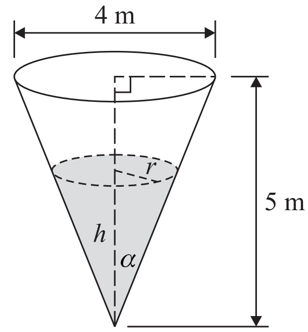
** Accept students' answers from part (ii) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* substitutes correctly x for a and $\sqrt{x^2+1}$ for b into $2(a^4 + 6a^2b^2 + b^4)$ <u>and stops</u>.
High partial credit: (4 marks)	–	Finds $2[x^4 + 6x^2(x^2 + 1) + (x^2 + 1)^2]$ <u>or</u> $2x^4 + 12x^2(x^2 + 1) + 2(x^2 + 1)^2$ correctly, but not fully simplified.

Question 8 **(50 marks)**

8(a) A grain silo is a tank used for the bulk storage of grain after it is harvested. A particular grain silo is in the shape of an inverted right cone, as shown. The vertical height of the cone is 5 m and the diameter of the base of the cone is 4 m.



Grain is pumped into an empty silo at a uniform rate of 4 m^3 per minute. Let h be the depth of the grain and r be the radius of the grain in the silo after t minutes.

(i) Using similar triangle, or otherwise, show that $r = \frac{2h}{5}$. **(5B)**

	Diameter of cone	=	4 m
⇒	radius of cone	=	2 m

From the diagram:

$$\frac{r}{2} = \frac{h}{5}$$

$$\Rightarrow \quad r = \frac{2h}{5}$$

... equiangular / similar triangles as both have common angle α, 90° angles and hence the third angles in both triangles are equal

Scale 5B (0, 2, 5)

Partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down $\tan\alpha = \frac{r}{h}$ <u>or</u> $\frac{2}{5}$ <u>and stops</u>.
	–	Explains why triangles are similar.

(ii) Find, in terms of π and h, the volume of grain in the silo after t minutes. **(5C)**

After t minutes:

$$V_{\text{grain}}(t) = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h$$

$$= \frac{1}{3}\pi\left(\frac{4h^2}{25}\right)h$$

$$= \frac{4\pi h^3}{75}\text{ m}^3$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for the volume of a cone with some substitution for r <u>and stops</u> [accept $r = 2$].
High partial credit: (4 marks)	–	Substitutes fully into volume formula *i.e.* $V_{\text{grain}}(t) = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h$, but fails to finish <u>or</u> finish incorrectly.

8(b) (i) Find, in terms of π, the rate at which the depth of grain is increasing when the depth of grain in the silo is 3 m. **(15D*)**

$$\frac{dV}{dt} = 4\text{ m}^3/\text{min}$$

$$V = \frac{4\pi h^3}{75}$$

$$\Rightarrow \frac{dV}{dh} = \frac{12\pi h^2}{75}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\Rightarrow 4 = \frac{12\pi h^2}{75} \times \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = 4 \times \frac{75}{12\pi h^2} = \frac{300}{12\pi h^2} = \frac{25}{\pi h^2}$$

@ $h = 3$

$$\Rightarrow \frac{dh}{dt} = \frac{25}{\pi(3)^2} = \frac{25}{9\pi}\text{ m/min}$$

** Accept students' answers from part (a)(ii) if not oversimplified.

Scale 15D* (0, 6, 10, 13, 15)

Low partial credit: (6 marks)	–	Any relevant first step, *e.g.* writes down $\frac{dV}{dt} = 4$ <u>or</u> $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ <u>and stops</u>.
	–	Some correct relevant differentiation *e.g.* $\frac{dV}{dh} = \frac{12\pi h^2}{75}$.
	–	Mentions a relevant rate of change *i.e.* $\frac{dV}{dt}$ <u>and/or</u> $\frac{dV}{dh}$ <u>and/or</u> $\frac{dh}{dt}$.
Mid partial credit: (10 marks)	–	Finds $4 = \frac{12\pi h^2}{75} \times \frac{dh}{dt}$ correctly, but fails to manipulate <u>or</u> manipulates incorrectly.
High partial credit: (13 marks)	–	Finds $\frac{dh}{dt} = \frac{75}{3\pi h^2}$, but fails to evaluate <u>or</u> evaluates incorrectly the rate of change when the depth of grain is 3 m.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('m/min') - apply only once in each section (a), (b), (c), *etc.* of question.

8(b) (cont'd.)

(ii) Find the rate at which the free surface of the grain is increasing when the radius is 1·5 m. **(10D*)**

Surface of the grain is a circle of radius r

$$S_{\text{grain}} = \pi r^2 = \pi\left(\frac{2h}{5}\right)^2 = \frac{4h^2\pi}{25}$$

$$\Rightarrow \frac{dS}{dh} = \frac{8h\pi}{25}$$

$$\frac{dh}{dt} = \frac{25}{\pi h^2} \quad \text{... answer from part (b)(i)}$$

$$\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt} = \frac{8h\pi}{25} \times \frac{25}{\pi h^2} = \frac{8}{h}$$

$$r = \frac{2h}{5} \quad \text{... given in part (a)}$$

$$\Rightarrow h = \frac{5r}{2}$$

$@\ r = 1\cdot 5$

$$h = \frac{5(1\cdot 5)}{2} = 3\cdot 75$$

$$\Rightarrow \frac{dS}{dt} = \frac{8}{3\cdot 75} = \frac{32}{15}\ \text{m}^2/\text{min} \quad \underline{\text{or}} \quad 2\cdot 1\dot{3}\ \text{m}^2/\text{min}$$

** Accept students' answers from part (b)(i) if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* substitutes $\frac{2h}{5}$ into area formula to find $S_{\text{grain}} = \frac{4h^2\pi}{25}$ <u>or</u> writes down $\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$ <u>and stops</u>.
	–	Mentions a relevant rate of change *i.e.* $\frac{dS}{dt}$ <u>and/or</u> $\frac{dS}{dh}$ <u>and/or</u> $\frac{dh}{dt}$.
Mid partial credit: (6 marks)	–	Correct relevant differentiation *e.g.* $\frac{dV}{dh} = \frac{12\pi h^2}{75}$ <u>and stops</u> <u>or</u> continues incorrectly.
High partial credit: (8 marks)	–	Finds $\frac{dS}{dt} = \frac{8h\pi}{25} \times \frac{25}{\pi h^2}$ <u>or</u> $\frac{8}{h}$, but fails to finish <u>or</u> finishes incorrectly.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('m/min') - apply only once in each section (a), (b), (c), *etc.* of question.

Question 8 (cont'd.)

8(c) The company which manufactures these grain silos wishes to minimise the amount of sheet metal required to produce each one while retaining the same capacity (volume) of the tank.

(i) Express the curved surface area of the silo in term of π and *h*. **(10D)**

$$V_{\text{silo}} = \frac{1}{3}\pi(2)^2(5)$$
$$= \frac{20\pi}{3}\text{ m}^3$$
$$\Rightarrow \quad V_{\text{optimum silo}} = \frac{1}{3}\pi R^2 H$$
$$= \frac{20\pi}{3}$$
$$\Rightarrow \quad \frac{1}{3}\pi R^2 H = \frac{20\pi}{3}$$
$$\Rightarrow \quad R^2 H = 20$$
$$\Rightarrow \quad R^2 H = 20$$
$$\Rightarrow \quad R^2 = \frac{20}{H}$$
$$\text{CSA} = \pi RL$$
$$= \pi R\sqrt{R^2 + H^2}$$
$$= \pi\sqrt{\frac{20}{H}}\sqrt{\frac{20}{H} + H^2}$$
$$= \pi\sqrt{\frac{400}{H^2} + \frac{20H^2}{H}}$$
$$= \pi\sqrt{\frac{400}{H^2} + 20H}$$
$$= \pi(400H^{-2} + 20H)^{\frac{1}{2}}$$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* calculates correct volume of cone [ans. $\frac{20\pi}{3}$].
	–	Equates volume of cone to optimum cone, but fails to find $R^2 = \frac{20}{H}$ <u>or</u> $R^2H = 20$.
Mid partial credit: (6 marks)	–	Equates volume of cone to optimum cone and find $R^2 = \frac{20}{H}$ <u>or</u> $R^2H = 20$ <u>and stops</u> <u>or</u> continues incorrectly.
High partial credit: (8 marks)	–	Finds CSA = $\pi\sqrt{\frac{20}{H}}\sqrt{\frac{20}{H} + H^2}$, but fails to finish <u>or</u> finishes incorrectly.

8(c) (cont'd.)

(ii) Hence, find the value of the radius that minimises the curved surface area of the grain silo, correct to two decimal places. **(5D*)**

$$\text{CSA} = \pi(400H^{-2} + 20H)^{\frac{1}{2}}$$

$$\Rightarrow \frac{d}{dh}(\text{CSA}) = \frac{1}{2}\pi(400H^{-2} + 20H)^{-\frac{1}{2}}.[(-2)400H^{-3} + 20]$$

$$= 0$$

$$\Rightarrow \frac{-400H^{-3} + 10}{\sqrt{400H^{-2} + 20H}} = 0 \quad \text{... for minimum surface}$$

$$\Rightarrow -400H^{-3} + 10 = 0$$

$$\Rightarrow \frac{400}{H^3} = 10$$

$$\Rightarrow H^3 = \frac{400}{10}$$

$$= 40$$

$$\Rightarrow H = \sqrt[3]{40}$$

$$\Rightarrow R^2 = \frac{20}{\sqrt[3]{40}}$$

$$= 5{\cdot}848035...$$

$$\Rightarrow R = 2{\cdot}418271...$$

$$\cong 2{\cdot}42 \text{ m}$$

** Accept students' answers from part (c)(i) if not oversimplified.

Scale 5D* (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down '$\frac{d}{dh}$(CSA) = 0 for minimum surface' or equivalent and stops.
Mid partial credit: (3 marks)		Differentiates correctly to find $\frac{d}{dh}$(CSA) and stops or continues incorrectly.
High partial credit: (4 marks)	–	Solves correctly for $H = \sqrt[3]{40}$, but fails to find or finds incorrect value for r.

* Deduct 1 mark off correct answer only ❶ if final answer(s) are not rounded or incorrectly rounded or ❷ for the omission of or incorrect use of units ('m') - apply only once to each section (a), (b), (c), *etc.* of question.

Question 9 **(50 marks)**

(a) The acceleration of a particle, in m s^{-2}, moving in a straight line during a particular time interval, is given by:

$$a = \frac{1}{t^2} + 3t, \quad \text{for } 1 \le t \le 5,$$

where t is the time, in seconds, from the instant the particle begins to move.

(i) Given that the speed of the particle is $\frac{1}{2}$ after 1 s, find its speed after 5 s. **(10D*)**

$$a = \frac{1}{t^2} + 3t$$
$$= t^{-2} + 3t$$
$$\Rightarrow \frac{dv}{dt} = t^{-2} + 3t$$
$$\Rightarrow \int \left(\frac{dv}{dt}\right)dt = \int (t^{-2} + 3t)\,dt$$
$$\Rightarrow \int dv = \int (t^{-2} + 3t)\,dt$$
$$v = \frac{t^{-1}}{-1} + \frac{3t^2}{2} + c$$
$$= \frac{3t^2}{2} - \frac{1}{t} + c$$

when $t = 1$, $v = \frac{1}{2}$

$$\Rightarrow \frac{1}{2} = \frac{3}{2}(1)^2 - \frac{1}{1} + c$$
$$\Rightarrow \frac{1}{2} = \frac{3}{2} - 1 + c$$
$$\Rightarrow c = \frac{1}{2} - \frac{3}{2} + 1$$
$$= 0$$

when $t = 5$

$$v(t) = \frac{3t^2}{2} - \frac{1}{t}$$
$$\Rightarrow v(5) = \frac{3}{2}(5)^2 - \frac{1}{5}$$
$$= \frac{75}{2} - \frac{1}{5}$$
$$= 37{\cdot}5 - 0{\cdot}2$$
$$= 37{\cdot}3 \text{ m/s}$$

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* replaces a with $\frac{dv}{dt}$ <u>and stops</u>.
	–	Some correct integration <u>and stops</u> <u>or</u> continues incorrectly.
Mid partial credit: (6 marks)	–	Finds $v = \frac{t^{-1}}{-1} + \frac{3t^2}{2} + c$ <u>or</u> $\frac{3t^2}{2} - \frac{1}{t} + c$ <u>and stops</u> <u>or</u> continues incorrectly.
High partial credit: (8 marks)	–	Finds correct expression for v, *i.e.* $v(t) = \frac{3t^2}{2} - \frac{1}{t}$, but fails to evaluate <u>or</u> evaluates incorrectly for $t = 5$.

* Note: If arithmetic error only, award 9 marks.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('m/s') - apply only once in each section (a), (b), (c), *etc*. of question.

Question 9 (cont'd.)

9(a) (cont'd.)

(ii) Find the average speed of the particle over the interval $1 \le t \le 5$.
Give your answer correct to two decimal places. **(10D*)**

Average value of $f(x)$ in the interval $[a, b]$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$v(t) = \frac{3t^2}{2} - \frac{1}{t} \quad \text{... from part (a)(i)}$$

$$\Rightarrow \text{Average speed} = \frac{1}{5-1}\int_1^5 \left(\frac{3t^2}{2} - \frac{1}{t}\right) dx$$

$$= \frac{1}{4}\left[\frac{3t^3}{6} - \ln|t|\right]\Big|_1^5$$

$$= \frac{1}{4}\left[\frac{3}{6}(5)^3 - \ln|5|\right] - \frac{1}{4}\left[\frac{3}{6}(1)^3 - \ln|1|\right]$$

$$= \frac{1}{4}\left[\frac{375}{6} - \ln|5|\right] - \frac{1}{4}\left[\frac{3}{6} - 0\right]$$

$$= \frac{1}{4}\left[\frac{372}{6} - \ln|5|\right]$$

$$= \frac{1}{4}[62 - \ln|5|]$$

$$= \frac{1}{4}[62 - \ln|5|]$$

$$= 15{\cdot}097640...$$

$$\cong 15{\cdot}10 \text{ m/s}$$

** Accept students' answers from part (a)(ii) if not oversimplified.

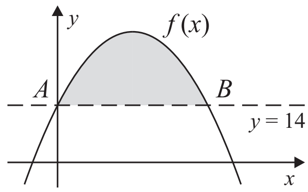
Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down relevant formula for the average value of a function. – Integrates one term correctly.
Mid partial credit: (6 marks)	–	Integrates both terms correctly, but excludes $\frac{1}{b-a}$ from calculation.
High partial credit: (8 marks)	–	Integrates correctly, *i.e.* average speed $= \frac{1}{4}[\frac{3t^3}{6} - \ln|t|]$ or $\frac{1}{4}[\frac{3t^3}{6} - \ln|t|]\Big|_1^5$, but fails to evaluate or evaluates incorrectly or evaluates using incorrect limits.

* Deduct 1 mark off correct answer only ❶ if final answer(s) are not rounded or incorrectly rounded or ❷ for the omission of or incorrect use of units ('m') - apply only once to each section (a), (b), (c), *etc.* of question.

Question 9 (cont'd.)

9(b) The proposed level of new roadways is achieved primarily through series of 'cuts' and 'fills', taking earth material from one area and using it somewhere else.



The diagram shows the vertical cross-section of a roadway through a particular terrain. The proposed elevation of the roadway is 14 m above sea-level and therefore a cut is required between points *A* and *B*.

Using the co-ordinate plane with the *y*-axis as the initial point of the cut and the *x*-axis as sea-level, the elevation of the terrain can be described by the function

$$f(x) = 32 - 2(x - 3)^2,$$

where both x and $f(x)$ are measured in metres.

(i) Find the co-ordinates of *A* and *B*. **(5C)**

$$f(x) = 32 - 2(x-3)^2 = 14$$
$$\Rightarrow 32 - 2(x-3)^2 = 14$$
$$\Rightarrow 2(x-3)^2 = 32 - 14 = 18$$
$$\Rightarrow (x-3)^2 = \frac{18}{2} = 9$$
$$\Rightarrow x - 3 = \sqrt{9}$$

$$\Rightarrow x - 3 = -3 \qquad \Rightarrow x - 3 = 3$$
$$\Rightarrow x = -3 + 3 = 0 \qquad x = 3 + 3 = 6$$
$$\Rightarrow A = (0, 14) \qquad \Rightarrow B = (6, 14)$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* equates $32 - 2(x-3)^2 = 14$ or similar <u>and stops</u>.
High partial credit: (4 marks)	–	Finds only one value of x correctly and hence finds only co-ordinates of *A* <u>or</u> *B*.
	–	Finds both values of x correctly, but fails to give co-ordinates of *A* and *B*.

Question 9 (cont'd.)

9(b) (cont'd.)

(ii) Use the trapezoidal rule and interval widths of 1 m to find the approximate area of the shaded cross-section of earth material to be excavated between the elevation of the terrain and the proposed elevation of the roadway. **(5C*)**

$f(x) = 32 - 2(x-3)^2$

x	0	1	2	3	4	5	6
$y = f(x)$	14	24	30	32	30	24	14

❶ Area under curve of $y = f(x)$

$= \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + y_4 + ... + y_{n-1})]$

$= \frac{1}{2}(1)[14 + 14 + 2(24 + 30 + 32 + 30 + 24)]$

$= (0 \cdot 5)[28 + 2(140)]$

$= (0 \cdot 5)[28 + 280)]$

$= (0 \cdot 5)[308]$

$= 154 \text{ m}^2$

❷ Shaded area = Area above – Area of rectangular box between $y = 14$ and x-axis

$= 154 - (6 \times 14)$

$= 154 - 84$

$= 70 \text{ m}^2$

Scale 5C* (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any work of merit, *e.g.* writes down correct formula for trapezoidal rule with some correct substitution <u>and stops</u>.
	–	Finds $f(x)$ for $x = 1, 2, 3, 4, 5, 6$ <u>and stops</u>.
Mid partial credit: (3 marks)	–	Fully correct substitution into trapezoidal rule, but fails to find correct value for area under the curve.
High partial credit: (4 marks)	–	Finds correct area under curve but fails to finish <u>or</u> finishes incorrectly.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect units - apply only once throughout the question.

Question 9 (cont'd.)

9(b) (cont'd.)

(iii) Use integration to find the actual area of the shaded cross-section. **(10D*)**

❶ Actual area under curve of $y = f(x)$

$$= \int_0^6 f(x)\,dx$$

$$\begin{aligned} f(x) &= 32 - 2(x-3)^2 \\ &= 32 - 2(x^2 - 6x + 9) \\ &= -2x^2 + 12x + 32 - 18 \\ &= -2x^2 + 12x + 14 \end{aligned}$$

Actual area under curve of $y = f(x)$

$$\begin{aligned} &= \int_0^6 (-2x^2 + 12x + 14)\,dx \\ &= -\frac{2x^3}{3} + \frac{12x^2}{2} + 14x \Big|_0^6 \\ &= -\frac{2}{3}(6)^3 + 6(6)^2 + 14(6) - 0 \\ &= -144 + 216 + 84 \\ &= -144 + 216 + 84 \\ &= 156 \text{ m}^2 \end{aligned}$$

❷ Shaded area = Area above – Area of rectangular box between $y = 14$ and x-axis

$$\begin{aligned} &= 156 - (6 \times 14) \\ &= 156 - 84 \qquad \text{... from part (b)(ii)} \\ &= 72 \text{ m}^2 \end{aligned}$$

** Accept students' answers from part (b)(ii) if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* simplifies correctly $f(x) = -2x^2 + 12x + 14$ <u>or</u> gives area = $\int_0^6 [32 - 2(x-3)^2]\,dx$ <u>and stops</u>.
Mid partial credit: (6 marks)	–	Simplifies and integrates $f(x)$ correctly, but fails to evaluate <u>or</u> evaluates incorrectly <u>or</u> evaluates using incorrect limits.
High partial credit: (8 marks)	–	Finds correct area under curve but fails to finish <u>or</u> finishes incorrectly.

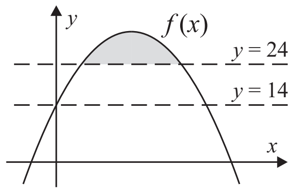
* Deduct 1 mark off correct answer only ❶ if final answer(s) are not rounded <u>or</u> incorrectly rounded <u>or</u> ❷ for the omission of <u>or</u> incorrect use of units ('m') - apply only once to each section (a), (b), (c), *etc*. of question.

Question 9 (cont'd.)

9(c) An alternative proposal is to construct the new roadway at an elevation of 24 m above sea-level. Find, correct to two decimal places, the percentage reduction in the cross-section of material to be excavated if this proposal was adopted. **(10D*)**

❶

$$\begin{aligned} f(x) &= 32 - 2(x-3)^2 \\ &= 24 \\ \Rightarrow 32 - 2(x-3)^2 &= 24 \\ \Rightarrow 2(x-3)^2 &= 32 - 24 \\ &= 8 \\ \Rightarrow (x-3)^2 &= 4 \\ \Rightarrow x - 3 &= \sqrt{4} \end{aligned}$$



$$\begin{aligned} \Rightarrow x - 3 &= -2 \qquad \Rightarrow \quad & x - 3 &= 2 \\ \Rightarrow x &= -2 + 3 & x &= 2 + 3 \\ &= 1 & &= 5 \end{aligned}$$

❷ Area under curve of $y = f(x)$ above $y = 24$

$$\begin{aligned} &= \int_1^5 f(x)\,dx - [24 \times (5-1)] \\ &= \int_1^5 (-2x^2 + 12x + 14)\,dx - 96 \qquad \text{... from part (b)(iii)} \\ &= -\frac{2x^3}{3} + \frac{12x^2}{2} + 14x \Big|_1^5 - 96 \\ &= -\frac{2}{3}(5)^3 + 6(5)^2 + 14(5) - [-\frac{2}{3}(1)^3 + 6(1)^2 + 14(1)] - 96 \\ &= -\frac{250}{3} + 150 + 70 + \frac{2}{3} - 6 - 14 - 96 \\ &= 104 - \frac{248}{3} \\ &= \frac{312 - 248}{3} \\ &= \frac{64}{3}\ \text{m}^2 \end{aligned}$$

❸ Percentage reduction in excavated material

$$\begin{aligned} \%\ \text{Reduction} &= \frac{72 - \frac{64}{3}}{72} \times \frac{100}{1} \\ &= \frac{\frac{152}{3}}{72} \times \frac{100}{1} \\ &= 70{\cdot}370370... \\ &\cong 70{\cdot}37\% \end{aligned}$$

** Accept students' answers from part (b)(ii) if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* equates $32 - 2(x-3)^2 = 24$ with work towards finding values of x.
Mid partial credit: (6 marks)	–	Simplifies and integrates $f(x)$ correctly, with some substitution of limits, but fails to evaluates correctly <u>or</u> evaluates using incorrect limits.
High partial credit: (8 marks)	–	Finds correct area under curve but fails to find <u>or</u> finishes incorrect % reduction.

* Deduct 1 mark off correct answer only if not rounded <u>or</u> incorrectly rounded - apply only once in each section (a), (b), (c), *etc.* of question.

Notes:

DEB exams

Pre-Leaving Certificate Examination, 2017

Mathematics

Higher Level – Paper 2
Marking Scheme (300 marks)

Structure of the Marking Scheme

Students' responses are marked according to different scales, depending on the types of response anticipated.
Scales labelled A divide students' responses into two categories (correct and incorrect).
Scales labelled B divide responses into three categories (correct, partially correct, and incorrect), and so on.
These scales and the marks that they generate are summarised in the following table:

Scale label	A	B	C	D
No. of categories	2	3	4	5
5 mark scale		**0, 2, 5**	**0, 2, 4, 5**	**0, 2, 3, 4, 5**
10 mark scale			**0, 4, 7, 10**	**0, 4, 6, 8, 10**
15 mark scale				**0, 6, 10, 13, 15**

A general descriptor of each point on each scale is given below. More specific directions in relation to interpreting the scales in the context of each question are given in the scheme, where necessary.

Marking scales – level descriptors

A-scales (two categories)
- incorrect response (no credit)
- correct response (full credit)

B-scales (three categories)
- response of no substantial merit (no credit)
- partially correct response (partial credit)
- correct response (full credit)

C-scales (four categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

D-scales (five categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response about half-right (middle partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

In certain cases, typically involving ❶ incorrect rounding, ❷ omission of units, ❸ a misreading that does not oversimplify the work or ❹ an arithmetical error that does not oversimplify the work, a mark that is one mark below the full-credit mark may also be awarded. Such cases are flagged with an asterisk.
Thus, for example, scale **10C*** indicates that 9 marks may be awarded.
- The * for units to be applied only if the student's answer is fully correct.
- The * to be applied once only **within each section (a), (b), (c), *etc.*** of all questions.
- The * penalty is not applied to currency solutions.

Unless otherwise specified, accept correct answer with or without work shown.

Accept students' work in one part of a question for use in subsequent parts of the question, unless this oversimplifies the work involved.

Summary of Marks – 2017 LC Maths (Higher Level, Paper 2)

Q	Part	Sub	Marks
Q.1	**(a)**		10D (0, 4, 6, 8, 10)
	(b)		10D (0, 4, 6, 8, 10)
	(c)		5B (0, 2, 5)
			25

Q	Part	Sub	Marks
Q.2	**(a)**	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5C (0, 2, 4, 5)
	(b)	**(i)**	5C (0, 2, 4, 5)
		(ii)	5C (0, 2, 4, 5)
			25

Q	Part	Sub	Marks
Q.3	**(a)**		5C (0, 2, 4, 5)
	(b)		10D (0, 4, 6, 8, 10)
	(c)		10D (0, 4, 6, 8, 10)
			25

Q	Part	Sub	Marks
Q.4	**(a)**	**(i)**	5C (0, 2, 4, 5)
		(ii)	5C (0, 2, 4, 5)
		(iii)	5C (0, 2, 4, 5)
	(b)		5C (0, 2, 4, 5)
	(c)		5C (0, 2, 4, 5)
			25

Q	Part	Sub	Marks
Q.5	**(a)**		5C (0, 2, 4, 5)
	(b)	**(i)**	5C (0, 2, 4, 5)
		(ii)	10D (0, 4, 6, 8, 10)
	(c)		5D (0, 2, 3, 4, 5)
			25

Q	Part	Sub	Marks
Q.6	**(a)**	**(i)**	5D* (0, 2, 3, 4, 5)
		(ii)	10D* (0, 4, 6, 8, 10)
	(b)	**(i)**	5C (0, 2, 4, 5)
		(ii)	5C (0, 2, 4, 5)
			25

Q	Part	Sub	Marks
Q.7	**(a)**	**(i)**	5B (0, 2, 5)
		(ii)	10C (0, 4, 7, 10)
		(iii)	5C (0, 2, 4, 5)
	(b)	**(i)**	5C (0, 2, 4, 5)
		(ii)	10D* (0, 4, 6, 8, 10)
		(iii)	5C* (0, 2, 4, 5)
	(c)		10D* (0, 4, 6, 8, 10)
			50

Q	Part	Sub	Marks
Q.8	**(a)**	**(i)**	5C (0, 2, 4, 5)
		(ii)	10C (0, 4, 7, 10)
	(b)	**(i)**	15D (0, 6, 10, 13, 15)
		(ii)	5C (0, 2, 4, 5)
	(c)	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5D (0, 2, 3, 4, 5)
			50

Q	Part	Sub	Marks
Q.9	**(a)**	**(i)**	10C (0, 4, 7, 10)
		(ii) **(iii)**	5C (0, 2, 4, 5)
		(iv)	10C (0, 4, 7, 10)
	(b)		15D (0, 6, 10, 13, 15)
	(c)		10D (0, 4, 6, 8, 10)
			50

Assumptions about these marking schemes on the basis of past SEC marking schemes should be avoided. While the underlying assessment principles remain the same, the exact details of the marking of a particular type of question may vary from a similar question asked by the SEC in previous years in accordance with the contribution of that question to the overall examination in the current year. In setting these marking schemes, we have strived to determine how best to ensure the fair and accurate assessment of students' work and to ensure consistency in the standard of assessment from year to year. Therefore, aspects of the structure, detail and application of the marking schemes for these examinations are subject to change from past SEC marking schemes and from one year to the next without notice.

General Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer all questions.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Pre-Leaving Certificate Examination, 2017

Mathematics

Higher Level – Paper 2
Marking Scheme (300 marks)

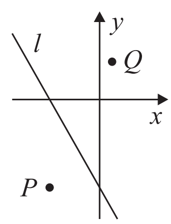
Section A	Concepts and Skills	150 marks

Answer **all six** questions from this section.

Question 1 **(25 marks)**

Two points $P(-4, -7)$ and $Q(1, 3)$ lie on opposite sides of the line l: $2x + y + 7 = 0$.

1(a) Calculate the ratio of the shortest distances from P and Q to line l. **(10D)**



$$|d_{\min}| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$P(-4, -7)$, l: $2x + y + 4 = 0$

$$\Rightarrow \quad |Pl_{\min}| = \frac{|2(-4) + 1(-7) + 7|}{\sqrt{(2)^2 + (1)^2}} = \frac{|-8 - 7 + 7|}{\sqrt{4 + 1}} = \frac{8}{\sqrt{5}}$$

$Q(1, 3)$, l: $2x + y + 7 = 0$

$$\Rightarrow \quad |Ql_{\min}| = \frac{|2(1) + 1(3) + 7|}{\sqrt{(2)^2 + (1)^2}} = \frac{|2 + 3 + 7|}{\sqrt{4 + 1}} = \frac{12}{\sqrt{5}}$$

$$\Rightarrow \quad |Pl_{\min}| : |Ql_{\min}| = \frac{8}{\sqrt{5}} : \frac{12}{\sqrt{5}} = 8 : 12 = 2 : 3$$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	– –	Any relevant first step, *e.g.* writes down shortest distance = ⊥ distance with formula stated. Some correct substitution into formula for ⊥ distance (a, b, c identified).
Mid partial credit: (6 marks)	–	Finds correct $\lvert Pl_{\min}\rvert$ <u>or</u> $\lvert Ql_{\min}\rvert$.
High partial credit: (8 marks)	–	Finds both $\lvert Pl_{\min}\rvert$ <u>or</u> $\lvert Ql_{\min}\rvert$, but fails to finish <u>or</u> finishes incorrectly.

1(b) Calculate the ratio of the distances from P and Q to line l along the line $[PQ]$. **(10D)**

❶ Slope of PQ

$P(-4, -7),\ Q(1, 3)$

$$\Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow m_{PQ} \text{ (slope of } PQ) = \frac{3-(-7)}{1-(-4)} = \frac{10}{5} = 2$$

❷ Equation of PQ

$Q(1, 3),\ m_{PQ} = 2$

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - (3) = 2(x - 1)$$
$$\Rightarrow y - 3 = 2x - 2$$
$$\Rightarrow 2x - y = -3 + 2$$
$$\Rightarrow 2x - y = -1$$

❸ $PQ \cap l$

$$2x + y = -7$$
$$2x - y = -1$$
$$\Rightarrow 4x = -8$$
$$\Rightarrow x = -2$$

$$2x + y = -7$$
$$\Rightarrow 2(-2) + y = -7$$
$$\Rightarrow y = -7 + 4 = -3$$

or

$$2x - y = -1$$
$$\Rightarrow 2(-2) - y = -1$$
$$\Rightarrow -y = -1 + 4 = 3$$
$$\Rightarrow y = -3$$

$\Rightarrow$ point of intersctoion $R(-2, -3)$

❹ Distances $|PR|$ and $|QR|$

$$\Rightarrow |d| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$P(-4, -7),\ R(-2, -3)$

$$\Rightarrow |PR| = \sqrt{(-2-(-4))^2 + (-3-(-7))^2} = \sqrt{(2)^2 + (4)^2} = \sqrt{20} = 2\sqrt{5}$$

$Q(1, 3),\ R(-2, -3)$

$$\Rightarrow |QR| = \sqrt{(-2-1)^2 + (-3-3)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45} = 3\sqrt{5}$$

❺ Ratio

$$|PR| : |QR| = 2\sqrt{5} : 3\sqrt{5} = 2 : 3$$

Question 1 (cont'd.)

1(b) (cont'd.)

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct relevant formula for slope, equation of a line <u>or</u> distance.
	–	Finds correct slope of PQ <u>and stops</u>.
	–	Finds incorrect slope of PQ and some correct substitution into the formula for the equation of the line PQ.
Mid partial credit: (6 marks)	–	Finds correct equation for PQ <u>and stops</u>.
	–	Finds incorrect equation for PQ and continues with substantial work towards finding point of intersection of PQ and l.
High partial credit: (8 marks)	–	Finds correct point of intersection of PQ and l <u>and</u> continues with substantial work towards finding distances of P and Q to l.
	–	Finds correct distances of P and Q to l, but fails to finish <u>or</u> finishes incorrectly, *e.g.* $\lvert PR\rvert : \lvert QR\rvert = \sqrt{20} : \sqrt{45}$.

1(c) What conclusion can you draw from your answers to parts **(a)** and **(b)** above?
Explain your answer with reference to a geometric theorem on your course. **(5B)**

❶ <u>Conclusion</u>

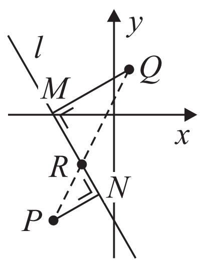
as $\lvert Pl_{\text{min}}\rvert : \lvert Ql_{\text{min}}\rvert = 2 : 3$

$\Rightarrow \dfrac{\lvert QM\rvert}{\lvert PN\rvert} = \dfrac{3}{2}$

and $\lvert PR\rvert : \lvert QR\rvert = 2 : 2$

$\Rightarrow \dfrac{\lvert QR\rvert}{\lvert PR\rvert} = \dfrac{3}{2}$

$\Rightarrow$ ΔQMR and ΔPNR are similar (equiangular)



❷ <u>Geometric theorem</u>

– if two triangles are similar, then their corresponding sides are in proportion (Theorem 13)

** Accept students' answers from parts (a) and (b) if not oversimplified.

Scale 5B (0, 2, 5)

Partial credit: (2 marks)	–	Any relevant first step, *e.g.* mentions similar triangles <u>and stops</u>.

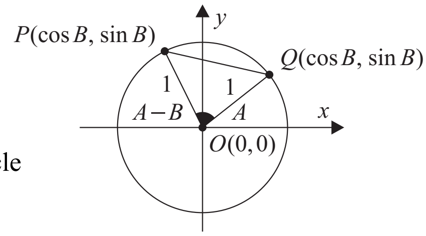
Question 2 **(25 marks)**

2(a) **(i)** Prove that $\cos(A - B) = \cos A \cos B + \sin A \sin B$. **(10D)**

s is a circle with centre $O(0, 0)$ and radius 1

Let $P(\cos A, \sin A)$ be any point on the circle such that $[OP]$ makes an angle A with the positive sense of the x-axis

Let $Q(\cos B, \sin B)$ be another point on the circle such that $[OQ]$ makes an angle B with the positive sense of the x-axis



❶ Using distance formula:

$\Rightarrow \quad |PQ| = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$

$\Rightarrow \quad |PQ|^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$

$= \cos^2 A - 2\cos A \cos B - \cos^2 B + \sin^2 A - 2\sin A \sin B - \sin^2 B$

$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2[\cos A \cos B + \sin A \sin B]$

$= 1 + 1 - 2[\cos A \cos B + \sin A \sin B]$

$= 2 - 2[\cos A \cos B + \sin A \sin B]$

❷ Using cosine rule:

$a^2 = b^2 + c^2 - 2bc \cos A$

$\Rightarrow \quad |PQ|^2 = |OP|^2 + |OQ|^2 - 2|OP|.|OQ|\cos|\angle POQ|$

$\Rightarrow \quad |PQ|^2 = 1^2 + 1^2 - 2(1)(1)\cos(A - B)$

$= 2 - 2\cos(A - B)$

❸ Equating results from ❶ and ❷:

$2 - 2\cos(A - B) = 2 - 2[\cos A \cos B + \sin A \sin B]$

$\Rightarrow \quad 2\cos(A - B) = 2[\cos A \cos B + \sin A \sin B]$

$\Rightarrow \quad \cos(A - B) = \cos A \cos B + \sin A \sin B$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* draws correct diagram with co-ordinates of P and/or Q indictaed and stops.
	–	Some correct substitution into either distance formula or cosine rule.
Mid partial credit: (6 marks)	–	Finds one correct expression for $\lvert PQ\rvert^2$ or $\lvert PQ\rvert$.
	–	Some correct substitution into both distance formula and cosine rule, but incomplete.
High partial credit: (8 marks)	–	Finds correct both expressions for $\lvert PQ\rvert^2$ and $\lvert PQ\rvert$, but fails to finish or finishes incorrectly.
	–	Proof complete with one critical step omitted or incorrect.

Question 2 (cont'd.)

2(a) (cont'd.)

(ii) Hence, show that $\cos 15° = \dfrac{\sqrt{2}+\sqrt{6}}{4}$, without using a calculator. **(5C)**

$$\begin{aligned}
\cos(A-B) &= \cos A\cos B + 2\sin A\sin B\\
\cos 15° &= \cos(60° - 45°)\\
&= \cos 60°\cos 45° + \sin 45°\sin 45°\\
&= \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}\\
&= \frac{1}{2}.\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}.\frac{\sqrt{2}}{2}\\
&= \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* identifies angles $A = 60°$ and $B = 45°$ <u>or</u> $A = 45°$ and $B = 30°$.
	–	Finds $\cos 15° = \cos(60° - 45°)$ <u>and stops</u>.
High partial credit: (4 marks)	–	Finds $\cos 15° = \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}$, <u>or equivalent</u>, but fails to finish <u>or</u> finishes incorrectly.
	–	Finds $\cos 15° = \frac{1+\sqrt{3}}{2\sqrt{2}}$ <u>and stops</u> <u>or</u> continues incorrectly.

(b) **(i)** Given that $\cos(A-B) = 2\cos(A+B)$, show that $3\tan A = \dfrac{1}{\tan B}$. **(5C)**

❶ $\cos(A-B) = \cos A\cos B + \sin A\sin B$ ... from part (a)(i)

❷
$$\begin{aligned}
\cos(A+B) &= \cos A\cos B - \sin A\sin B\\
2\cos(A+B) &= 2(\cos A\cos B - \sin A\sin B)\\
&= 2\cos A\cos B - 2\sin A\sin B
\end{aligned}$$

❸ Equating results from ❶ and ❷:

$$\begin{aligned}
\cos A\cos B + \sin A\sin B &= 2\cos A\cos B - 2\sin A\sin B\\
\Rightarrow\quad 3\sin A\sin B &= \cos A\cos B\\
\Rightarrow\quad 3\frac{\sin A\sin B}{\cos A\cos B} &= 1\\
\Rightarrow\quad 3\tan A\tan B &= 1\\
\Rightarrow\quad 3\tan A &= \frac{1}{\tan B}
\end{aligned}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* expands $2\cos(A+B)$ correctly <u>and stops</u>.
High partial credit: (4 marks)	–	Equates correctly both sides, *i.e.* finds $3\sin A\sin B = \cos A\cos B$, but fails to finish <u>or</u> finishes incorrectly.

Question 2 (cont'd.)

2(b) (cont'd.)

(ii) Hence, solve the equation $\cos(\theta - \frac{\pi}{6}) = 2\cos(\theta + \frac{\pi}{6})$, where $0 \leq \theta \leq 2\pi$. **(5C)**

For $\cos(A - B) = 2\cos(A + B)$

$3\tan A = \frac{1}{\tan B}$ ... from part (b)(i)

Let $A = \theta$ and $B = \frac{\pi}{6}$

$\Rightarrow \quad 3\tan\theta = \frac{1}{\tan\frac{\pi}{6}}$

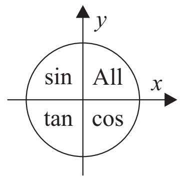
$= \frac{1}{\frac{1}{\sqrt{3}}}.$

$= \sqrt{3}$

$\Rightarrow \quad \tan\theta = \frac{\sqrt{3}}{3}$

$= \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \tan^{-1}\frac{1}{\sqrt{3}}$



as $0 \leq \theta \leq 2\pi$.

$\theta = 30°$ or $\frac{\pi}{6}$ ... 1st quadrant

and $\theta = 180° + 30°$

$= 210°$ or $\frac{7\pi}{6}$ ... 3rd quadrant

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down $A = \theta$ and/or $B = \frac{\pi}{6}$ and stops.
	–	Finds $3\tan\theta = \frac{1}{\tan\frac{\pi}{6}}$ and stops or continues incorrectly.
High partial credit: (4 marks)	–	Finds $\tan\theta = \frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$, but fails to finish or finishes incorrectly. Finds one solution only.

Question 3 **(25 marks)**

Circle s: $x^2 + y^2 + 2gx + 2fy + c = 0$ touches the y-axis at the point $A(0, -2)$.

3(a) Write down the value of f and hence, show that c is equal to 4. **(5C)**

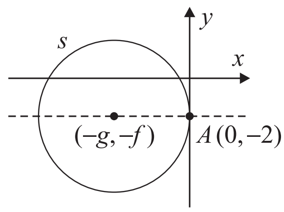
❶ s: $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $(-g, -f)$

as y-axis is a tangent at $A(0, -2)$

$\Rightarrow$ centre lies on the line $y = -2$

$\Rightarrow$ $-f = -2$

$\Rightarrow$ $f = 2$



❷ $A(0, -2) \in s$: $x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow$ $0^2 + (-2)^2 + 2g(0) + 2(2)(-2) + c = 0$

$\Rightarrow$ $4 - 8 + c = 0$

$\Rightarrow$ $c = 8 - 4$

$= 4$

Scale 5C (0, 2, 4, 5)

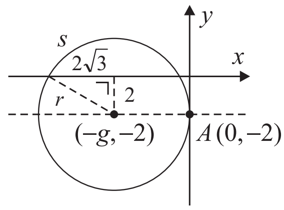
Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* substitutes $A(0, -2)$ into s, but fails to find correct value of f [ans. $4 - 4f + c = 0$].
	–	Finds $f = 2$ <u>and stops</u>.
High partial credit: (4 marks)	–	Finds $f = 2$ and substitutes $A(0, -2)$ into s, but fails to finish <u>or</u> finishes incorrectly.
	–	Finds $f = -2$ and substitutes $A(0, -2)$ into s <u>and</u> finishes correctly [ans. $c = 12$].

Question 3 (cont'd.)

3(b) The centre of s lies in the third quadrant and s makes a chord of length $4\sqrt{3}$ on the x-axis.
Find the value of g and hence, write down the equation of s. **(10D)**

❶ Let $[PQ]$ be the chord of circle on the x-axis

$\Rightarrow$ $|PQ| = 4\sqrt{3}$



<u>Using Pythagoras' theorem</u>

$$r^2 = (2)^2 + \left(\frac{4\sqrt{3}}{2}\right)^2$$
$$= 4 + (2\sqrt{3})^2$$
$$= 4 + 12$$
$$= 16$$
$$r = 4$$

❷ $s: x^2 + y^2 + 2gx + 2fy + c = 0$ has centre $(-g, -f)$ and radius r

$\Rightarrow$ $r = \sqrt{g^2 + f^2 - c}$

$\Rightarrow$ $r^2 = g^2 + f^2 - c$

$\Rightarrow$ $4^2 = g^2 + (-2)^2 - 4$

$\Rightarrow$ $16 = g^2 + 4 - 4$

$= g^2$

$\Rightarrow$ $g = \sqrt{16}$

$= \pm 4$

as centre of circle lies in 3rd quadrant [centre $(-g, -f)$]

$\Rightarrow$ $g = 4$

❸ $s: x^2 + y^2 + 2gx + 2fy + c = 0$

centre $(-g, -f) = (-4, -2)$

$c = 4$

$\Rightarrow$ $s: x^2 + y^2 + 2(4)x + 2(2)y + 4 = 0$

$\Rightarrow$ $s: x^2 + y^2 + 8x + 4y + 4 = 0$

** Accept students' answers from parts (a) if not oversimplified.

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* some correct use of Pythagoras' theorem with 2 <u>and</u> $2\sqrt{3}$, but fails to find correct value of r.
Mid partial credit: (6 marks)	–	Finds $r = 4$ and substitutes into formula $r = \sqrt{g^2 + f^2 - c}$ <u>or</u> $r^2 = g^2 + f^2 - c$, but fails to find correct value of g.
High partial credit: (8 marks)	– –	Finds $g = 4$, but fails to find <u>or</u> finds incorrect equation of s. Finds $g = -4$ <u>and</u> finishes correctly [ans. $c = 12$].

Question 3 (cont'd.)

3(c) Find the equations of the two tangents from the origin to s. **(10D)**

❶ Equation of first tangent

y-axis passes through (0, 0) and is a tangent to s at $A(0, -2)$

$\Rightarrow \quad t_1: x = 0$

❷ Equation of second tangent

point (0, 0), slope m

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 0 = m(x - 0)$$
$$\Rightarrow y = mx$$
$$\Rightarrow t_2: mx - y = 0$$

❸ $\perp$distance from centre of circle to tangent

centre of $s(-4, -2)$, $r = 4$

$\Rightarrow \quad \perp$distance to $t_2 = 4$

Perpendicular distance from a point (x_1, y_1) to line $ax + by + c = 0$

$$|d| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$
$$\Rightarrow 4 = \frac{|m(-4) - (-2) + 0|}{\sqrt{(m)^2 + (-1)^2}}$$
$$= \frac{|-4m + 2|}{\sqrt{m^2 + 1}}$$
$$\Rightarrow 4\sqrt{m^2 + 1} = |-4m + 2|$$
$$\Rightarrow 4^2(m^2 + 1) = (-4m + 2)^2$$
$$\Rightarrow 16m^2 + 16 = 16m^2 - 16m + 4$$
$$\Rightarrow 16m = 4 - 16$$
$$= -12$$
$$\Rightarrow m = -\frac{12}{16}$$
$$= -\frac{3}{4}$$

❹ Equation of t_2: $mx - y = 0$

$$\Rightarrow -\frac{3}{4}x - y = 0$$
$$\Rightarrow t_2: y = -\frac{3}{4}x \quad \underline{\text{or}} \quad 3x + 4y = 0$$

** Accept students' answers from parts (a) and (b) if not oversimplified.

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down formula for $\perp$distance <u>and stops</u>. Substitutes (0, 0) correctly into equation of a line to find t_2, *i.e.* $y - 0 = m(x - 0)$ <u>and stops</u>.
	–	Finds $t_1: x = 0$ <u>and stops</u>.
Mid partial credit: (6 marks)	–	Substitutes fully into $\perp$distance formula, *i.e.* $4 = \frac{\|m(-4) - (-2) + 0\|}{\sqrt{(m)^2 + (-1)^2}}$, but fails to find correct value of m.
High partial credit: (8 marks)	–	Finds correct value of m, but fails to find <u>or</u> finds incorrect equation of t_2.
	–	Substantially correct work with one error and equation of both tangents found.

** Award full marks for $t_1: x = 0$ <u>and</u> $t_2: m = -\frac{3}{4}$.

Question 4 **(25 marks)**

Pat and Mark are playing against each other in a darts match. The winner is the first player to win two of three *legs* (games). Pat is a better player and the probability of him winning an individual leg against Mark is $\frac{3}{5}$.

4(a) (i) Find the probability that Pat wins the match after just two legs. **(5C)**

$$P(\text{Pat wins}) = \frac{3}{5}$$

P(Pat wins the match after just two legs)

$$= P(\text{Pat wins 1st leg}) + P(\text{Pat wins 2nd leg})$$
$$= \frac{3}{5} \times \frac{3}{5}$$
$$= \frac{9}{25} \quad \underline{\text{or}} \quad 0{\cdot}36$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct explanation of probability that Pats wins after 2 legs, *e.g.* 'P(wins 1st leg) + P(wins 2nd leg)'.
	–	Correct probabilities chosen, but incorrect operator used.
High partial credit: (4 marks)	–	Correct probabilities chosen and correct operator, *i.e.* $P(\text{wins}) = \frac{3}{5} \times \frac{3}{5}$, but fails to express as a single fraction <u>or equivalent</u>.

(ii) Find the probability that Pat wins the match. **(5C)**

$$P(\text{Pat wins}) = \frac{3}{5}$$
$$\Rightarrow \quad P(\text{Mark wins}) = 1 - \frac{3}{5}$$
$$= \frac{2}{5}$$

P(Pat wins the match)

$$= P(\text{Pat wins in 2 legs}) + P(\text{Pat wins in 3 legs})$$
$$= \frac{9}{25} + \left(\frac{3}{5} \times \frac{2}{5} \times \frac{3}{5}\right) + \left(\frac{2}{5} \times \frac{3}{5} \times \frac{3}{5}\right)$$
$$= \frac{9}{25} + \frac{18}{125} + \frac{18}{125}$$
$$= \frac{81}{125} \quad \underline{\text{or}} \quad 0{\cdot}648$$

** Accept students' answers from part (a)(i) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct explanation of probability that Pat wins in 3 legs, *e.g.* 'P(W × L × W)' <u>or</u> 'P(L × W × W)'.
	–	Finds one correct probability of winning in 3 legs, *i.e.* $\left(\frac{3}{5} \times \frac{2}{5} \times \frac{3}{5}\right)$ <u>or</u> $\left(\frac{2}{5} \times \frac{3}{5} \times \frac{3}{5}\right)$.
High partial credit: (4 marks)	–	Finds all probabilities and operators correct, *i.e.* $\frac{9}{25} + \left(\frac{3}{5} \times \frac{2}{5} \times \frac{3}{5}\right) + \left(\frac{2}{5} \times \frac{3}{5} \times \frac{3}{5}\right)$, but fails to express result as a single fraction <u>or equivalent</u>.

4(a) (cont'd.)

(iii) Find the probability that Pat wins exactly one leg. **(5C)**

$P(\text{Pat wins}) = \frac{3}{5}$

$\Rightarrow P(\text{Mark wins}) = \frac{2}{5}$

$P(\text{Pat wins exactly one leg})$

$= P(\text{Pat wins first leg only}) \text{ or } P(\text{Pat wins 2nd leg only})$

$= \left(\frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}\right) + \left(\frac{2}{5} \times \frac{3}{5} \times \frac{2}{5}\right)$

$= \frac{12}{125} + \frac{12}{125}$

$= \frac{24}{125}$ or $0 \cdot 192$

** Explanation: Pat must win either the first or second leg of the match as it will be over if Mark wins the first two legs (no requirement to play the third leg).

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct explanation of probability that Pat wins only 1 leg, *e.g.* '$P(W \times L \times L)$' or '$P(L \times W \times L)$'.
	–	Finds one correct probability of winning 1 leg, *i.e.* $\left(\frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}\right)$ or $\left(\frac{2}{5} \times \frac{3}{5} \times \frac{2}{5}\right)$.
High partial credit: (4 marks)	–	Finds all probabilities and operators correct, *i.e.* $\frac{9}{25} + \left(\frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}\right) + \left(\frac{2}{5} \times \frac{3}{5} \times \frac{2}{5}\right)$, but fails to express result as a single fraction or equivalent.

Question 4 (cont'd.)

4(b) Find the probability that the match requires three legs to decide the winner. **(5C)**

P(match requires three legs)

$= 1 - [P(\text{Pat wins after 2 legs}) + P(\text{Mark wins after 2 legs})]$

$= 1 - \left(\frac{3}{5} \times \frac{3}{5}\right) - \left(\frac{2}{5} \times \frac{2}{5}\right)$

$= 1 - \frac{9}{25} - \frac{4}{25}$

$= \frac{12}{25}$ <u>or</u> $0{\cdot}48$

** Accept students' answers from part (a)(i) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct explanation of probability that match requires 3 legs, *e.g.* '1 – P(Pat <u>and/or</u> Mark wins after 2 legs)'.
	–	Finds one correct probability of winning 1 leg, *i.e.* $\left(\frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}\right)$ <u>or</u> $\left(\frac{2}{5} \times \frac{3}{5} \times \frac{2}{5}\right)$.
High partial credit: (4 marks)	–	Finds all probabilities and operators correct, *i.e.* $1 - \left(\frac{3}{5} \times \frac{3}{5}\right) - \left(\frac{2}{5} \times \frac{2}{5}\right)$, but fails to express result as a single fraction <u>or equivalent</u>.

(c) Given that Pat wins the match, find the probability that he wins the first leg. **(5C)**

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow$ P(Pat wins first leg | Pat wins the match)

$= \dfrac{\frac{9}{25} + \frac{18}{125}}{\frac{81}{125}}$

$= \frac{63}{125} \times \frac{125}{81}$

$= \frac{63}{81}$ <u>or</u> $\frac{7}{9}$ <u>or</u> $0{\cdot}777777...$

** Accept students' answers from parts (a)(i) and (a)(ii) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down explanation of <u>or</u> defines conditional probability, *i.e.* $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
	–	Finds P(Pat wins first leg and wins match), *i.e.* $\left(\frac{3}{5} \times \frac{2}{5} \times \frac{3}{5}\right) + \left(\frac{3}{5} \times \frac{3}{5}\right)$.
High partial credit: (4 marks)	–	Finds all probabilities and operators correct, *i.e.* $\left(\frac{9}{25} \times \frac{18}{125}\right) / \frac{81}{125}$, but fails to express result as a single fraction <u>or equivalent</u>.

Question 5 **(25 marks)**

In the standard game of poker, each player receives five cards, called a *hand*.
The player with the best hand, the best combination of cards, is the winner.
The game is normally played with a pack consisting of 52 cards in four suits:

13 hearts (♥)	:	2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
13 diamonds (♦)	:	2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
13 clubs (♣)	:	2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
13 spades (♠)	:	2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

5(a) **(i)** Find the number of possible hands a player can receive. **(5C)**

$$\# \text{hands} = \binom{52}{5} = {}^{52}C_5 = \frac{52!}{5!\,(52-5)!} = 2{,}598{,}960$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	– Any relevant first step, *e.g.* writes down '$\# \text{hands} = \binom{52}{5}$' <u>or</u> ${}^{52}C_5$' <u>and stops</u>. – Writes downs <u>or</u> evaluates correctly ${}^{52}P_5$, *i.e.* $52 \times 51 \times 50 \times 49 \times 48$ <u>or</u> 311,875,200.
High partial credit: (4 marks)	– Finds $\frac{52!}{5!\,(52-5)!}$, but fails to evaluate <u>or</u> evaluates incorrectly.

5(b) **(i)** The best hand is a 'royal flush', which consists of 10, J, Q, K, A of the same suit.
Find the probability of a royal flush, as a fraction. **(5C)**

$$P(\text{Royal flush}) = \frac{1}{2{,}598{,}960} \times 4 = \frac{1}{649{,}740}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	– Any relevant first step, *e.g.* writes down '4 different royal flushes' <u>and stops</u>. – Writes down $\frac{1}{52} \times \frac{1}{51} \times \frac{1}{50} \times \frac{1}{49} \times \frac{1}{48} \times 4$ <u>or</u> evaluates correctly [ans. $\frac{4}{311{,}875{,}200}$ <u>or</u> $\frac{1}{77{,}968{,}800}$]. – Writes down $\frac{1}{2{,}598{,}960}$ <u>and stops</u>.
High partial credit: (4 marks)	– Finds $\frac{4}{2{,}598{,}960}$ <u>or equivalent</u>, but fails to express in its singlest form.

Question 5 (cont'd.)

5(b) (cont'd.)

(ii) The next most valuable hand is a 'straight flush', which is five cards in sequential order, all of the same suit. As part of a straight flush, an ace can rank either above a King or below a 2 (e.g. 7, 8, 9, 10, J or A, 2, 3, 4, 5 is a straight flush).

List all the ways that a straight flush can be achieved in the same suit and hence, find the probability of a straight flush. Give your answer as a fraction. **(10D)**

❶ <u>List ways to achieved straight flush (same suit)</u>

Straight flush –	A, 2, 3, 4, 5	× 9
–	2, 3, 4, 5, 6	
–	3, 4, 5, 6, 7	
–	4, 5, 6, 7, 8	
–	5, 6, 7, 8, 9	
–	6, 7, 8, 9, 10	
–	7, 8, 9, 10, J	
–	8, 9, 10, J, Q	
–	9, 10, J, Q, K	
–	10, J, Q, K, A	Excluded as hand is a 'royal flush'

❷ $P(\text{Straight flush}) = \dfrac{9}{2{,}598{,}960} \times 4$

$= \dfrac{36}{2{,}598{,}960}$

$= \dfrac{3}{216{,}580}$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down at least 5 ways in which a straight flush can be achieved.
Mid partial credit: (6 marks)	–	Lists all <u>nine</u> ways in which a straight flush can be achieved.
	–	Fails to list ways to achieve straight flush, but finds $P(\text{Straight flush}) = \dfrac{10}{2{,}598{,}960} \times 4$.
High partial credit: (8 marks)	–	Fails to list ways to achieve straight flush, but finds $P(\text{Straight flush}) = \dfrac{9}{2{,}598{,}960} \times 4$.
	–	Lists all ways to achieve straight flush and finds $\dfrac{9}{2{,}598{,}960}$, but fails to multiply by 4 (different possible suits).
	–	Lists 10 ways (including 'royal flush') in which a straight flush can be achieved and evaluates probability correctly, i.*e.* $P(\text{Straight flush}) = \dfrac{10}{2{,}598{,}960} \times 4$.

Question 5 (cont'd.)

5(c) Another valuable hand is a 'full house', which is three cards of one denomination and two cards of another denomination (e.g. three Jacks and two 5s is a full house).

Find the probability of a full house, as a fraction. **(5D)**

❷ $P(\text{Full house}) = \dfrac{\left[\binom{4}{3}\times 13\right]\times\left[\binom{4}{2}\times 12\right]}{\binom{52}{5}}$

$= \dfrac{[4\times 13]\times[6\times 12]}{2{,}598{,}960}$

$= \dfrac{52\times 72}{2{,}598{,}960}$

$= \dfrac{3{,}744}{2{,}598{,}960}$

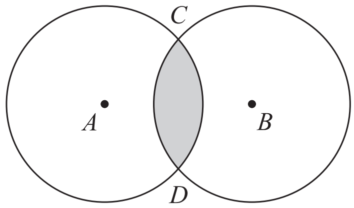
$= \dfrac{6}{4{,}165}$

Scale 5D (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down $\binom{4}{3}\times 13$, ${}^4C_3\times 13$, $\binom{4}{2}\times 12$ or ${}^4C_2\times 12$ (evaluated or not).
	–	Writes down $\frac{1}{52}\times\frac{1}{51}\times\frac{1}{50}\times\frac{1}{49}\times\frac{1}{48}\times 4$ or evaluates correctly [ans. $\frac{4}{311{,}875{,}200}$ or $\frac{1}{77{,}968{,}800}$].
	–	Writes down $\frac{1}{2{,}598{,}960}$ and stops.
Mid partial credit: (3 marks)	–	Finds $\left[\binom{4}{3}\times 13\right]\times\left[\binom{4}{2}\times 12\right]$, buts fails to divide by $\binom{52}{5}$.
High partial credit: (4 marks)	–	Finds $\left[\binom{4}{3}\times 13\right]\times\left[\binom{4}{2}\times 12\right]/\binom{52}{5}$ or equivalent, but fails to express in its singlest form.

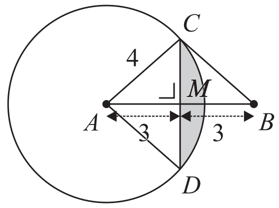
Question 6 **(25 marks)**

6(a) Two circles, each of radius 4 units, intersect at the points C and D, as shown.
The distance between the centres of the circles, A and B, is 6 units.



(i) Find $|\angle CAD|$, correct to two decimal places. **(5D*)**

❶ <u>Trigonometry using ΔCAM</u>



$$\cos|\angle CAM| = \frac{|AM|}{|AC|} = \frac{3}{4}$$

$$\Rightarrow \quad |\angle CAM| = \cos^{-1} 0{\cdot}75 = 41{\cdot}409622...$$

$$|\angle CAD| = 2|\angle CAM|$$

$$\Rightarrow \quad |\angle CAD| = 2(41{\cdot}409622...) = 82{\cdot}819244... \cong 82{\cdot}82°$$

<u>or</u>

❷ <u>Cosine rule using ΔCAB</u>

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Rightarrow \quad |CB|^2 = |AC|^2 + |AB|^2 - 2|AC|.|AB|\cos|\angle CAB|$$

$$\Rightarrow \quad (4)^2 = (4)^2 + (6)^2 - 2(4)(6)\cos|\angle CAB|$$

$$\Rightarrow \quad 16 = 16 + 36 - 48\cos|\angle CAB|$$

$$\Rightarrow \quad 48\cos|\angle CAB| = 32$$

$$\Rightarrow \quad \cos|\angle CAB| = \frac{36}{48} = \frac{3}{4}$$

$$\Rightarrow \quad |\angle CAB| = \cos^{-1} 0{\cdot}75 = 41{\cdot}409622...$$

$$|\angle CAD| = 2|\angle CAB|$$

$$\Rightarrow \quad |\angle CAD| = 2(41{\cdot}409622...) = 82{\cdot}819244... \cong 82{\cdot}82°$$

Scale 5D* (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* draws <u>or</u> indicates on diagram ΔCAM <u>or</u> ΔCAB with correct lengths of sides shown [*i.e.* hypotenuse, adjacent and relevant angle in method ❶; sides a, b, c and relevant angle in method ❷].
	–	Some correct substitution into trigonometric ratio <u>or</u> cosine rule, but fails to finish <u>or</u> finishes incorrectly.
	–	Finds $|CM|$ [ans. $\sqrt{7}$].
Mid partial credit: (3 marks)	–	Finds $|\angle CAM| = \cos^{-1}\frac{3}{4}$ <u>or</u> $\cos^{-1} 0{\cdot}75$, but fails to finish <u>or</u> finishes incorrectly.
	–	Fully correct substitution into formula for cosine rule [method ❷], but fails to finish <u>or</u> finishes incorrectly.

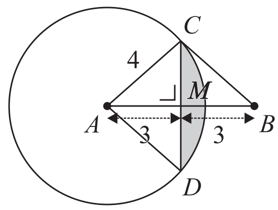
Question 6 (cont'd.)

6(a) (i) (cont'd.)

High partial credit: (4 marks)	–	Finds $\lvert\angle CAM\rvert = 41{\cdot}409622...°$ or $41{\cdot}41°$ [method ❶ or ❷], but fails to find or finds incorrect $\lvert\angle CAD\rvert$.

* Deduct 1 mark off correct answer only if ❶ not rounded or incorrectly rounded or ❷ for the omission of or incorrect use of units ('°') - apply only once in each section (a), (b), (c), *etc*. of question.

(ii) Hence, find the area of the shaded region, correct to one decimal place. **(10D*)**



Area of the shaded region = 2 × [Area of sector ADC – area of ΔADC]

area of sector = $\dfrac{\theta}{360}\pi r^2$

area of triangle = $\dfrac{1}{2}ab\sin C$

$\Rightarrow$ Area of the shaded region = $2 \times [\dfrac{82{\cdot}82}{360}\pi(4)^2 - \dfrac{1}{2}(4)(4)\sin 82{\cdot}82]$

= $2 \times [11{\cdot}563853... - 7{\cdot}937267...]$

= $2 \times [3{\cdot}626585...]$

= $7{\cdot}253171...$

= $7{\cdot}3$ units^2

** Accept students' answers from part (a)(i) if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

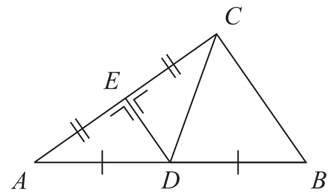
Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down '2 × [Area of sector ADC – area of ΔADC]', '4 × [Area of sector CAM' – area of ΔCAM]' or similar and stops.
	–	Some correct substitution into formula for area of a sector or area of triangle (formula stated or not).
Mid partial credit: (6 marks)	–	Finds correct area of sector CAD or ΔCAD [or sector CAM or ΔCAM] and stops or continues incorrectly.
High partial credit: (8 marks)	–	Finds correct areas of sector CAD and ΔCAD [or sector CAM and ΔCAM], but fails to finish (multiply by 2 or 4) or finishes incorrectly.

* Deduct 1 mark off correct answer only if ❶ not rounded or incorrectly rounded or ❷ for the omission of or incorrect use of units ('units^2') - apply only once in each section (a), (b), (c), *etc*. of question.

Question 6 (cont'd.)

6(b) In the diagram, [CD] is a median of triangle ABC, [DE] is a median of triangle ADC and DE is perpendicular to AC.

(i) Show that DBC is an isosceles triangle. **(5C)**



Consider ΔAED and ΔCED

❶ [DE] is a median of ΔADC

$\Rightarrow |AE| = |EC|$

❷ Also $DE \perp AC$

$\Rightarrow |\angle AED| = |\angle CED|$

$= 90°$

❸ Also $|ED| = |ED|$ ... common side of both triangles

$\Rightarrow \Delta AED \equiv \Delta CED$ ... SAS

$\Rightarrow |AD| = |DC|$ ① ... other corresponding sides of Δs

❹ [CD] is a median of ΔABC

$\Rightarrow |AD| = |DB|$ ②

Equating ① and ②:

$\Rightarrow |DC| = |DB|$

$\Rightarrow \Delta DBC$ is isosceles

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down one correct step such as '[DE] is a median of ΔADC], $\Rightarrow |AE| = |EC|$' and stops.
High partial credit: (4 marks)	–	Shows that $\Delta AED \equiv \Delta CED$, *i.e.* identifies three pairs of corresponding angles or sides correctly (with brief explanations), thereby showing ΔAED and ΔCED are congruent (must be stated), but fails to finish [step ❹] or finishes incorrectly.

(ii) Given that the area of triangle EDC is 5 square units, find the area of triangle ABC. Explain the reasoning for your answer. **(5C)**

Area of ΔADE = 5 units^2

Area of ΔADE = Area of ΔEDC ... as [DE] is a median of ΔADC

= 5 units^2

$\Rightarrow$ Area of ΔADC = Area of ΔADE + Area of ΔADE

= 5 + 5

= 10 units^2

Area of ΔDBC = Area of ΔADC ... as [CD] is a median of ΔABC

= 10 units^2

$\Rightarrow$ Area of ΔABC = Area of ΔADC + Area of ΔDBC

= 10 + 10

= 20 units^2

Scale 5C (0, 2, 4, 5)

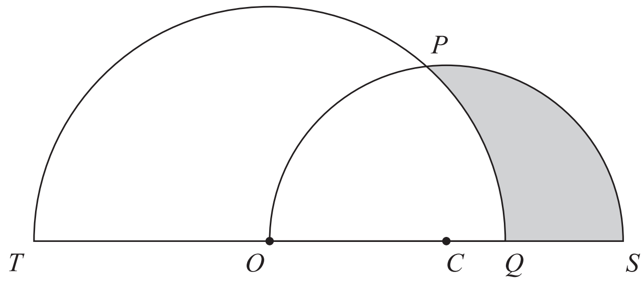
Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down 'Area of ΔADE = Area of ΔEDC' or 'Area of ΔEDC = 5' and stops.
	–	Finds correct answer [ans. 20 units^2], but no justifications given.
High partial credit: (4 marks)	–	Finds correct answer [ans. 20 units^2], but incomplete justifications given.

Section B — Contexts and Applications — 150 marks

Answer **all three** questions from this section.

Question 7 **(50 marks)**

The diagram below shows two semi-circles of different radii that intersect at the point *P*. The larger semi-circle has centre *O* and radius 4 cm. The smaller semi-circle has centre *C* and radius 3 cm. The line through the centres, *OC*, intersects the smaller semi-circle at the point *S* and the larger semi-circle at the points *Q* and *T*.



7(a) **(i)** Name two triangles of equal area in the diagram above and give a reason for your answer. **(5B)**

Any 1:

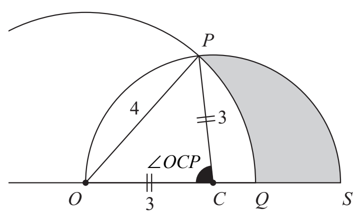
ΔPTO	=	ΔPOQ	... both the perpendicular height and the length of the bases of each triangle (both radii of the semi-circle) are the same
or			
ΔPOC	=	ΔPCS	... same reason

Scale 5B (0, 2, 5)

Partial credit: (2 marks)	–	Identifies two triangles of equal area, but no reason or incomplete reason given.

(ii) Using the cosine rule, or otherwise, show that $\cos|\angle OCP| = \dfrac{1}{9}$. **(10C)**

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\Rightarrow |OP|^2 = |OC|^2 + |CP|^2 - 2|OC|.|CP|\cos|\angle OCP|$$
$$\Rightarrow (4)^2 = (3)^2 + (3)^2 - 2(3)(3)\cos|\angle OCP|$$
$$\Rightarrow 16 = 9 + 9 - 18\cos|\angle OCP|$$
$$\Rightarrow 18\cos|\angle OCP| = 18 - 16 = 2$$
$$\Rightarrow \cos|\angle OCP| = \frac{2}{18} = \frac{1}{9}$$
$$\Rightarrow |\angle OCP| = \cos^{-1}\frac{1}{9}$$



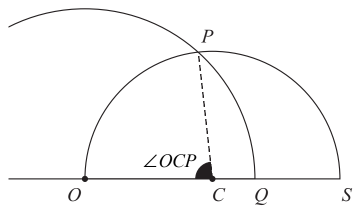
Scale 10C (0, 4, 7, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct formula for cosine rule with some correct substitution, but fails to finish or finishes incorrectly.
High partial credit: (7 marks)	–	Fully correct substitution into formula for cosine rule, *i.e.* $(4)^2 = (3)^2 + (3)^2 - 2(3)(3)\cos|\angle OCP|$, but fails to finish or finishes incorrectly.

Question 7 (cont'd.)

7(a) (cont'd.)

(iii) Hence, show that $\sin|\angle OCP| = \frac{4\sqrt{5}}{9}$. **(5C)**



$$\cos|\angle OCP| = \frac{1}{9}$$

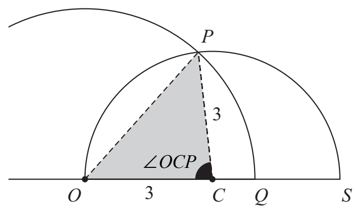
Using Pythagoras' theorem:

$$|\text{Hyp}|^2 = |\text{Opp}|^2 + |\text{Adj}|^2$$
$$\Rightarrow (9)^2 = |\text{Opp}|^2 + (1)^2$$
$$\Rightarrow |\text{Opp}|^2 = 81 - 1 = 80$$
$$\Rightarrow |\text{Opp}| = \sqrt{80} = 4\sqrt{5}$$
$$\sin|\angle OCP| = \frac{|\text{Opp}|}{|\text{Hyp}|} = \frac{4\sqrt{5}}{9}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* draws right-angle triangle with lengths of hypotenuse and adjacent marked. – Some correct use of Pythagoras' theorem, but fails to find $|\text{Opp}|$.
High partial credit: (4 marks)	–	Finds correct $|\text{Opp}|$, [ans. $\sqrt{80}$ <u>or</u> $4\sqrt{5}$], but fails to show that $\sin|\angle OCP| = \frac{4\sqrt{5}}{9}$.

7(b) **(i)** Find the area of triangle *OCP*, giving your answer in surd form. **(5C)**



$$\text{Area of triangle} = \frac{1}{2}ab\sin C$$
$$\Rightarrow \text{Area of } \Delta OCP = \frac{1}{2}|OC|.|CP|ab\sin|\angle OCP|$$
$$= \frac{1}{2}(3)(3)\frac{4\sqrt{5}}{9} \quad \text{... given in part (a)(iii)}$$
$$= 2\sqrt{5} \text{ units}^2$$

Scale 5C (0, 2, 4, 10)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for the area of a triangle with some correct substitution, but fails to finish <u>or</u> finishes incorrectly.
High partial credit: (4 marks)	–	Fully correct substitution into formula, *i.e.* area of $\Delta OCP = \frac{1}{2}(3)(3)\,\frac{4\sqrt{5}}{9}$, but fails to give final answer in surd form.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('units^2') - apply only once in each section (a), (b), (c), *etc*. of question.

7(b) (cont'd.)

(ii) Calculate the areas of the two sectors *OCP* and *POQ*.
Give your answers correct to two decimal places. **(10D*)**

Area of sector $= \dfrac{\theta}{360}\pi r^2$

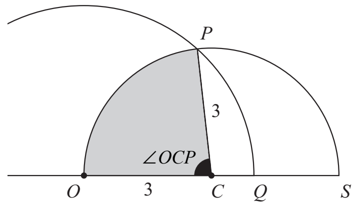
❶ <u>Sector *OCP*</u>

$\cos|\angle OCP| = \dfrac{1}{9}$ ... given in part (a)(ii)

$\Rightarrow \quad |\angle OCP| = \cos^{-1}\dfrac{1}{9}$

$= \cos^{-1} 0{\cdot}111111...$

$= 83{\cdot}620629...°$



$\Rightarrow$ Area of sector *OCP* $= \dfrac{83{\cdot}620629...}{360}\pi(3)^2$

$= 6{\cdot}567548...$

$\cong 6{\cdot}57$ units^2

❶ <u>Sector *POQ*</u>

Consider ΔPOC

$|OC| = |CP|$

$\Rightarrow \Delta POC$ is isosceles

$\Rightarrow |\angle POC| = |\angle COP|$

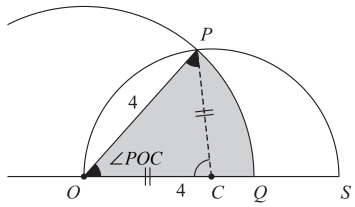
$|\angle OCP| = 180° - |\angle POC| - |\angle COP|$

$= 180° - 2|\angle POC|$

$\Rightarrow 2|\angle POC| = 180° - |\angle OCP|$

$\Rightarrow |\angle POC| = 180° - |\angle OCP|$

$= \dfrac{180° - |\angle OCP|}{2}$



$|\angle POC| = \dfrac{180° - 83{\cdot}620629...}{2}$

$= 48{\cdot}189685...°$

$\Rightarrow$ Area of sector *POQ* $= \dfrac{48{\cdot}189685...}{360}\pi(4)^2$

$= 6{\cdot}728549...$

$\cong 6{\cdot}73$ units^2

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct formula for the area of a sector with some correct substitution, but fails to finish <u>or</u> finishes incorrectly.
	–	Finds correct value for $|\angle OCP|$ <u>and stops</u> <u>or</u> continues incorrectly.
Mid partial credit: (6 marks)	–	Fully correct substitution into area formula for sector *OCP* (evaluated <u>or</u> not) *i.e.* area of $OCP = \dfrac{83{\cdot}620629...}{360}\pi(3)^2$.
High partial credit: (8 marks)	–	Finds correct area of sector *OCP* <u>and</u> finds correct $|\angle POC|$, but fails to find area of sector *POQ* <u>or</u> finds incorrect area.
	–	Finds correct area of sector *OCP* <u>and</u> some correct substitution into formula for area of sector *OCP* (no value <u>or</u> incorrect value for $|\angle POC|$ found).

* Deduct 1 mark off correct answer only if ❶ not rounded <u>or</u> incorrectly rounded <u>or</u> ❷ for the omission of <u>or</u> incorrect use of units ('units^2') - apply only once in each section (a), (b), (c), *etc*. of question.

Question 7 (cont'd.)

7(b) (cont'd.)

(iii) Hence, find the area of the shaded region, correct to one decimal place. **(5C*)**

❶ $\Rightarrow$ Area of PCQ $=$ Area of sector POQ – Area of ΔOCP

$= 6{\cdot}73 - 2\sqrt{5}$ ... answers from parts (b)(i) and (b)(ii)

$= 6{\cdot}73 - 2(2{\cdot}236067...)$

$= 6{\cdot}73 - 4{\cdot}472135...$

$= 2{\cdot}257864...$

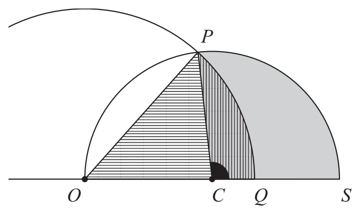
$\Rightarrow$ Area of shaded area $=$ Area of semi-circle – (Area of sector OCP + Area of PCQ)

$= \frac{1}{2}\pi(3)^2 - (6{\cdot}57 - 2{\cdot}257864...)$ ... answer from part (b)(ii)

$= 14{\cdot}137166... - 8{\cdot}827864...$

$= 5{\cdot}309302...$

$\cong 5{\cdot}3$ units^2



or

❷ $|\angle PCS| = 180° - |\angle OCP|$

$|\angle OCP| = \cos^{-1}\frac{1}{9}$

$= 83{\cdot}620629...°$ ... given in part (a)(ii)

$\Rightarrow |\angle PCS| = 180 - 83{\cdot}620629...$

$= 96{\cdot}379370...°$

$\Rightarrow$ area of sector PCS $= \frac{96{\cdot}379370...}{360}\pi(3)^2$

$= 7{\cdot}569618...$

Area of PCQ $=$ Area of sector POQ – Area of ΔOCP

$= 6{\cdot}73 - 2\sqrt{5}$ ... answers from parts (b)(i) and (b)(ii)

$= 6{\cdot}73 - 2(2{\cdot}236067...)$

$= 6{\cdot}73 - 4{\cdot}472135...$

$= 2{\cdot}257864...$

$\Rightarrow$ Area of shaded area $=$ Area of sector PCS – Area of PCQ

$= 7{\cdot}569618... - 2{\cdot}257864...$

$= 5{\cdot}311754...$

$\cong 5{\cdot}3$ units^2

** Accept students' answers from parts (b)(i) and (b)(ii) if not oversimplified.

Scale 5C* (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g*. finds correct area of semi-circle <u>or</u> circle with correct radius (3 units).
	–	Finds correct area of PCQ <u>and stops</u>.
	–	Finds correct area of sector PCS <u>and stops</u>.
High partial credit: (4 marks)	–	Fully correct substitution into correct area of shaded region (evaluated <u>or</u> not), *i.e.* $\frac{1}{2}\pi(3)^2 - (6{\cdot}57 - 2{\cdot}257864...)$, $\frac{96{\cdot}379370...}{360}\pi(3)^2 - (6{\cdot}73 - 2\sqrt{5})$ or equivalents, but fails to finish of finishes incorrectly.

* Deduct 1 mark off correct answer only if ❶ not rounded <u>or</u> incorrectly rounded <u>or</u> ❷ for the omission of <u>or</u> incorrect use of units ('units^2') - apply only once in each section (a), (b), (c), *etc*. of question.

Question 7 (cont'd.)

7(c) Find the perimeter of the shaded region, correct to one decimal place. **(10D*)**

Perimeter of shaded region

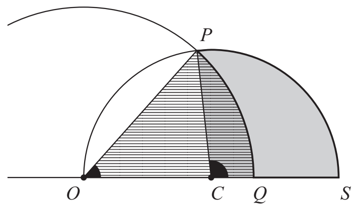
$= |\text{arc } PQ| + |\text{arc } PS| + |QS|$

$|\text{arc } PQ| = \dfrac{48{\cdot}189685...}{360}(2\pi)(4)$ ... answer from part (b)(ii)

$= 3{\cdot}364274...$

$|\text{arc } PS| = \dfrac{96{\cdot}379370...}{360}(2\pi)(3)$ ... answer from part (b)(iii)

$= 5{\cdot}047162...$



$|QS| = |OS| - |OQ|$

$= 2|OC| - |OQ|$

$= 2(3) - 4$

$= 2$

$\Rightarrow$ Perimeter of shaded region

$= 3{\cdot}364274... + 5{\cdot}047162... + 2$

$= 10{\cdot}411436...$

$\cong 10{\cdot}4$ units

** Accept students' answers from parts (b)(ii) and (b)(iii) if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct formula for the length of arc with some correct substitution to find one arc, but fails to finish or finishes incorrectly.
	–	Finds correct value for $\|QS\|$ and stops.
Mid partial credit: (6 marks)	–	Finds correct value for either $\|\text{arc } PQ\|$ or $\|\text{arc } PS\|$, and stops or continues incorrectly.
High partial credit: (8 marks)	–	Finds correct value for $\|\text{arc } PQ\|$ and $\|\text{arc } PS\|$, but fails to finish or finishes incorrectly.
	–	Finds correct value for either $\|\text{arc } PQ\|$ or $\|\text{arc } PS\|$ and $\|QS\|$, but fails to finish or finishes incorrectly.

* Deduct 1 mark off correct answer only if ❶ not rounded or incorrectly rounded or ❷ for the omission of or incorrect use of units ('units^2') - apply only once in each section (a), (b), (c), *etc*. of question.

Question 8 **(50 marks)**

8(a) Figures on the numbers of people passing their driving test are published annually. On analysis of the data, a researcher found that the probability of a person passing his/her test in a particular test centre on the first attempt was $\frac{2}{3}$. Six individuals take their driving test for the first time.

(i) Find the probability that at least one of the individuals passes the test. **(5C)**

<u>Bernoulli trial</u>

p (probability of success) $= \frac{2}{3}$

$\Rightarrow$ q (probability of failure) $= 1 - \frac{2}{3}$

$= \frac{1}{3}$

n (number of individuals) $= 6$

$P(k) = \binom{n}{k} p^k q^{n-k}$

P(at least one passes) $= 1 - P$(no individual passes)

$$= 1 - \binom{6}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^6$$

$$= 1 - (1)(1)\left(\frac{1}{3}\right)^6$$

$$= 1 - \frac{1}{729}$$

$$= \frac{728}{729} \text{ or } 0{\cdot}998628...$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down explanation, *i.e.* P(at least one passes) = 1 – P(no individual passes) <u>and stops</u>.
	–	Finds p (success) = $\frac{2}{3}$ <u>and</u> q (failure) = $\frac{1}{3}$.
High partial credit: (4 marks)	–	Finds correct P(no individual passes) $= \left(\frac{1}{3}\right)^6$ <u>or</u> $\frac{1}{729}$, but fails to finish correctly.
	–	Finds P(at least one passes) $= 1 - \left(\frac{1}{3}\right)^6$, but fails to finish <u>or</u> finishes incorrectly.

Question 8 (cont'd.)

8(a) (cont'd.)

(ii) Find the probability that at most four individuals pass the test. **(10C)**

$$
\begin{aligned}
P(\text{at most four pass}) &= 1 - [P(\text{five pass}) + P(\text{six pass})] \\
&= 1 - \left[\binom{6}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1 + \binom{6}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0\right] \\
&= 1 - \left[6\left(\frac{32}{243}\right)\left(\frac{1}{3}\right) + 1\left(\frac{64}{729}\right)(1)\right] \\
&= 1 - \left[\frac{192}{729} + \frac{64}{729}\right] \\
&= 1 - \frac{256}{729} \\
&= \frac{473}{729} \ \underline{\text{or}}\ 0\cdot648834...
\end{aligned}
$$

Scale 10C (0, 4, 7, 10)			
	Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct explanation, *i.e.* P(at most four pass) $= 1 - [P(\text{five pass}) + P(\text{six pass})]$ <u>and stops</u>.
		–	Some correct substitution into binomial formula, <u>and stops</u> <u>or</u> continues incorrectly, *e.g.* $\binom{6}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1$ <u>or</u> $\binom{6}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0$.
	High partial credit: (7 marks)	–	Fully correct substitution into binomial formula, but fails to evaluate correctly, *i.e.* $1 - [6\left(\frac{32}{243}\right)\left(\frac{1}{3}\right) + 1\left(\frac{64}{729}\right)(1)]$.

Question 8 (cont'd.)

8(b) A reputable driving school claims on its website that 80% of its students pass their driving test on their first attempt. In order to test this claim, a sample of 900 people who used the school and who had taken their test for the first time are chosen at random. The number of people who passed the driving test on their first attempt was 675.

(i) Conduct a hypothesis test at the 5% level of significance to decide whether there is sufficient evidence to justify the driving school's claim. Write the null hypothesis and the alternative hypothesis and state your conclusion clearly. **(15D)**

❶ $H_0 : p = 0{\cdot}8$ – percentage of students who used the driving school passed their driving test on their first attempt is 80%

$H_1 : p \neq 0{\cdot}8$ – percentage of students who used the driving school passed their driving test on their first attempt is not 80%

❷ A confidence interval for a population proportion, p, is

$$= \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$$\hat{p} = \frac{675}{900} = 0{\cdot}75$$

At 95% confidence interval
z-value $= 1{\cdot}96$

$\Rightarrow$ The 95% confidence interval for this population proportion, p, is

$$= \left[0{\cdot}75 - 1{\cdot}96\sqrt{\frac{0{\cdot}75(1-0{\cdot}75)}{900}},\ 0{\cdot}75 + 1{\cdot}96\sqrt{\frac{0{\cdot}75(1-0{\cdot}75)}{900}}\right]$$

$$= [0{\cdot}75 - 0{\cdot}028290, \cdot 0{\cdot}75 - 0{\cdot}028290]$$

$$= [0{\cdot}72171, \cdot 0{\cdot}77829]$$

❸ Conclusion

As $p = 0{\cdot}8$ is outside this interval, the result is significant.
There is evidence to reject the null hypothesis (H_0) and accept the alternative hypothesis (H_1), *i.e.* the percentage of students who took the driving test for the first time and who passed the test is not 80%

Scale 15D (0, 6, 10, 13, 15)

Low partial credit: (6 marks)	–	Any relevant first step, *e.g.* writes down null hypothesis and/or alternative hypothesis only.
	–	Finds correct value for observed population, $\hat{p}$ and stops.
	–	Mention of 5% level of significance and therefore comparing to z-value of $\pm 1{\cdot}96$.
Mid partial credit: (10 marks)	–	Finds correct value for $\hat{p}$ and some correct substitution into 95% confidence interval for population proportion.
High partial credit: (13 marks)	–	Finds correct confidence interval and compares to correctly calculated value for $\hat{p}$ but: - fails to state the null and/or alternative hypothesis correctly, - fails to accept or rejecting hypothesis. - fails to contextualise answer properly. *i.e.* stops at rejects null hypothesis.

8(b) (cont'd.)

(ii) Find, using a 5% level of significance, the least number of people in that sample required to have passed the driving test in order to accept the driving school's claim. **(5C)**

$\hat{p} = 0{\cdot}8$

95% confidence interval for the population proportion, p, to accept the driving school's claim, is

$$= \left[0{\cdot}80 - 1{\cdot}96\sqrt{\frac{0{\cdot}8(1-0{\cdot}8)}{900}},\ + 1{\cdot}96\sqrt{\frac{0{\cdot}8(1-0{\cdot}8)}{900}}\right]$$

$$= [\,0{\cdot}80 - 0{\cdot}026133,\ 0{\cdot}80 - 0{\cdot}026133\,]$$

$$= [\,0{\cdot}773867,\ 0{\cdot}826133\,]$$

$\Rightarrow$ Least number of people in sample

$$= 0{\cdot}773867 \times 900$$

$$= 696{\cdot}4803$$

$$\cong 697 \text{ people}$$

* Accept either 696 or 697 as correct final answer.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* formulates confidence interval with some correct substitution.
High partial credit: (4 marks)	–	Finds correct confidence interval, but fails to find <u>or</u> finds incorrect least number of people in sample.

8(c) In a random sample of 200 drivers from all parts of the country, the 95% confidence interval for the mean number of penalty points received was $4{\cdot}1921 \le \mu \le 4{\cdot}6079$.

(i) Assuming that the number of penalty points received follows a normal distribution, find the standard deviation of this sample. **(10D)**

❶ 95% confidence interval for the population mean:

$$\bar{x} - 1{\cdot}96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + 1{\cdot}96\frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \quad \bar{x} - 1{\cdot}96\frac{\sigma}{\sqrt{n}} = 4{\cdot}1921$$

$$\text{and} \quad \bar{x} + 1{\cdot}96\frac{\sigma}{\sqrt{n}} = 4{\cdot}6079$$

$$\Rightarrow \quad 2\bar{x} = 8{\cdot}8$$

$$\Rightarrow \quad \bar{x} = 4{\cdot}4$$

$$\bar{x} + 1{\cdot}96\frac{\sigma}{\sqrt{n}} = 4{\cdot}6079$$

$$\Rightarrow \quad 4{\cdot}4 + 1{\cdot}96\frac{\sigma}{\sqrt{n}} = 4{\cdot}6079$$

$$\Rightarrow \quad 1{\cdot}96\frac{\sigma}{\sqrt{n}} = 4{\cdot}6079 - 4{\cdot}4$$

$$= 0{\cdot}2079$$

$$\Rightarrow \quad \frac{\sigma}{\sqrt{200}} = \frac{0{\cdot}2079}{1{\cdot}96}$$

$$\Rightarrow \quad \sigma = \sqrt{200}\,(0{\cdot}106071...)$$

$$= 1{\cdot}500076...$$

$$\cong 1{\cdot}5$$

8(c) (i) (cont'd.)

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* formulates confidence interval for population mean, *i.e.* $\bar{x} - 1{\cdot}96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 1{\cdot}96\frac{\sigma}{\sqrt{n}}$.
	–	Formulates two simultaneous equations in terms of $\bar{x}$ and $\frac{\sigma}{\sqrt{n}}$.
Mid partial credit: (6 marks)	–	Finds correct value for $\bar{x}$, but fails to find correct value for $\frac{\sigma}{\sqrt{n}}$.
High partial credit: (8 marks)	–	Finds correct value for $\bar{x}$ and $\frac{\sigma}{\sqrt{n}}$, but fails to finish or finishes incorrectly.

(ii) How many drivers in this sample can be expected to have more than 7 penalty points? **(5D)**

$$z = \frac{x - \bar{x}}{\sigma}$$

$$x = 7$$
$$\bar{x} = 4{\cdot}4$$
$$\sigma = 1{\cdot}5$$

$$\Rightarrow P(x > 7) = P\left(z > \frac{7 - 4{\cdot}4}{1{\cdot}5}\right)$$
$$= P(z > 1{\cdot}7333)$$
$$\cong P(z > 1{\cdot}73)$$
$$= 1 - P(z < 1{\cdot}73)$$
$$= 1 - 0{\cdot}9582 \quad \text{... from } z\text{-tables}$$
$$= 0{\cdot}0418$$

$\Rightarrow$ # drivers expected to have more than 7 penalty points
$$= 200 \times 0{\cdot}0418$$
$$= 8{\cdot}36$$
$$\cong 9 \text{ drivers}$$

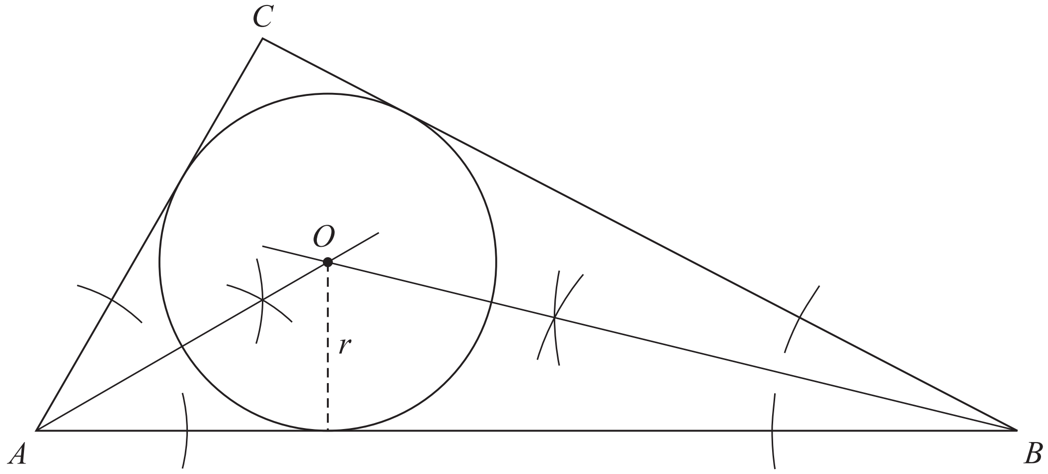
** Accept students' answers from part (c)(i) if not oversimplified.

Scale 5D (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for z with some correct substitution into formula.
	–	Finds correct value for z, but fails to find z-value from tables.
Mid partial credit: (3 marks)	–	Finds correct $P(z < 1{\cdot}73)$ [ans. 0·9582], but fails to finish or finishes incorrectly.
High partial credit: (4 marks)	–	Finds correct probability, *i.e.* $P(x > 7)$ [ans. 0·0418], but fails to find or finds incorrect expected # driver.

Question 9 **(50 marks)**

9(a) **(i)** Construct the incircle of the triangle ABC below using only a compass and a straight edge.
Show all construction lines clearly. **(10C)**



Scale 10C (0, 4, 7, 10)

Low partial credit: (4 marks)	– –	Any relevant first step, *e.g*. construct one bisector correctly. Draws circle by trial and error, but within tolerance of 2 mm.
High partial credit: (7 marks)	– –	Finds incentre correctly, but fails to draw incircle. Draws incircle correctly using method shown, but outside tolerance of 2 mm.

(ii) On the diagram above, mark the point O, the centre of the incircle, and the perpendicular distance from O to $[AB]$, r, the radius of the incircle. **(5C)**

– see diagram above

(iii) Let $|AB| = c$, $|BC| = a$ and $|AC| = b$.
Find an expression for the area of triangle ABO, in terms of r and one of these constants.

$$\text{Area of } \Delta ABO = \frac{1}{2}(\text{base} \times \perp\text{height})$$
$$= \frac{1}{2}(c \times r)$$
$$= \frac{cr}{2}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	– – –	Any relevant first step, *e.g*. writes down correct formula for area of a triangle. Shows centre O and radius r correctly on diagram. Some correct substitution into formula for area of a triangle (<u>not</u> stated), *e.g.* $\frac{1}{2} \times \lvert AB \rvert \times r$ <u>or</u> $\frac{1}{2} \times c \times h$.
High partial credit: (4 marks)	–	Shows centre O and radius r correctly on diagram <u>and</u> finds area of triangle as $\frac{1}{2} \times \lvert AB \rvert \times r$ <u>or</u> $\frac{1}{2} \times c \times h$.

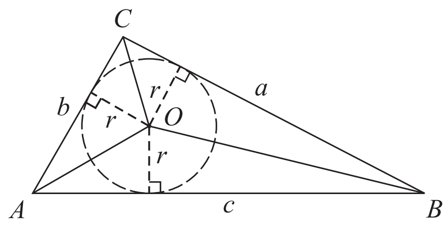
Question 9 (cont'd.)

9(a) (cont'd.)

(iv) Hence, or otherwise, show that, if p is the length of the perimeter of triangle ABC, the area of triangle ABC is equal to $\frac{1}{2}pr$. **(10C)**

Area of ΔABC = Area of ΔABO + Area of ΔBCO + Area of ΔCAO

Area of $\Delta ABO = \frac{cr}{2}$



<u>Similarly</u>

Area of $\Delta BCO = \frac{ar}{2}$

Area of $\Delta ABO = \frac{br}{2}$

$\Rightarrow$ Area of $\Delta ABC = \frac{cr}{2} + \frac{ar}{2} + \frac{br}{2}$

$= \frac{r}{2}(a + b + c)$

$= \frac{1}{2}pr$

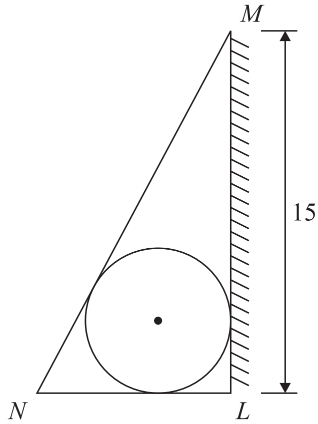
Scale 10C (0, 4, 7, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down 'Area of ΔABC = Area of ΔABO + Area of ΔBCO + Area of ΔCAO' <u>or equiavlent</u>.
	–	Writes down perimeter, $p = a + b + c$.
	–	Finds correct area of ΔBCO <u>or</u> ΔABO, *i.e.* Area $\Delta BCO = \frac{ar}{2}$ <u>or</u> Area $\Delta ABO = \frac{br}{2}$, <u>and stops</u> <u>or</u> continues incorrectly.
High partial credit: (7 marks)	–	Adds areas of all three smaller triangles, *i.e.* Area of $\Delta ABC = \frac{cr}{2} + \frac{ar}{2} + \frac{br}{2}$, but fails to finish <u>or</u> finishes incorrectly.

Question 9 (cont'd.)

9(b) A wheel of radius 3 units rests against a vertical wall of height 15 units. A straight thin board leans against the wheel with one end of the board touching the top of the wall, M, and the other end resting on the ground, N, as shown.

Using the result from part **(a)(iv)** above, or otherwise, find $|NL|$, the distance from the bottom of the board to the foot of the wall.
[Hint: Let $x = |NL|$ and find $|MN|$ in terms of x.] **(15D)**



Let $x = |NL|$

❶ Using area formula:

$$\text{Area of } \Delta = \frac{1}{2}(\text{base} \times \perp\text{height})$$

$$\Rightarrow \text{Area of } \Delta NLM = \frac{1}{2}(x)(15) \quad ①$$

❷ Using Pythagoras' theorem

$$|MN|^2 = |NL|^2 + |LM|^2$$
$$\Rightarrow |MN|^2 = x^2 + 15^2$$
$$\Rightarrow |MN| = \sqrt{x^2 + 225}$$
$$\Rightarrow \text{perimeter, } p = x + 15 + \sqrt{x^2 + 225}$$

❸ Using result from part (a)(iv):

$$\text{Area of } \Delta = \frac{1}{2}pr$$

$$\Rightarrow \text{Area of } \Delta NLM = \frac{1}{2}(x + 15 + \sqrt{x^2 + 225})(3) \quad ②$$

❹ Equating ① and ②:

$$\frac{1}{2}(x)(15) = \frac{1}{2}(x + 15 + \sqrt{x^2 + 225})(3)$$
$$\Rightarrow 5x = x + 15 + \sqrt{x^2 + 225}$$
$$\Rightarrow 4x - 15 = \sqrt{x^2 + 225}$$
$$\Rightarrow (4x - 15)^2 = x^2 + 225$$
$$\Rightarrow 16x^2 - 120x + 225 = x^2 + 225$$
$$\Rightarrow 15x^2 - 120x = 0$$
$$\Rightarrow x^2 - 8x = 0$$
$$\Rightarrow x(x - 8) = 0$$
$$\Rightarrow x - 8 = 0$$
$$\Rightarrow x = 8$$
$$\Rightarrow |NL| = 8 \text{ units}$$

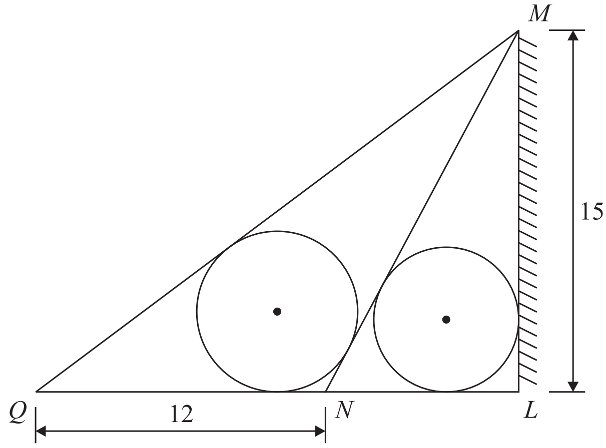
Scale 15D (0, 6, 10, 13, 15)

Low partial credit: (6 marks)	–	Any relevant first step, *e.g.* finds, using Pythagoras' theorem, $|MN|^2 = x^2 + 15^2$ or $|MN| = \sqrt{x^2 + 225}$ and stops.
	–	Finds Area of $\Delta NLM = \frac{1}{2}(x)(15)$ and stops or continues incorrectly.
High partial credit: (10 marks)	–	Finds $\frac{1}{2}pr = \frac{1}{2}(x + 15 + \sqrt{x^2 + 225})$ and stops or continues incorrectly.
	–	Finds $p = x + 15 + \sqrt{x^2 + 225}$ and Area of $\Delta NLM = \frac{1}{2}(x)(15)$, and stops or continues incorrectly.
High partial credit: (13 marks)	–	Equates both expressions for area, *i.e.* $\frac{1}{2}(x)(15) = \frac{1}{2}(x + 15 + \sqrt{x^2 + 225})(3)$, but fails to finish or finishes incorrectly.

* No deduction applied for the omission of or incorrect use of units ('units').

Question 9 (cont'd.)

9(c) Another wheel rests on the ground, touching the board $[MN]$. A second straight thin board $[MQ]$ leans against this wheel with one end touching the top of the wall, M, and the other end resting on the ground, Q, a distance of 12 units further away from the wall than N, as shown.



Find, by calculation, the radius of this wheel. . **(10D)**

❶ Using Pythagoras' theorem

$$|MN|^2 = |NL|^2 + |LM|^2$$
$$\Rightarrow |MN|^2 = 8^2 + 15^2 = 64 + 225 = 289$$
$$\Rightarrow |MN| = \sqrt{289} = 17 \text{ units}$$

❷ Using Pythagoras' theorem

$$|QM|^2 = (|QN| + |NL|)^2 + |LM|^2$$
$$\Rightarrow |QM|^2 = (12+8)^2 + 15^2 = 400 + 225 = 625$$
$$\Rightarrow |QM| = \sqrt{625} = 25 \text{ units}$$

❸ Perimeter of ΔQNM

$$p = 12 + 17 + 25 = 54 \text{ units}$$

❹ Using area formula:

$$\text{Area of } \Delta QNM = \frac{1}{2}(12)(15) = 90 \text{ units}^2$$

❺ Using result from part (a)(iv):

$$\text{Area of } \Delta = \frac{1}{2}pr$$
$$\Rightarrow \text{Area of } \Delta QNM = \frac{1}{2}(54)r = 90$$
$$\Rightarrow \frac{1}{2}(54)r = 90$$
$$\Rightarrow r = \frac{90}{27} = \frac{10}{3} \text{ units}$$

Question 9 (cont'd.)

9(c) (cont'd.)

** Accept students' answers from part (b) if not oversimplified.

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* finds, using Pythagoras' theorem, value for $\lvert MN \rvert$ and stops [allow use of students' answers from part (i)].
	–	Finds correct Area of ΔQNM and stops or continues incorrectly.
High partial credit: (6 marks)	–	Finds correct values of $\lvert MN \rvert$ and $\lvert QM \rvert$ and stops or continues incorrectly.
	–	Finds correct value of $\lvert MN \rvert$ and correct Area of ΔQNM and stops or continues incorrectly.
High partial credit: (8 marks)	–	Finds correct perimeter of ΔQNM [ans. 54] and Area of ΔQNM [ans. 90], but fails to finish or finishes incorrectly.

* No deduction applied for the omission of or incorrect use of units ('units').

Notes:

Notes:

Notes: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