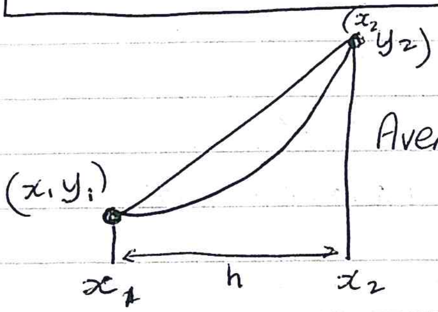
Calculus - Differentiation :

$\dfrac{dy}{dx}$ = Slope of tangent at x !!!



Average rate of change = $\dfrac{y_2-y_1}{x_2-x_1}$ or $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

Make x_2-x_1 very small (h) and the line l becomes a tangent at x_1. This is what we do when we differentiate.

$$\boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_1+h) - f(x_1)}{h}} = \lim_{\substack{h \to 0 \\ x_2 \to x_1}} \text{ of slope } \frac{y_2-y_1}{x_2-x_1}$$

Differentiate from 1st principles e.g. 1 pg 62 $f(x) = 3x+8$:

find $f(x+h)$	$\rightarrow$	$3(x+h) + 8$
Subtract $f(x)$		$-\ 3x+8$
	=	$3h$
$\div$ by h	=	3
Take Limit as $h \to 0$		= 3

$x^2 + 6x = f(x)$

$\underbrace{(x+h)^2 + 6(x+h)}_{f(x+h)} - \underbrace{(x^2+6x)}_{f(x)}$

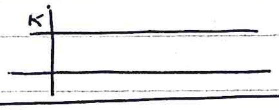
$= 2hx + h^2 - 6h \quad \div h$

$= 2x + h - 6$

$\lim_{h \to 0} = 2x - 6$

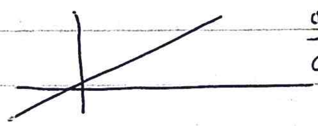
$\dfrac{dy}{dx}$ = $\underline{\underline{\text{slope}}}$ of tangent at x (it is NOT equation of tangent !!!)

if $y = k$ (constant) :



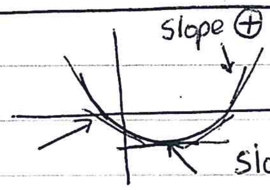
$\dfrac{dy}{dx}$ = slope $= 0 \ \forall x$

if $y = 3x+2$ (line)



$\dfrac{dy}{dx}$ = slope = constant $= 3, \forall x$

if $y = ax^2 + bx + c$



$\dfrac{dy}{dx}$ = slope $\rightarrow$ changes as x changes

slope = 0 @ min or max

$\dfrac{dy}{dx} = 2ax + b$ (varies with x)

$y = u(x) + v(x)$ | $y = k\,u(x)$ | $y = u(x)\,v(x)$

$\frac{dy}{dx} = u'(x) + v'(x)$ | $\frac{dy}{dx} = k\,u'(x)$ | $\frac{dy}{dx} = u(x)\,v'(x) + v(x)\,u'(x)$ (product rule)

* (product rule): $y = x \sin x$ | $\sqrt{x} = x^{1/2}$ | $\frac{1}{x^2} = x^{-2}$

$y = \frac{u(x)}{v(x)}$ $\quad \frac{dy}{dx} = \frac{v(x)\,u'(x) - u(x)\,v'(x)}{v^2(x)}$ (quotient rule)

Chain rule: $y = f(u(x))$ $\quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$y = e^{x^2} \rightarrow y = e^u$

$dy/du = e^u$

$dy/dx = e^u \cdot du/dx$

$dy/dx = e^{x^2} \times 2x$

pg 74 : Q11, 13, 17, 9(ii)

$y \longrightarrow \frac{dy}{dx} \longrightarrow d^2y/dx^2$
(differentiate, differentiate)

$y \longrightarrow f'(x) \longrightarrow f''(x)$

differentiate, differentiate with respect to x, find $f'(x)$, find slope of tangent

chain rule with trignometric functions: $y = \sin 6x$, $y = \sin^4 x$

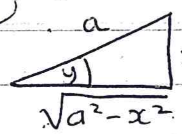
$y = \sin^6 x^2$

$y = \sin u$, $\quad y = u^4$, $\quad y = (\sin x^2)^6 \rightarrow y = u^6$

$\frac{dy}{dx} = \cos u \frac{du}{dx}$, $\quad \frac{dy}{dx} = 4u^3 \frac{du}{dx}$, $\quad$ (double chain rule)

P 80 Q10, 16 $\quad$ Inverse Trig functions:

$y = \sin^{-1} x/a \Rightarrow \sin y = \frac{x}{a}$



$\Rightarrow x = a \sin y$

$\frac{dx}{dy} = a \cos y$

$\frac{dy}{dx} = \frac{1}{a \cos y} = \frac{1}{\frac{\sqrt{a^2-x^2}}{a}}$

$\frac{dy}{dx} = \frac{a}{\sqrt{a^2-x^2}}$

log tables: $y = \sin^{-1}\frac{x}{a}$, $dy/dx = \frac{a}{\sqrt{a^2-x^2}}$

$y = \sin^{-1}\frac{x}{1}$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$y = \sin^{-1}\left(\frac{5x}{3}\right) \Rightarrow y = \sin^{-1} u$ (chain rule)

$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

$= \frac{1}{\sqrt{1-\left(\frac{5x}{3}\right)^2}} \cdot \frac{5}{3} = \frac{1}{\sqrt{\frac{9-25x^2}{9}}} \cdot \frac{5}{3} = \frac{\cancel{3}}{\sqrt{9-25x^2}} \cdot \frac{5}{\cancel{3}}$

pg 83 Q10, 5(ii),

$y = e^x \quad \frac{dy}{dx} = e^x$

means the slope of the tangent $= f(x)$!!

$y = e^{ax} \quad \frac{dy}{dx} = a e^{ax}$

$= ay$!!!

Q: $y = (e^x + 1)^4$

product rule:

$y = e^{2x} \cdot \cos 2x$

show

$\frac{dy}{dx} = 0$ when $x = \frac{\pi}{8}$

P 86 Q8 $\quad$ LOGS $\quad y = \ln x$

$dy/dx = \frac{1}{x}$

$\log_e x = y$

$x = e^y \rightarrow \frac{dx}{dy} = e^y$

$\rightarrow dy/dx = \frac{1}{e^y} = \frac{1}{x}$

LOGS: $\ln e^{x} = x \ln e = x \log_e e = x$ so $\ln e^{x} = x$

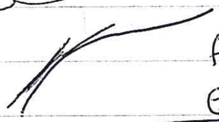
differentiate $y = \log_e(4x^2+1) \rightarrow y = \log_e u \quad dy/dx = \frac{1}{u} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{4x^2+1} \cdot 8x = \frac{8x}{4x^2+1}$$

P88, Q16, 17

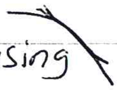
APPLICATIONS of DIFFERENTIAL CALCULUS: $\frac{dy}{dx}$ = slope of tangent

$\frac{dy}{dx} = \oplus$



$f(x)$ is increasing
$\oplus$ slope

$\frac{dy}{dx} = \ominus$ $f(x)$ is decreasing
$\ominus$ slope

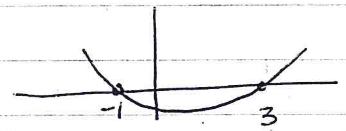


Q Find the interval for which $f(x) = x^3 - 3x^2 - 9x + 9$ is increasing?

$f'(x)$ is $\oplus$, $f'(x) > 0$, $3x^2 - 6x - 9 > 0$

$x^2 - 2x - 3 > 0$ DRAW $x^2 - 2x - 3 = 0$

where (for what values of x) is it > 0



$x < -1$, $x > 3$

if an $f(x)$ is always increasing $f'(x)$ is always $\oplus$
- can never be $\ominus$

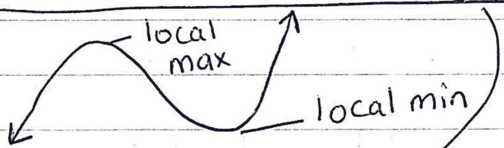
$y = x^3 + 3x^2 + 6x$

$\frac{dy}{dx} = 3x^2 + 6x + 6$ – show this is always > 0 ?

$= 3(x^2 + 2x + 2) = 3((x+1)^2 + 1) \leftarrow$ always $\oplus$

Pg 97 Q8 !! Q15, 16 p98 P98 Q23

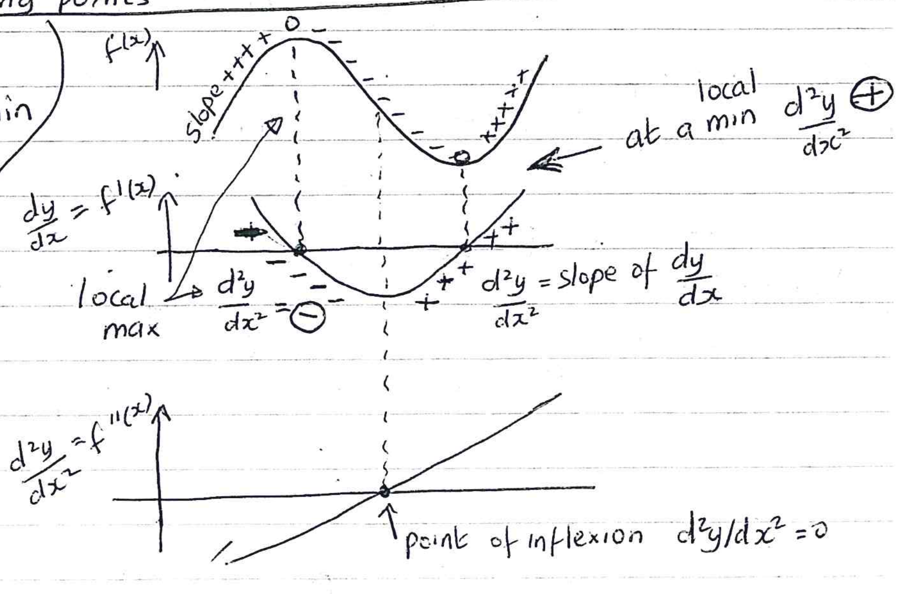
STATIONARY POINTS = max, min points where $dy/dx = 0$ slope $= 0$
= turning points



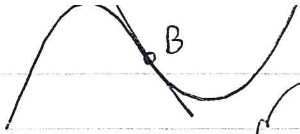
local max if $\frac{d^2y}{dx^2} = \ominus$

local min if $\frac{d^2y}{dx^2} = \oplus$

Q5 p104
Q7 p104



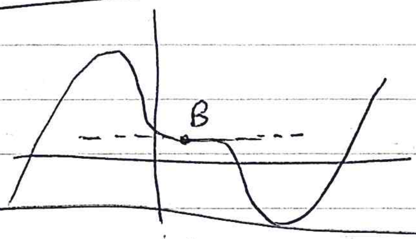
Points of inflexion: (B)



* Tangent to curve crosses curve *

$\frac{d^2y}{dx^2} = 0$ at point of inflexion

if the tangent at the point of inflexion is // to x axis then pt of inflexion = saddle pt or horizontal pt of inflexion



Q: Verify that $y = \frac{x+3}{2x-3}$ has no turning points OR pts of inflexion

- show $\frac{dy}{dx} \neq 0$ (ever), $\frac{d^2y}{dx^2} \neq 0$ ever

Verify that curve is always decreasing (show dy/dx always $\ominus$)

Q17 p105!!!

GRAPHS & DERIVATIVES: p106 Q Example 2, P107 Q2

MAX and MIN problems: Q7, 8, 13 p116

chain rule

$$\frac{dA}{dr} = \frac{dA}{d\circ}\frac{d\circ}{dr}$$

RATES OF CHANGE:

x = Distance

$\frac{dx}{dt}$ = Velocity (speed)

$\frac{d^2x}{dt^2}$ = acceleration

$$\frac{dt}{dr} = \frac{1}{\frac{dr}{dt}}$$

p 121, Q12, 13

p 124 Q12, 13,
p 131 Q5, 6, 7!

Q9 p132!

INTEGRATION: (Antidifferentiation)

$y = f(x)$ differentiate ↓ ↑ integrate
$\frac{dy}{dx} = f'(x)$

$\frac{dy}{dx} = f'(x) \rightarrow dy = f'(x)dx$

$$\int dy = \int f'(x)dx$$

$$y = \int f'(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$\int \sqrt{x}(x+4)\,dx$

$= \int (x^{3/2} + 4x^{1/2})\,dx$

$= \int x^{3/2}dx + \int 4x^{1/2}dx$

Constant of integration (+C)

DONT FORGET IT!!

Q7 page 138!!

Pg 138 Q13, 14

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$ gives

$y = 5^x$

$\frac{dy}{dx} = ?$

$\ln y = \ln 5^x$

$x = \frac{\ln y}{\ln 5}$, $\frac{dx}{dy} = \frac{1}{\ln 5}\cdot\frac{1}{y}$

$dy/dx = \ln 5 \cdot y$

$= \ln 5 \cdot 5^x$

$= 5^x \ln 5$

$\int 5^x \ln 5 =$

$\int 5^x d$

INTEGRATION ...

$y = \sin ax$

$\frac{dy}{dx} = a\cos ax$ (chain rule) so $\int a\cos ax\, dx = \sin ax$

$\int \cos ax\, dx = \frac{\sin ax}{a}$ or $\frac{1}{a}\sin ax$

... same for $\tan ax$, ...

$h(x) = x\ln x$

$h'(x) = x \cdot \frac{1}{x} + \ln x \cdot 1 = 1 + \ln x = \frac{dh}{dx}$

So $\int (1 + \ln x)\, dx = x\ln x$

Q15 p142

Q17 p142

$\int 1\, dx + \int \ln x\, dx = x\ln x$

$\int \ln x\, dx = x\ln x - \int 1\, dx = x\ln x - x + c$

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x = distance $\xleftarrow{\text{integrate}}$

diff $\frac{dx}{dt}$ = speed/velocity

diff $\frac{d^2x}{dt^2}$ = acceleration $= \frac{dv}{dt}$ integrate

acceleration a

speed/vel $\int a\, dt = v$

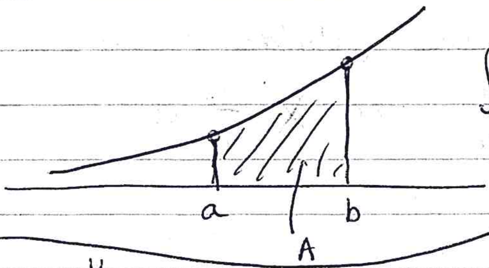
distance $\int v\, dt$

Do Q6, 8, 5 pg 145

Definite integrals $\Rightarrow \int_a^b$

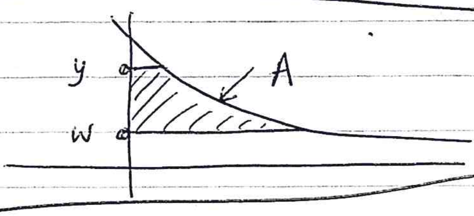
No Constant C!

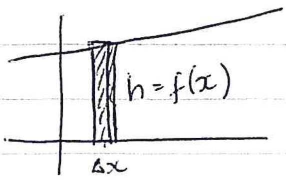
$\int_4^9 \frac{1}{\sqrt{x}}\, dx =$



$\int_a^b f(x)\, dx = \int_a^b y\, dx$ = Area between $f(x)$ and x axis. from $[a, b]$

$\int_w^y x\, dy$ = Area between $f(x)$ and y axis from $[a, b]$





Area of $\| = f(x)\Delta x = \Delta A$

as $\Delta x \to 0$ $\quad f(x)\,dx = dA \Rightarrow f(x) = dA/dx$

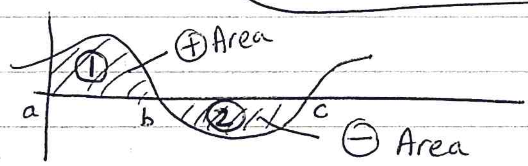
$\int f(x)\, dx = A$

$\int_0 \frac{\quad}{2x+8}\,dx$ FACTORISE top / bottom & divide

dx hence evaluate $\int_0 3x\cos 3x\,dx$

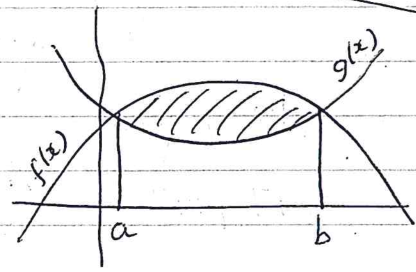
Find $\int_{-2}^{2} \frac{e^x+e^{-x}}{2}\,dx$

AREA



to find area you must
$\int_a^b f(x)dx + \left|\int_b^c f(x)dx\right|$

if you $\int_a^c f(x)\,dx$, the 'maths' subtracts A2 from A1
⇒ wrong



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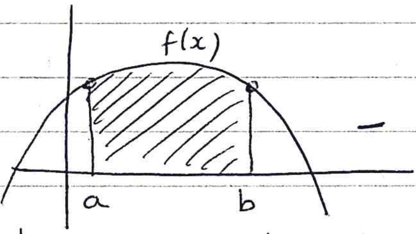

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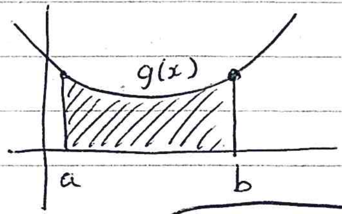
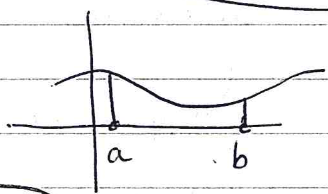


$= \int_a^b f(x)dx - \int_a^b g(x)dx$

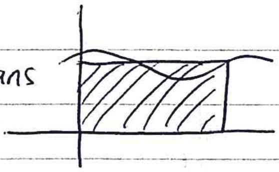
Q11 p154, Q18 p155

Average Value of a function f(x) between $x=a$, $x=b$

$= \frac{1}{b-a}\int_a^b f(x)\,dx$



Average value means equivalent ~~area~~ height of ▭ which gives same area under curve



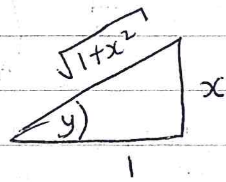
Q13 p161

(*) $dy/dx = dy/dt \times dt/dx$!!! Q5 papers

Paper Q's:

Q4 (c) snowball
Q5 (b, c)
Q7 (b i, ii) all)
Q8 (b, c)
Q13 → * $y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$
Q11
Q10

* show two tangents are ⊥ to each other (dy/dx = slope so $\frac{dy}{dx}$ of one $\times$ $\frac{dy}{dx}$ of other $= -1$



~~Stan~~ $\tan y = \frac{x}{1}$ $\quad y = \tan^{-1}x$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

*