Chapter 12 Exercise 12A

Q. 1. (i) $\log_e 4 + \log_e 3 = \log_e 12$

(ii) $\log_e 6 - \log_e 7 = \log_e \frac{6}{7}$

(iii) $2\log_e 3 + 3\log_e 2 = \log_e 3^2 + \log_e 2^3$

$= \log_e 9 + \log_e 8$

$= \log_e 72$

(iv) $5\log_e 2 - 2\log_e 5 = \log_e 2^5 - \log_e 5^2$

$= \log_e 32 - \log_e 25$

$= \log_e \frac{32}{25}$

(v) $\frac{1}{2}\log_e 4 + \frac{1}{3}\log_e 27$

$= \log_e 4^{\frac{1}{2}} + \log_e 27^{\frac{1}{3}}$

$= \log_e 2 + \log_e 3$

$= \log_e 6$

(vi) $\frac{1}{2}\log_e 64 - \frac{1}{3}\log_e 64$

$= \log_e 8 - \log_e 4$

$= \log_e 2$

(vii) $2\log_e 10 + \log_e 6 - 3\log_e 4$

$= \log_e 100 + \log_e 6 - \log_e 64$

$= \log_e \left(\frac{100 \times 6}{64}\right)$

$= \log_e \left(\frac{75}{8}\right)$

(viii) $\log_e x^2 + \log_e x = \log_e x^3$

(ix) $\frac{1}{2}\log_e x - \log_e 7 + \log_e 2$

$= \log_e \sqrt{x} - \log_e 7 + \log_e 2$

$= \log_e \frac{2\sqrt{x}}{7}$

Q. 2. (i) $e^x = e$

$\Rightarrow x = 1$

(ii) $x = e^2$

(iii) $e^x = \frac{1}{e}$

$\Rightarrow x = -1$

(iv) $e^x = \sqrt[3]{e}$

$\Rightarrow x = \frac{1}{3}$

(v) $e^{\log_e x} = 8$

$\Rightarrow x = 8$ (since e and $\log_e$ are inverse functions).

(vi) $\log_e(e^4) = x$

$\Rightarrow e^x = e^4$

$\Rightarrow x = 4$

(vii) $e^{\log_e 2} = x$

$\Rightarrow 2 = x$

(viii) $e^{\frac{1}{2}\log_e x} = 7$

$\Rightarrow e^{\log_e \sqrt{x}} = 7$

$\Rightarrow \sqrt{x} = 7$

$\Rightarrow x = 49$

(ix) $e^{2\log_e x} = 9$

$\Rightarrow e^{\log_e x^2} = 9$

$\Rightarrow x^2 = 9$

$\Rightarrow x = +3$

(-3 is not possible since $\log_e(-3)$ doesn't exist).

(x) $e^{3\log_e 4} = x$

$\Rightarrow e^{\log_e 64} = x$

$\Rightarrow x = 64$

(xi) $\log_e x = 1 + \log_e 2$

$\Rightarrow \log_e x - \log_e 2 = 1$

$\Rightarrow \log_e \frac{x}{2} = 1$

$\Rightarrow \frac{x}{2} = e^1$

$\Rightarrow x = 2e$

(xii) $\log_e x = 3 - \log_e 5$

$\Rightarrow \log_e x + \log_e 5 = 3$

$\Rightarrow \log_e 5x = 3$

$\Rightarrow 5x = e^3$

$\Rightarrow x = \frac{1}{5}e^3$

(xiii) $2\log_e x = 1 - \log_e 3$

$\Rightarrow \log_e x^2 = 1 - \log_e 3$

$\Rightarrow \log_e x^2 + \log_e 3 = 1$

$\Rightarrow \log_e 3x^2 = 1$

$\Rightarrow 3x^2 = e$

$\Rightarrow x = \sqrt{\frac{e}{3}}$

(xiv) $\frac{1}{2}\log_e x + 1 = \frac{3}{2}\log_e 3$

$\Rightarrow \log_e x + 2 = 3\log_e 3$

$\Rightarrow \log_e x - 3\log_e 3 = -2$

$\Rightarrow \log_e\left(\frac{x}{27}\right) = -2$

$\Rightarrow \frac{x}{27} = e^{-2}$

$\Rightarrow x = 27e^{-2}$

Exercise 12B

Q. 1. (i) $\int x^4\,dx = \frac{x^5}{5} + c$

(ii) $\int 3x^4\,dx = \frac{3x^5}{5} + c$

Q. 2. (i) $\int \cos x\,dx = \sin x + c$

(ii) $\int \cos 3x\,dx = \frac{\sin 3x}{3} + c$

Q. 3. (i) $\int \sin x\,dx = -\cos x + c$

(ii) $\int \sin 4x\,dx = -\frac{\cos 4x}{4} + c$

Q. 4. (i) $\int \frac{1}{x}\,dx = \ln x + c$

(ii) $\int \frac{1}{2x+3}\,dx = \frac{\ln(2x+3)}{2} + c$

Q. 5. (i) $\int e^x\,dx = e^x + c$

(ii) $\int e^{8x}\,dx = \frac{e^{8x}}{8} + c$

Q. 6. (i) $\int \frac{1}{\sqrt{49 - x^2}}\,dx = \sin^{-1}\frac{x}{7} + c$

(ii) $\int \frac{dx}{\sqrt{100 - x^2}} = \sin^{-1}\frac{x}{10} + c$

Q. 7. (i) $\int \frac{1}{x^2 + 25}\,dx = \frac{1}{5}\tan^{-1}\frac{x}{5} + c$

(ii) $\int \frac{dx}{x^2 + 625} = \int \frac{dx}{x^2 + (25)^2}$

$= \frac{1}{25}\tan^{-1}\frac{x}{25} + c$

Q. 8. (i) $\int \frac{x+1}{x}\,dx = \int\left(1 + \frac{1}{x}\right)dx$

$= x + \ln x + c$

(ii) $\int \frac{1}{x+1}dx = \ln(x+1) + c$

Q. 9. (i) $\int (2x+1)\,dx = x^2 + x + c$

(ii) $\int (2x+1)^2 = \int (4x^2 + 4x + 1)dx$

$= \frac{4x^3}{3} + 2x^2 + x + c$

Q. 10. (i) $\int \frac{1}{5x+1}\,dx = \frac{\ln(5x+1)}{5} + c$

(ii) $\int \frac{1}{\sqrt{x}}\,dx = \int \frac{1}{x^{\frac{1}{2}}}\,dx$

$= \int x^{\frac{-1}{2}}\,dx$

$= 2x^{\frac{1}{2}} + c$

$= 2\sqrt{x} + c$

Exercise 12C

Q. 1. (i) $\int_1^3 4x\,dx = 2\left[x^2\right]_1^3 = 2[9 - 1] = 16$

(ii) $\int_0^1 x^2\,dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}$

(iii) $\int_1^2 (2x+1)dx = \left[x^2 + x\right]_1^2$

$= (4+2) - (1+1) = 4$

(iv) $\int_1^4 \left(2 - x^{\frac{1}{2}}\right)dx = \left[2x - \frac{2x^{\frac{3}{2}}}{3}\right]_1^4$

$= 2\left[x - \frac{x\sqrt{x}}{3}\right]_1^4$

$= 2\left[\left(4 - \frac{4\sqrt{4}}{3}\right) - \left(1 - \frac{1\sqrt{1}}{3}\right)\right]$

$= \frac{4}{3}$

(v) $\int_2^6 \frac{1}{x}\,dx = \left[\ln x\right]_2^6 = \ln 6 - \ln 2$

$= \ln\frac{6}{2} = \ln 3$

(vi) $\int_{-1}^1 x^3\,dx = \frac{1}{4}\left[x^4\right]_{-1}^1$

$= \frac{1}{4}[(1)^4 - (-1)^4] = 0$

Q. 2. $\int_0^{\ln 4} e^x\,dx = \left[e^x\right]_0^{\ln 4}$

$= e^{\ln 4} - e^0$

$= 4 - 1$

$= 3$

Q. 3. $\int_0^{\frac{\pi}{2}} \cos x\,dx = \left[\sin x\right]_0^{\frac{\pi}{2}}$

$= \sin\frac{\pi}{2} - \sin 0$

$= 1$

Q. 4. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \, dx = -\left[\cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= -\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right]$

$= \frac{1}{\sqrt{2}}$

Q. 5. $\int_{\ln 2}^{\ln 5} e^x \, dx = \left[e^x\right]_{\ln 2}^{\ln 5}$

$= e^{\ln 5} - e^{\ln 2}$

$= 5 - 2$

$= 3$

Q. 6. $2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x \, dx = 2\left[\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= 2\left[\sin\frac{\pi}{3} - \sin\frac{\pi}{6}\right]$

$= \sqrt{3} - 1$

Q. 7. $4\int_0^{\frac{\pi}{3}} \sin x \, dx = -4\left[\cos x\right]_0^{\frac{\pi}{3}}$

$= -4\left[\cos\frac{\pi}{3} - \cos 0\right]$

$= 2$

Q. 8. $\int_0^{\frac{\pi}{4}} (\cos x + \sin x) \, dx$

$= [\sin x - \cos x]_0^{\frac{\pi}{4}}$

$= \left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) - (\sin 0 - \cos 0) = 1$

Q. 9. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x) \, dx$

$= \left[\sin x + \cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \left(\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)$

$= 1 - \sqrt{2}$

Q. 10. $\int_2^4 \frac{dx}{\sqrt{16 - x^2}}$

$= \left[\sin^{-1}\frac{x}{4}\right]_2^4$

$= \sin^{-1}1 - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

Q. 11. $\int_0^{\ln 2} e^{3x} \, dx$

$= \frac{1}{3}\left[e^{3x}\right]_0^{\ln 2}$

$= \frac{1}{3}[e^{3 \ln 2} - e^0]$

$= \frac{1}{3}[e^{\ln(2)^3} - 1]$

$= \frac{1}{3}[e^{\ln 8} - 1] = \frac{1}{3}[8 - 1] = \frac{7}{3}$

Q. 12. $\int_0^{\frac{\pi}{16}} \cos 4x \, dx$

$= \frac{1}{4}[\sin 4x]_0^{\frac{\pi}{16}}$

$= \frac{1}{4}\left[\sin\left(\frac{\pi}{4}\right) - \sin 0\right]$

$= \frac{1}{4\sqrt{2}}$

Q. 13. $\int_0^{\frac{\pi}{6}} \sin 2x \, dx$

$= -\frac{1}{2}[\cos 2x]_0^{\frac{\pi}{6}}$

$= -\frac{1}{2}\left[\cos\frac{\pi}{3} - \cos 0\right]$

$= \frac{1}{4}$

Q. 14. $\int_0^{\frac{\pi}{6}} \cos 3x \, dx$

$= +\frac{1}{3}[\sin 3x]_0^{\frac{\pi}{6}}$

$= +\frac{1}{3}\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= +\frac{1}{3}$

Q. 15. $\int_0^{\frac{\pi}{12}} \sin 6x \, dx$

$= -\frac{1}{6}[\cos 6x]_0^{\frac{\pi}{12}}$

$= -\frac{1}{6}\left[\cos\frac{\pi}{2} - \cos 0\right]$

$= \frac{1}{6}$

Q. 16. (i) $\int_0^2 \frac{dx}{x^2 + 4}$

$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$

$= \frac{1}{2}[\tan^{-1} 1 - \tan^{-1} 0]$

$= \frac{\pi}{8}$

(ii) $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2x) \, dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin \pi}{2}\right]$

$= \frac{\pi}{4}$

Q. 17. (i) $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4 - x^2}}$

$= \left[\sin^{-1} \frac{x}{2}\right]_1^{\sqrt{3}}$

$= \left[\sin^{-1} \frac{\sqrt{3}}{2} - \sin^{-1} \frac{1}{2}\right]$

$= \frac{\pi}{3} - \frac{\pi}{6}$

$= \frac{\pi}{6}$

(ii) $\int_0^{\frac{\pi}{3}} \sin^2 x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}}(1 - \cos 2x)dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\sin \frac{2\pi}{3}}{2}\right]$

$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

Q. 18. (i) $\int_0^9 \frac{dx}{x^2 + 81}$

$= \frac{1}{9}\left[\tan^{-1} \frac{x}{9}\right]_0^9$

$= \frac{1}{9}[\tan^{-1} 1 - \tan^{-1} 0]$

$= \frac{\pi}{36}$

(ii) $\int_0^{\frac{\pi}{4}} \cos^2 2x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}}(1 + \cos 4x)dx$

$= \frac{1}{2}\left[x + \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\frac{\pi}{4} + 0\right]$

$= \frac{\pi}{8}$

Q. 19. (i) $\int_0^1 \frac{dx}{\sqrt{2 - x^2}}$

$= \int_0^1 \frac{dx}{\sqrt{(\sqrt{2})^2 - x^2}}$

$= \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1$

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4}$

(ii) $\int_0^{\frac{\pi}{8}} \sin^2(2x) \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{8}}(1 - \cos 4x) \, dx$

$= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{8}}$

$= \frac{1}{2}\left[\frac{\pi}{8} - \frac{1}{4}\right]$

$= \frac{1}{8}\left[\frac{\pi}{2} - 1\right]$

Q. 20. (i) $\int_1^{\sqrt{3}} \frac{dx}{x^2 + 3}$

$= \int_1^{\sqrt{3}} \frac{dx}{x^2 + (\sqrt{3})^2}$

$= \frac{1}{\sqrt{3}}\left[\tan^{-1} \frac{x}{\sqrt{3}}\right]_1^{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}\left[\tan^{-1} 1 - \tan^{-1} \frac{1}{\sqrt{3}}\right]$

$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$

$= \frac{\pi}{12\sqrt{3}}$

(ii) $\int_0^{\sqrt{2}} \frac{dx}{\sqrt{8 - x^2}}$

$= \int_0^{\sqrt{2}} \frac{dx}{\sqrt{(\sqrt{8})^2 - x^2}}$

$= \left[\sin^{-1} \frac{x}{\sqrt{8}}\right]_0^{\sqrt{2}}$

$= \sin^{-1} \frac{\sqrt{2}}{8}$

$= \sin^{-1} \frac{1}{2}$

$= \frac{\pi}{6}$

Exercise 12D

Q. 1. $\frac{dy}{dx} = 6y$

$\Rightarrow \int \frac{dy}{y} = 6 \int dx$

$\Rightarrow \ln y = 6x + c$

$\Rightarrow y = e^{6x + c}$

Q. 2. $\frac{dy}{dx} = 2xy$

$\Rightarrow \int \frac{dy}{y} = 2 \int x \, dx$

$\Rightarrow \ln y = x^2 + c$

$\Rightarrow y = e^{x^2 + c}$

Q. 3. $\frac{dy}{dx} = \cos x$

$\Rightarrow \int dy = \int \cos x \, dx$

$\Rightarrow y = \sin x + c$

Q. 4. $\frac{dy}{dx} = \frac{4x^3}{y}$

$\Rightarrow \int y \, dy = 4\int x^3 \, dx$

$\Rightarrow \frac{y^2}{2} = x^4 + c$

$\Rightarrow y^2 = 2(x^4 + c)$

$\Rightarrow y = \pm\sqrt{2(x^4 + c)}$

Q. 5. $\frac{dy}{dx} = 3x^2y$

$\Rightarrow \int \frac{dy}{y} = 3\int x^2 \, dx$

$\Rightarrow \ln y = x^3 + c$

$\Rightarrow y = e^{(x^3 + c)}$

Q. 6. $\frac{dy}{dx} + \frac{\sin 2x}{y} = 0$

$\Rightarrow \int \frac{dy}{dx} = -\frac{\sin 2x}{y}$

$\Rightarrow \int y \, dy = -\int \sin 2x \, dx$

$\Rightarrow \frac{y^2}{2} = \frac{\cos 2x}{2} + c$

$\Rightarrow y^2 = \cos 2x + 2c$

$\Rightarrow y = \pm\sqrt{\cos 2x + 2c}$

Q. 7. $\frac{dy}{dx} = 2x(y^2 + 1)$

$\Rightarrow \int \frac{dy}{y^2 + 1} = 2\int x \, dx$

$\Rightarrow \tan^{-1} y = x^2 + c$

$\Rightarrow y = \tan(x^2 + c)$

Q. 8. $\frac{dy}{dx} = 2\sqrt{1 - y^2}$

$\Rightarrow \int \frac{dy}{\sqrt{1 - y^2}} = 2\int dx$

$\Rightarrow \sin^{-1} y = 2x + c$

$\Rightarrow y = \sin(2x + c)$

Exercise 12E

Q. 1. $\frac{dy}{dx} = 3y$

$\Rightarrow \frac{1}{y} dy = 3 \, dx$

$\Rightarrow \int_1^y \frac{1}{y} dy = \int_0^x 3 \, dx$

$\Rightarrow \log_e y|_1^y = 3x|_0^x$

$\Rightarrow \log_e y - \log_e 1 = 3x - 3(0)$

$\Rightarrow \log_e y = 3x$

$\Rightarrow y = e^{3x}$

Q. 2. $\frac{dy}{dx} = 5y$

$\Rightarrow \frac{1}{y} dy = 5 \, dx$

$\Rightarrow \int_2^y \frac{1}{y} dy = \int_0^x 5 \, dx$

$\Rightarrow \log_e y|_2^y = 5x|_0^x$

$\Rightarrow \log_e y - \log_e 2 = 5x - 5(0)$

$\Rightarrow \log_e \frac{y}{2} = 5x$

$\Rightarrow \frac{y}{2} = e^{5x}$

$\Rightarrow y = 2e^{5x}$

Q. 3. $\frac{dy}{dx} = 2x\sqrt{1 - y^2}$

$\Rightarrow \frac{1}{\sqrt{1 - y^2}} dy = 2x \, dx$

$\Rightarrow \int_1^y \frac{1}{\sqrt{1 - y^2}} dy = \int_0^x 2x \, dx$

$\Rightarrow \sin^{-1} y|_1^y = x^2|_0^x$

$\Rightarrow \sin^{-1} y - \sin^{-1} 1 = x^2 - 0^2$

$\Rightarrow \sin^{-1} y - \frac{\pi}{2} = x^2$

$\Rightarrow \sin^{-1} y = x^2 + \frac{\pi}{2}$

$\Rightarrow y = \sin\left(x^2 + \frac{\pi}{2}\right)$

Q. 4. $\frac{dx}{dt} = tx$

$\Rightarrow \frac{1}{x} dx = t \, dt$

$\Rightarrow \int_{\sqrt{e}}^x \frac{1}{x} dx = \int_1^t t \, dt$

$\Rightarrow \log_e x|_{\sqrt{e}}^x = \frac{t^2}{2}\Big|_1^t$

$\Rightarrow \log_e x - \log_e \sqrt{e} = \frac{t^2}{2} - \frac{1}{2}$

$\Rightarrow \log_e x - \frac{1}{2}\log_e e = \frac{t^2 - 1}{2}$

$$\Rightarrow 2\log_e x - 1 = t^2 - 1$$
$$\Rightarrow \log_e x = \frac{t^2}{2}$$
$$\Rightarrow x = e^{\frac{t^2}{2}}$$

Q. 5. $\frac{dy}{dx} = \frac{y}{x}$

$$\Rightarrow \frac{1}{y}\,dy = \frac{1}{x}\,dx$$
$$\Rightarrow \int_3^y \frac{1}{y}\,dy = \int_1^x \frac{1}{x}\,dx$$
$$\Rightarrow \log_e y\Big|_3^y = \log_e x\Big|_1^x$$
$$\Rightarrow \log_e y - \log_e 3 = \log_e x - \log_e 1$$
$$\Rightarrow \log_e \frac{y}{3} = \log_e x$$
$$\Rightarrow \frac{y}{3} = x \quad \text{...let } y = 21$$
$$\Rightarrow x = 7$$

Q. 6. $\frac{ds}{dt} + \frac{\sin t}{s} = 0$

$$\Rightarrow \frac{ds}{dt} = \frac{-\sin t}{s}$$
$$\Rightarrow s\,ds = -\sin t\,dt$$
$$\Rightarrow \int_{\sqrt{2}}^s s\,ds = \int_{\frac{\pi}{3}}^t (-\sin t)\,dt$$
$$\Rightarrow \frac{s^2}{2}\Big|_{\sqrt{2}}^s = \cos t\Big|_{\frac{\pi}{3}}^t$$
$$\Rightarrow \frac{s^2}{2} - 1 = \cos t - \cos\frac{\pi}{3}$$
$$\Rightarrow \frac{s^2}{2} - 1 = \cos t - \frac{1}{2}$$
$$\Rightarrow s^2 - 2 = 2\cos t - 1$$
$$\Rightarrow s^2 = 2\cos t + 1$$
$$\Rightarrow s = \sqrt{2\cos t + 1}$$

Q. 7. $\frac{dy}{dx} - 4x^3y = 0$

$$\Rightarrow \frac{dy}{dx} = 4x^3y$$
$$\Rightarrow \frac{1}{y}\,dy = 4x^3\,dx$$
$$\Rightarrow \int_3^y \frac{1}{y}\,dy = \int_0^x 4x^3\,dx$$
$$\Rightarrow \log_e y\Big|_3^y = x^4\Big|_0^x$$
$$\Rightarrow \log_e y - \log_e 3 = x^4 - 0^4$$
$$\Rightarrow \log_e \frac{y}{3} = x^4$$
$$\Rightarrow \frac{y}{3} = e^{x^4}$$
$$\Rightarrow y = 3e^{x^4}$$

Q. 8. $\frac{y}{x^2}\frac{dy}{dx} = 1$

$$\Rightarrow y\,dy = x^2\,dx$$
$$\Rightarrow \int_0^y y\,dy = \int_1^x x^2\,dx$$
$$\Rightarrow \frac{y^2}{2}\Big|_0^y = \frac{x^3}{3}\Big|_1^x$$
$$\Rightarrow \frac{y^2}{2} = \frac{x^3}{3} - \frac{1}{3}$$
$$\Rightarrow 3y^2 = 2x^3 - 2$$
$$\Rightarrow y^2 = \frac{2}{3}(x^3 - 1)$$
$$\Rightarrow y = \sqrt{\frac{2}{3}(x^3 - 1)}$$

Q. 9. $\frac{dy}{dx} - \frac{\cos x}{y} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{y}$$
$$\Rightarrow y\,dy = \cos x\,dx$$
$$\Rightarrow \int_1^y y\,dy = \int_{\frac{\pi}{2}}^x \cos x\,dx$$
$$\Rightarrow \frac{y^2}{2}\Big|_1^y = \sin x\Big|_{\frac{\pi}{2}}^x$$
$$\Rightarrow \frac{y^2}{2} - \frac{1}{2} = \sin x - \sin\frac{\pi}{2}$$
$$\Rightarrow \frac{y^2}{2} - \frac{1}{2} = \sin x - 1$$
$$\Rightarrow y^2 - 1 = 2\sin x - 2$$
$$\Rightarrow y^2 = 2\sin x - 1$$
$$\Rightarrow y = \sqrt{2\sin x - 1}$$

Q. 10. $\frac{dy}{dx} - \frac{y}{x} = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{y+1}{x}$$
$$\Rightarrow \frac{1}{y+1}\,dy = \frac{1}{x}\,dx$$
$$\Rightarrow \int_0^y \frac{1}{y+1}\,dy = \int_4^x \frac{1}{x}\,dx$$
$$\Rightarrow \log_e |y+1|\Big|_0^y = \log_e x\Big|_4^x$$
$$\Rightarrow \log_e |y+1| - \log_e 1 = \log_e x - \log_e 4$$
$$\Rightarrow \log_e |y+1| = \log_e \frac{x}{4}$$
$$\Rightarrow y + 1 = \frac{x}{4}$$
$$\Rightarrow y = \frac{x}{4} - 1$$

Q. 11. $\dfrac{dy}{dx} - y^2 \sin x = 0$

$$\Rightarrow \frac{dy}{dx} = y^2 \sin x$$
$$\Rightarrow \frac{1}{y^2} dy = \sin x \, dx$$
$$\Rightarrow \int_1^y \frac{1}{y^2} dy = \int_\pi^x \sin x \, dx$$
$$\Rightarrow \left[-\frac{1}{y}\right]_1^y = -\cos x \Big|_\pi^x$$
$$\Rightarrow \left[\frac{1}{y}\right]_1^y = \cos x \Big|_\pi^x$$
$$\Rightarrow \frac{1}{y} - 1 = \cos x - \cos \pi$$
$$\Rightarrow \frac{1}{y} - 1 = \cos x - (-1)$$
$$\Rightarrow \frac{1}{y} = \cos x + 2$$
$$\Rightarrow y = \frac{1}{\cos x + 2}$$

Q. 12. $xy \dfrac{dy}{dx} = y^2$

$$\Rightarrow \frac{1}{y} dy = \frac{1}{x} dx$$
$$\Rightarrow \int_2^y \frac{1}{y} dy = \int_1^x \frac{1}{x} dx$$
$$\Rightarrow \log_e y \Big|_2^y = \log_e x \Big|_1^x$$
$$\Rightarrow \log_e y - \log_e 2 = \log_e x - \log_e 1$$
$$\Rightarrow \log_e \frac{y}{2} = \log_x x$$
$$\Rightarrow \frac{y}{2} = x$$
$$\Rightarrow y = 2x$$

Q. 13. $v \dfrac{dv}{dx} = \cos^2 x$

$$\Rightarrow v \, dv = \cos^2 x \, dx$$
$$\Rightarrow \int_1^v v \, dv = \int_0^x \cos^2 x \, dx$$
$$\Rightarrow \frac{v^2}{2}\Big|_1^v = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)_0^x$$
$$\Rightarrow v^2\Big|_1^v = \left(x + \frac{1}{2}\sin 2x\right)_0^x$$
$$\Rightarrow v^2 - 1 = x + \frac{1}{2}\sin 2x$$
$$\Rightarrow v^2 = x + \frac{1}{2}\sin 2x + 1$$
$$\Rightarrow v = \sqrt{x + \frac{1}{2}\sin 2x + 1}$$

Q. 14. $\dfrac{dy}{dx} = y \sin x$

$$\Rightarrow \frac{1}{y} dy = \sin x \, dx$$
$$\Rightarrow \int_{\sqrt{e}}^y \frac{1}{y} dy = \int_{\frac{\pi}{3}}^x \sin x \, dx$$
$$\Rightarrow \log_e y \Big|_{\sqrt{e}}^y = -\cos x \Big|_{\frac{\pi}{3}}^x$$
$$\Rightarrow \log_e y - \log_e \sqrt{e} = -\cos x + \cos \frac{\pi}{3}$$
$$\Rightarrow \log_e y - \frac{1}{2} = -\cos x + \frac{1}{2}$$
$$\Rightarrow \log_e y = 1 - \cos x$$
$$\Rightarrow y = e^{1 - \cos x}$$

Q. 15. (i) $\dfrac{dy}{dx} = y \cos x$

$$\Rightarrow \frac{1}{y} dy = \cos x \, dx$$
$$\Rightarrow \int_2^y \frac{1}{y} dy = \int_{\frac{\pi}{6}}^x \cos x \, dx$$
$$\Rightarrow \log_e y \Big|_2^y = \sin x \Big|_{\frac{\pi}{6}}^x$$
$$\Rightarrow \log_e y - \log_e 2 = \sin x - \sin \frac{\pi}{6}$$
$$\Rightarrow \log_e \frac{y}{2} = \sin x - \frac{1}{2}$$
$$\Rightarrow \frac{y}{2} = e^{\sin x - \frac{1}{2}}$$
$$\Rightarrow y = 2e^{\sin x - \frac{1}{2}}$$

(ii) Let $x = \dfrac{\pi}{2}$

$$\Rightarrow y = 2e^{1 - \frac{1}{2}} = 2e^{\frac{1}{2}} = 2\sqrt{e}$$

Q. 16. $2\dfrac{dy}{dx} = xy + x$

$$\Rightarrow 2\frac{dy}{dx} = x(y + 1)$$
$$\Rightarrow \frac{2}{y + 1} dy = x \, dx$$
$$\Rightarrow \int_2^y \frac{2}{y + 1} dy = \int_1^x x \, dx$$
$$\Rightarrow 2 \log_e |y + 1| \Big|_2^y = \frac{x^2}{2}\Big|_1^x$$
$$\Rightarrow 2(\log_e |y + 1| - \log_e 3) = \frac{x^2}{2} - \frac{1}{2}$$
$$\Rightarrow 2 \log_e \left|\frac{y + 1}{3}\right| = \frac{x^2 - 1}{2}$$
$$\Rightarrow \log_e \left|\frac{y + 1}{3}\right| = \frac{x^2 - 1}{4}$$
$$\Rightarrow \frac{y + 1}{3} = e^{\frac{x^2 - 1}{4}}$$

$$\Rightarrow y + 1 = 3e^{\frac{x^2 - 1}{4}}$$

$$\Rightarrow y = 3e^{\frac{x^2 - 1}{4}} - 1 \quad \ldots\text{let } x = 2$$

$$\Rightarrow y = 3e^{\frac{3}{4}} - 1$$

$$\Rightarrow y = 5.35$$

Q. 17. $x\dfrac{dy}{dx} = \sqrt{4 - y^2}$

$$\Rightarrow \frac{1}{\sqrt{4 - y^2}}dy = \frac{1}{x}dx$$

$$\Rightarrow \int_0^y \frac{1}{\sqrt{4 - y^2}}dy = \int_1^x \frac{1}{x}dx$$

$$\Rightarrow \sin^{-1}\frac{y}{2}\Big|_0^y = \log_e x\Big|_1^x$$

$$\Rightarrow \sin^{-1}\frac{y}{2} - \sin^{-1}0 = \log_e x - \log_e 1$$

$$\Rightarrow \sin^{-1}\frac{y}{2} = \log_e x$$

$$\Rightarrow \frac{y}{2} = \sin(\log_e x)$$

$$\Rightarrow y = 2\sin(\log_e x)$$

...general solution

$$\text{Let } x = \sqrt[3]{e^{\frac{\pi}{2}}}$$

$$\Rightarrow y = 2\sin\left(\log_e \sqrt[3]{e^{\frac{\pi}{2}}}\right)$$

$$\Rightarrow y = 2\sin\left(\frac{\pi}{6}\log_e e\right)$$

$$\Rightarrow y = 2\sin\frac{\pi}{6}$$

$$\Rightarrow y = 2\left(\frac{1}{2}\right)$$

$$\Rightarrow y = 1$$

Exercise 12F

Q. 1. $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = 2v, \text{ where } v = \frac{dy}{dx}$$

$$\Rightarrow \int_1^v \frac{dv}{v} = \int_0^x 2\,dx$$

$$\Rightarrow \log_e v\Big|_1^v = 2x\Big|_0^x$$

$$\Rightarrow \log_e v - \log_e 1 = 2x - 2(0)$$

...$\log_e 1 = 0$

$$\Rightarrow \log_e v = 2x$$

$$\Rightarrow v = e^{2x} \quad \ldots\text{End of Step 1}$$

$$\Rightarrow \frac{dy}{dx} = e^{2x}$$

$$\Rightarrow \int_0^y dy = \int_0^x e^{2x}\,dx$$

$$\Rightarrow y\Big|_0^y = \frac{1}{2}e^{2x}\Big|_0^x$$

$$\Rightarrow y - 0 = \frac{1}{2}[e^{2x} - e^0]$$

$$\Rightarrow y = \frac{1}{2}[e^{2x} - 1]$$

Q. 2. $\dfrac{\frac{d^2y}{dt^2}}{\left(\frac{dx}{dt}\right)^2}$

$$\Rightarrow \frac{dv}{dt} = v^2, \text{ where } v = \frac{dx}{dt}$$

$$\Rightarrow \int_1^v \frac{dv}{v^2} = \int_0^t dt$$

$$\Rightarrow -\frac{1}{v}\Big|_1^v = t\Big|_0^t$$

$$\Rightarrow -\frac{1}{v} - (-1) = t - 0$$

$$\Rightarrow 1 - \frac{1}{v} = t$$

$$\Rightarrow \frac{1}{v} = 1 - t$$

$$\Rightarrow v = \frac{1}{1 - t} \quad \ldots\text{End of Step 1}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{1 - t}$$

$$\Rightarrow \int_0^x dx = \int_0^t \frac{dt}{1 - t}$$

$$\Rightarrow x\Big|_0^x = -\log_e(1 - t)\Big|_0^t$$

$$\Rightarrow x - 0 = -\log_e(1 - t) - (-\log_e 1)$$

...$\log_e 1 = 0$

$$\Rightarrow x = \log_e(1 - t)^{-1}$$

$$\Rightarrow x = \log_e\left(\frac{1}{1 - t}\right)$$

Q. 3. $\dfrac{d^2s}{dt^2} = 6$

$$\Rightarrow \frac{dv}{dt} = 6, \text{ where } v = \frac{ds}{dt}$$

$$\Rightarrow \int_4^v dv = \int_0^t 6\,dt$$

$$\Rightarrow v\Big|_4^v = 6t\Big|_0^t$$

$$\Rightarrow v - 4 = 6t$$

$$\Rightarrow v = 6t + 4 \quad \ldots\text{End of Step 1}$$

$$\Rightarrow \frac{ds}{dt} = 6t + 4$$

$$\Rightarrow \int_0^s ds = \int_0^t (6t + 4)dt$$

$$\Rightarrow s\Big|_0^s = (3t^2 + 4t)\Big|_0^t$$

$$\Rightarrow s = 3t^2 + 4t$$

Q. 4. $\dfrac{d^2s}{dt^2} = -\left(\dfrac{ds}{dt}\right)^2$

$$\Rightarrow \frac{dv}{dt} = -v^2, \quad \text{where } v = \frac{ds}{dt}$$

$$\Rightarrow \int_{\frac{1}{2}}^{v} \frac{dv}{v^2} = -\int_0^t dt$$

$$\Rightarrow -\frac{1}{v}\Big|_{\frac{1}{2}}^{v} = -t\big|_0^t$$

$$\Rightarrow -\frac{1}{v} - (-2) = -t$$

$$\Rightarrow 2 - \frac{1}{v} = -t$$

$$\Rightarrow \frac{1}{v} = t + 2$$

$$\Rightarrow v = \frac{1}{t+2} \quad \text{...End of Step 1}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{t+2}$$

$$\Rightarrow \int_0^s ds = \int_0^t \frac{dt}{t+2}$$

$$\Rightarrow s\big|_0^s = \log_e (t+2)\big|_0^t$$

$$\Rightarrow s - 0 = \log_e (t+2) - \log_e 2$$

$$\Rightarrow s = \log_e \left(\frac{t+2}{2}\right)$$

Q. 5. $\dfrac{d^2x}{dt^2} = \left(\dfrac{dx}{dt}\right)^2 + 1$

$$\Rightarrow \frac{dv}{dt} = v^2 + 1, \quad \text{where } v = \frac{dx}{dt}$$

$$\Rightarrow \int_0^v \frac{dv}{v^2+1} = \int_0^t dt$$

$$\Rightarrow \tan^{-1} v\big|_0^v = t\big|_0^t$$

$$\Rightarrow \tan^{-1} v - \tan^{-1} 0 = t - 0 \quad \text{... } \tan^{-1} 0 = 0$$

$$\Rightarrow \tan^{-1} v = t$$

$$\Rightarrow v = \tan t \quad \text{...End of Step 1}$$

$$\Rightarrow \frac{dx}{dt} = \tan t$$

$$\Rightarrow \int_0^x dx = \int_0^t \tan t \, dt$$

$$\Rightarrow x\big|_0^x = -\log_e |\cos t|\big|_0^t$$

$$\Rightarrow x - 0 = -\log_e |\cos t| - (-\log_e |\cos 0|)$$

$$\text{...}\log_e |\cos 0| = \log_e 1 = 0$$

$$\Rightarrow x = -\log_e |\cos t|$$

$$\Rightarrow x = \log_e |\cos t|^{-1}$$

$$\Rightarrow x = \log_e \left|\frac{1}{\cos t}\right| \quad \textbf{OR} \quad x = \log_e |\sec t|$$

Q. 6. $\dfrac{d^2x}{dt^2} = 2x \quad \text{...let } v = \dfrac{dy}{dx} \Rightarrow \dfrac{dv}{dx} = \dfrac{d^2y}{dx^2}$

$$\Rightarrow \frac{dv}{dx} = 2x$$

$$\Rightarrow \int_1^v dv = \int_0^x 2x \, dx$$

$$\Rightarrow v\big|_1^v = x^2\big|_0^x$$

$$\Rightarrow v - 1 = x^2 - 0$$

$$\Rightarrow v = x^2 + 1 \quad \text{...End of Step 1}$$

$$\Rightarrow \frac{dy}{dx} = x^2 + 1$$

$$\Rightarrow \int_{10}^y dy = \int_0^x (x^2 + 1) \, dx$$

$$\Rightarrow y\big|_{10}^y = \left(\frac{x^3}{3} + x\right)\Big|_0^x$$

$$\Rightarrow y - 10 = \frac{x^3}{3} + x$$

$$\Rightarrow y = \frac{x^3}{3} + x + 10$$

Exercise 12G

Q. 1. $\dfrac{d^2y}{dx^2} = y$

$$\Rightarrow v\frac{dv}{dy} = y, \text{ where } v = \frac{dy}{dx}$$

$$\Rightarrow \int_1^v v \, dv = \int_1^y y \, dy$$

$$\Rightarrow \frac{v^2}{2}\Big|_0^v = \frac{y^2}{2}\Big|_0^y$$

$$\Rightarrow \frac{v^2}{2} = \frac{y^2}{2}$$

$$\Rightarrow v^2 = y^2$$

$$\Rightarrow v = y$$

$(v = -y \text{ won't work})$

$$\Rightarrow \frac{dy}{dx} = y$$

$$\Rightarrow \int_1^y \frac{dy}{y} = \int_0^x dx$$

$$\Rightarrow \log_e y\big|_1^y = x\big|_0^x$$

$$\Rightarrow \log_e y - \log_e 1 = x - 0 \quad \text{...}\log_e 1 = 0$$

$$\Rightarrow \log_e y = x$$

$$\Rightarrow y = e^x$$

Q. 2. $\dfrac{d^2s}{dt^2} = 9s$

$$\Rightarrow v\frac{dv}{ds} = 9s, \text{ where } v = \frac{ds}{dt}$$

$$\Rightarrow \int_{-6}^{v} v\,dv = \int_{2}^{s} 9s\,ds$$

$$\Rightarrow \frac{v^2}{2}\Big|_{-6}^{v} = \frac{9s^2}{2}\Big|_{2}^{s}$$

$$\Rightarrow \frac{v^2}{2} - 18 = \frac{9s^2}{2} - 18$$

$$\Rightarrow v^2 = 9s^2$$

$$\Rightarrow v = -3s$$

$(v = 3s$ won't work$)$

$$\Rightarrow \frac{ds}{dt} = -3s$$

$$\Rightarrow \int_{e}^{s} \frac{ds}{s} = \int_{0}^{t} -3dt$$

$$\Rightarrow \log_e s\Big|_{e}^{s} = -3t\Big|_{0}^{t}$$

$$\Rightarrow \log_e s - \log_e e = -3t$$

$$\Rightarrow \log_e s - 1 = -3t$$

$$\Rightarrow \log_e s = 1 - 3t$$

$$\Rightarrow s = e^{1-3t}$$

Q. 3. $\dfrac{d^2y}{dx^2} = -\dfrac{2}{y^5}$

$$\Rightarrow v\frac{dv}{dy} = -\frac{2}{y^5}, \text{ where } v = \frac{dy}{dx}$$

$$\Rightarrow \int_{1}^{v} v\,dv = \int_{1}^{y} -\frac{2}{y^5}\,dy$$

$$\Rightarrow \frac{v^2}{2}\Big|_{1}^{v} = \frac{1}{2y^4}\Big|_{1}^{y}$$

$$\Rightarrow \frac{v^2}{2} - \frac{1}{2} = \frac{1}{2y^4} - \frac{1}{2}$$

$$\Rightarrow v^2 = \frac{1}{y^4}$$

$$\Rightarrow v = \frac{1}{y^2} \quad \left(v = -\frac{1}{y^2} \text{ won't work}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{y^2}$$

$$\Rightarrow \int_{1}^{y} y^2\,dy = \int_{\frac{1}{3}}^{x} dx$$

$$\Rightarrow \frac{y^3}{3}\Big|_{1}^{y} = x\Big|_{\frac{1}{3}}^{x}$$

$$\Rightarrow \frac{y^3}{3} - \frac{1}{3} = x - \frac{1}{3}$$

$$\Rightarrow y^3 = 3x$$

$$\Rightarrow y = \sqrt[3]{3x}$$

Q. 4. $\dfrac{d^2y}{dx^2} = -y$

$$\Rightarrow v\frac{dv}{dy} = -y, \text{ where } v = \frac{dy}{dx}$$

$$\Rightarrow \int_{2}^{v} v\,dv = \int_{0}^{y} -y\,dy$$

$$\Rightarrow \frac{v^2}{2}\Big|_{2}^{v} = -\frac{y^2}{2}\Big|_{0}^{y}$$

$$\Rightarrow \frac{v^2}{2} - 2 = -\frac{y^2}{2}$$

$$\Rightarrow v^2 - 4 = -y^2$$

$$\Rightarrow v^2 = 4 - y^2$$

$$\Rightarrow v = \sqrt{4 - y^2}$$

$(v = -\sqrt{4 - y^2}$ won't work$)$

$$\Rightarrow \frac{dy}{dx} = \sqrt{4 - y^2}$$

$$\Rightarrow \int_{0}^{y} \frac{dy}{\sqrt{4 - y^2}} = \int_{0}^{x} dx$$

$$\Rightarrow \sin^{-1}\frac{y}{2}\Big|_{0}^{y} = x\Big|_{0}^{x}$$

$$\Rightarrow \sin^{-1}\frac{y}{2} - \sin^{-1} 0 = x - 0$$

$\ldots\sin^{-1} 0 = 0$

$$\Rightarrow \sin^{-1}\frac{y}{2} = x$$

$$\Rightarrow \frac{y}{2} = \sin x$$

$$\Rightarrow y = 2\sin x$$

Q. 5. $\dfrac{d^2x}{dt^2} = 2x(9 + x^2)$

$$\Rightarrow v\frac{dv}{dx} = 2x(9 + x^2), \text{ where } v = \frac{dx}{dt}$$

$$\Rightarrow \int_{9}^{v} v\,dv = \int_{0}^{x} 2x(9 + x^2)\,dx$$

$$\Rightarrow \frac{v^2}{2}\Big|_{9}^{v} = \frac{1}{2}(9 + x^2)^2\Big|_{0}^{x}$$

(using the substitution: $u = 9 + x^2$)

$$\Rightarrow \frac{v^2}{2} - \frac{81}{2} = \frac{1}{2}(9 + x^2)^2 - \frac{81}{2}$$

$$\Rightarrow v^2 = (9 + x^2)^2$$

$$\Rightarrow v = 9 + x^2$$

$(v = -(9 + x^2)$ won't work$)$

$$\Rightarrow \frac{dx}{dt} = 9 + x^2$$

$$\Rightarrow \int_{3}^{x} \frac{dx}{9 + x^2} = \int_{0}^{t} dt$$

$$\Rightarrow \frac{1}{3}\tan^{-1}\frac{x}{3}\Big|_{3}^{x} = t\Big|_{0}^{t}$$

$$\Rightarrow \frac{1}{3}\left(\tan^{-1}\frac{x}{3} - \tan^{-1}1\right) = t - 0$$

$$\Rightarrow \frac{1}{3}\left(\tan^{-1}\frac{x}{3} - \frac{\pi}{4}\right) = t$$

$$\Rightarrow \tan^{-1}\frac{x}{3} - \frac{\pi}{4} = 3t$$

$$\Rightarrow \tan^{-1}\frac{x}{3} = 3t + \frac{\pi}{4}$$

$$\Rightarrow \frac{x}{3} = \tan\left(3t + \frac{\pi}{4}\right)$$

$$\Rightarrow x = 3\tan\left(3t + \frac{\pi}{4}\right)$$

Q. 6. $\dfrac{d^2x}{dt^2} = \dfrac{3x^2}{2}$

$$\Rightarrow v\frac{dv}{dx} = \frac{3x^2}{2}, \text{ where } v = \frac{dx}{dt}$$

$$\Rightarrow \int_{-8}^{v} 2v\, dv = \int_{4}^{x} 3x^2\, dx$$

$$\Rightarrow v^2\Big|_{-8}^{v} = x^3\Big|_{4}^{x}$$

$$\Rightarrow v^2 - 64 = x^3 - 64$$

$$\Rightarrow v^2 = x^3$$

$$\Rightarrow v = -\sqrt{x^3}$$

$(v = \sqrt{x^3}$ won't work)

$$\Rightarrow \frac{dx}{dt} = -x^{\frac{3}{2}}$$

$$\Rightarrow \int_{4}^{x} x^{-\frac{3}{2}}\, dx = \int_{0}^{t} -dt$$

$$\Rightarrow \frac{-2}{\sqrt{x}}\Big|_{4}^{x} = -t\Big|_{0}^{t}$$

$$\Rightarrow \frac{-2}{\sqrt{x}} - (-1) = -t$$

$$\Rightarrow \frac{2}{\sqrt{x}} = t + 1$$

$$\Rightarrow \frac{\sqrt{x}}{2} = \frac{1}{t+1}$$

$$\Rightarrow \sqrt{x} = \frac{2}{t+1}$$

$$\Rightarrow x = \left(\frac{2}{t+1}\right)^2$$

Q. 7. $\dfrac{d^2y}{dx^2} + \dfrac{2}{y^3} = 0$

$$\Rightarrow \frac{d^2y}{dx^2} = -2y^{-3}$$

$$\text{Let } \frac{dy}{dx} = v \Rightarrow \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\cdot\frac{dy}{dx} = v\frac{dv}{dy}$$

$$\Rightarrow v\frac{dv}{dy} = -2y^{-3}$$

$$\Rightarrow \int_{\sqrt{2}}^{v} v\, dv = \int_{1}^{y} -2y^{-3}\, dy$$

$$\Rightarrow \frac{v^2}{2}\Big|_{\sqrt{2}}^{v} = \frac{1}{y^2}\Big|_{1}^{y}$$

$$\Rightarrow \frac{v^2}{2} - 1 = \frac{1}{y^2} - 1$$

$$\Rightarrow v^2 = \frac{2}{y^2}$$

$$\Rightarrow v = \frac{\sqrt{2}}{y} \quad \left(v = -\frac{\sqrt{2}}{y} \text{ won't work}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{2}}{y}$$

$$\Rightarrow \int_{1}^{y} y\, dy = \int_{\sqrt{2}}^{\sqrt{18}} \sqrt{2}\, dx$$

$$\Rightarrow \frac{y^2}{2}\Big|_{1}^{y} = \sqrt{2}\,x\Big|_{\sqrt{2}}^{\sqrt{18}}$$

$$\Rightarrow \frac{y^2}{2} - \frac{1}{2} = 6 - 2$$

$$\Rightarrow y^2 - 1 = 8$$

$$\Rightarrow y^2 = 9$$

$$\Rightarrow y = \pm 3$$

Exercise 12H

Q. 1. (i)

$$F = ma$$

$$\Rightarrow -\frac{1}{3}mv = ma$$

$$\Rightarrow a = -\frac{1}{3}v$$

$$\Rightarrow \frac{dv}{dt} = -\frac{1}{3}v$$

$$\Rightarrow \int_{10}^{v} \frac{dv}{v} = \int_{0}^{t} -\frac{1}{3}\, dt$$

$$\Rightarrow \log_e v\Big|_{10}^{v} = -\frac{1}{3}t\Big|_{0}^{t}$$

$$\Rightarrow \log_e v - \log_e 10 = -\frac{1}{3}t$$

$$\Rightarrow \log_e \frac{v}{10} = -\frac{1}{3}t$$

$$\Rightarrow \frac{v}{10} = e^{-\frac{1}{3}t}$$

$$\Rightarrow v = 10e^{-\frac{1}{3}t}$$

...Equation 1

When $t = 3$, $v = 10e^{-1} = \dfrac{10}{e}$ m/s

(ii) $\dfrac{ds}{dt} = 10e^{-\frac{1}{3}t}$ ...from Equation 1

$$\Rightarrow \int_{0}^{s} ds = \int_{0}^{t} 10e^{-\frac{1}{3}t}\, dt$$

$$\Rightarrow s\Big|_{0}^{s} = -30e^{-\frac{1}{3}t}\Big|_{0}^{t}$$

$$\Rightarrow s - 0 = -30e^{-\frac{1}{3}t} - (-30)$$

$$\Rightarrow s = 30\left(1 - e^{-\frac{1}{3}t}\right)$$

When $t = 3$

$$s = 30(1 - e^{-1}) = 30\left(1 - \frac{1}{e}\right)$$

(iii) As $t \to \infty$, $e^{-\frac{1}{3}t} \to 0$

$$\therefore s \to 30(1 - 0) = 30 \text{ m}$$

Q. 2. $a = 25v + v^3$

$$\Rightarrow \frac{\not{v}dv}{ds} = \not{v}(25 + v^2)$$

$$\Rightarrow dv = (25 + v^2)ds \quad \textbf{QED}$$

$$\Rightarrow \int_0^v \frac{dv}{25 + v^2} = \int_0^{0.01} ds$$

$$\Rightarrow \frac{1}{5}\left[\tan^{-1}\frac{v}{5}\right]_0^v = [s]_0^{0.01}$$

$$\Rightarrow \tan^{-1}\frac{v}{5} = 0.05$$

$$\Rightarrow \frac{v}{5} = \tan 0.05$$

$$\Rightarrow v = 5 \tan 0.05$$

$$\Rightarrow v = 0.25 \text{ m/s}$$

Q. 3. $a = v^2 + 100$

$$\Rightarrow \frac{dv}{dt} = v^2 + 100$$

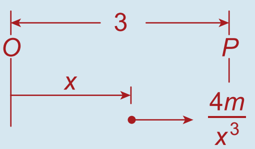
$$\Rightarrow \int_0^{20} \frac{dv}{v^2 + 100} = \int_0^t dt$$

$$\Rightarrow \frac{1}{10}\left[\tan^{-1}\frac{v}{10}\right]_0^{20} = t$$

$$\Rightarrow t = \frac{1}{10}[\tan^{-1} 2]$$

$$\Rightarrow t = 0.11 \text{ s}$$

Q. 4. **Forces**



NZL: $\Sigma F = ma$

$$\underset{\rightarrow}{+} \frac{4m}{x^3} = mv\frac{dv}{dx}$$

(i) $\Rightarrow v\,dv = 4x^{-3}\,dx$ **QED**

$$\Rightarrow \int_{-\frac{\sqrt{5}}{3}}^{v} v\,dv = 4\int_3^x x^{-3}\,dx$$

$$\Rightarrow \frac{1}{2}\left[v^2\right]_{-\frac{\sqrt{5}}{3}}^{v} = -2\left[\frac{1}{x^2}\right]_3^x$$

$$\Rightarrow v^2 - \frac{5}{9} = 4\left[\frac{1}{9} - \frac{1}{x^2}\right]$$

$$\Rightarrow v^2 = 1 - \frac{4}{x^2}$$

$$\Rightarrow v = \sqrt{1 - \frac{4}{x^2}}$$

(ii) For $v = 0$, $\frac{4}{x^2} = 1 \Rightarrow x = 2$ m

(iii) When $x = \frac{5}{2}$

$$v = \sqrt{1 - \frac{4}{\left(\frac{25}{4}\right)}}$$

$$= \sqrt{1 - \frac{16}{25}}$$

$$= \sqrt{\frac{9}{25}}$$

$$\Rightarrow v = \frac{3}{5} \text{ m/s}$$

Q. 5. (i) $F = ma$

$$\Rightarrow 8a = (40 - 3\sqrt{x})$$

$$\Rightarrow a = \frac{40 - 3\sqrt{x}}{8}$$

$$\Rightarrow v\frac{dv}{dx} = \frac{40 - 3\sqrt{x}}{8}$$

$$\Rightarrow \int_0^v 8v\,dv = \int_0^x \left(40 - 3x^{\frac{1}{2}}\right)dx$$

$$\Rightarrow 4v^2\Big|_0^v = \left(40x - 2\sqrt{x^3}\right)\Big|_0^x$$

$$\Rightarrow 4v^2 = 40x - 2\sqrt{x^3}$$

$$\Rightarrow v^2 = 10x - \frac{1}{2}\sqrt{x^3}$$

$$\Rightarrow v = \sqrt{10x - \frac{1}{2}\sqrt{x^3}}$$

...let $x = 100$

$$\Rightarrow v = \sqrt{1000 - \frac{1}{2}(1000)}$$

$$= \sqrt{500}$$

$$= 22.36 \text{ m/s}$$

(ii) $v^2 = 10x - \frac{1}{2}\sqrt{x^3}$ ...let $v = 0$

$$\Rightarrow 10x - \frac{1}{2}\sqrt{x^3} = 0$$

$$\Rightarrow 20x - \sqrt{x^3} = 0$$

$$\Rightarrow 20x - x\sqrt{x} = 0$$

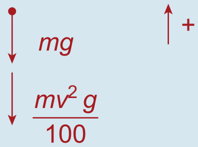
$$\Rightarrow x(20 - \sqrt{x}) = 0$$

$$\Rightarrow 20 - \sqrt{x} = 0$$

$$\Rightarrow \sqrt{x} = 20$$

$$\Rightarrow x = 400 \text{ m}$$

Q. 6. **Forces**



NZL: $\Sigma F = ma$

$$+\uparrow -mg\left[1+\frac{v^2}{100}\right] = m\frac{dv}{dt}$$

$$\Rightarrow -g\left[\frac{100+v^2}{100}\right] = \frac{dv}{dt}$$

$$\Rightarrow \int_{120}^{0}\frac{dv}{v^2+100} = -\frac{g}{100}\int_0^t dt$$

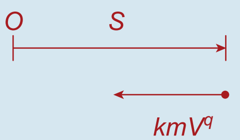
$$\Rightarrow \frac{1}{10}\left[\tan^{-1}\frac{v}{10}\right]_{120}^{0} = -\frac{g}{100}t$$

$$\Rightarrow \frac{gt}{10} = \tan^{-1} 12$$

$$\Rightarrow t = 1.518$$

$$\Rightarrow t = 1.5 \text{ s}$$

Q. 7. **Forces**



NZL: $\Sigma F = ma$

$$\rightarrow -kmv^q = mv\frac{dv}{ds}$$

$$\Rightarrow \int_u^0 v^{1-q}\,dv = -k\int_0^s ds$$

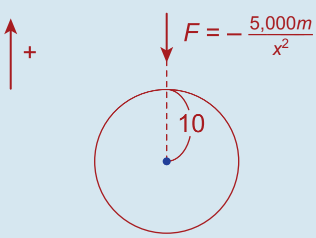
$$\Rightarrow \frac{1}{2-q}\left[v^{2-q}\right]_u^0 = -ks$$

$$\Rightarrow \frac{1}{2-q}\left[0^{2-q} - u^{2-q}\right] = -ks$$

$$\Rightarrow ks = \frac{u^{2-q}}{2-q}$$

$$\Rightarrow s = \frac{u^{2-q}}{(2-q)^k} \quad \textbf{QED}$$

Q. 8.



(i) $F = ma$

$$\Rightarrow -\frac{5,000\,m}{x^2} = ma$$

$$\Rightarrow a = -\frac{5,000}{x^2}$$

$$\Rightarrow v\frac{dv}{dx} = -\frac{5,000}{x^2}$$

$$\Rightarrow \int_{20}^{v} v\,dv = \int_{10}^{x} -\frac{5,000}{x^2}\,dx$$

$$\Rightarrow \frac{v^2}{2}\Big|_{20}^{v} = \frac{5,000}{x}\Big|_{10}^{x}$$

$$\Rightarrow \frac{v^2}{2} - 200 = \frac{5,000}{x} - 500$$

$$\Rightarrow \frac{v^2}{2} = \frac{5,000}{x} - 300$$

$$\Rightarrow v^2 = \frac{10,000}{x} - 600$$

$$\Rightarrow v = \sqrt{\frac{10,000}{x} - 600}$$

$$\ldots\text{let } x = \frac{30}{7} + 10 = \frac{100}{7}$$

$$\Rightarrow v = \sqrt{700 - 600}$$

$$= \sqrt{100}$$

$$= 10 \text{ m/s}$$

(ii) At its greatest height, $v = 0$

$$\Rightarrow \frac{10,000}{x} - 600 = 0$$

$$\Rightarrow 10,000 - 600x = 0$$

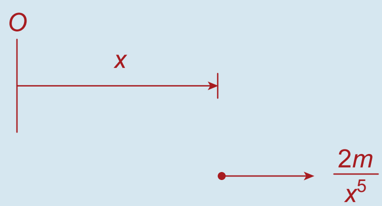
$$\Rightarrow x = \frac{10,000}{600}$$

$$= \frac{50}{3} \text{ m}$$

$$\therefore \text{ height above surface} = \frac{50}{3} - 10$$

$$= \frac{20}{3} \text{ m}$$

Q. 9. **Forces**



NZL: $\Sigma F = ma$

$$\overset{+}{\rightarrow} \frac{2m}{x^5} = mv\frac{dv}{dx}$$

$$\Rightarrow \int_0^v v\,dv = 2\int_d^x x^{-5}\,dx$$

$$\Rightarrow \frac{1}{2}\left[v^2\right]_0^v = -\frac{2}{4}\left[\frac{1}{x^4}\right]_d^x$$

$$\Rightarrow v^2 = \left[\frac{1}{d^4} - \frac{1}{x^4}\right]$$

$$\Rightarrow v = \sqrt{\frac{1}{d^4} - \frac{1}{x^4}} \quad \ldots(1)$$

(i) When $x = \sqrt{3}d$,

$$v = \sqrt{\frac{1}{d^4} - \frac{1}{9d^4}}$$

$$\Rightarrow v = \sqrt{\frac{9-1}{9d^4}}$$

$$\Rightarrow v = \frac{2\sqrt{2}}{3d^2} \quad \textbf{QED}$$

(ii) From ①, $v_T = \sin_{x \to \infty} \sqrt{\frac{1}{d^4} - \frac{1}{x^4}}$

where v_T = Terminal Velocity

$$\Rightarrow v_T = \frac{1}{d^2} \quad \textbf{QED}$$

Q. 10. $\frac{d^2s}{dt^2} = a$, Let $\frac{ds}{dt} = v$

$$\Rightarrow \frac{dv}{dt} = a$$

$$\Rightarrow \int_u^v dv = a\int_0^t dt$$

$$\Rightarrow v - u = at$$

$$\Rightarrow v = u + at$$

$$\Rightarrow \frac{ds}{dt} = u + at$$

$$\Rightarrow \int_0^s ds = \int_0^t (u + at)\, dt$$

$$\Rightarrow s = ut + \frac{at^2}{2} \quad \textbf{QED}$$

Q. 11. $\frac{d^2s}{dt^2} = a$

$$\Rightarrow v\frac{dv}{ds} = a$$

$$\Rightarrow \int_u^v v\, dv = a\int_0^s ds$$

$$\Rightarrow \frac{1}{2}\left[v^2\right]_u^v = as$$

$$\Rightarrow v^2 - u^2 = 2as$$

$$\Rightarrow v^2 = u^2 + 2as \quad \textbf{QED}$$

Exercise 12I

Q. 1. $\frac{dv}{dt} = v^2 + 4$

Note: the integration does require a substitution.

$$\Rightarrow \int_2^v \frac{dv}{v^2 + 4} = \int_0^t dt$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\frac{v}{2}\Big|_2^v = t\Big|_0^t$$

$$\Rightarrow \frac{1}{2}\left[\tan^{-1}\frac{v}{2} - \tan^{-1}1\right] = t - 0$$

$\ldots\tan^{-1} 1 = \frac{\pi}{4}$

$$\Rightarrow t = \frac{1}{2}\left[\tan^{-1}\frac{v}{2} - \frac{\pi}{4}\right]$$

$\ldots$let $v = 6$

$$\Rightarrow t = \frac{1}{2}\left[\tan^{-1}3 - \frac{\pi}{4}\right]$$

$$= 0.23 \text{ s}$$

Q. 2. $\frac{d^2s}{dt^2} = -\left(\frac{ds}{dt}\right)^2$

$$\Rightarrow \frac{dv}{dt} = -v^2$$

where $v = \frac{ds}{dt}$

$$\Rightarrow \int_1^v \frac{dv}{v^2} = -\int_0^t dt$$

$$\Rightarrow -\frac{1}{v}\Big|_1^v = -t\Big|_0^t$$

$$\Rightarrow \frac{1}{v}\Big|_1^v = t\Big|_0^t$$

$$\Rightarrow \frac{1}{v} - 1 = t - 0$$

$$\Rightarrow \frac{1}{v} = t + 1$$

$$\Rightarrow v = \frac{1}{t+1}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{t+1}$$

$$\Rightarrow \int_0^s ds = \int_0^1 \frac{dt}{t+1}$$

$$\Rightarrow s\Big|_0^s = \log_e (t+1)\Big|_0^1$$

$$\Rightarrow s - 0 = \log_e 2 - \log_e 1$$

$\ldots\log_e 1 = 0$

$$\Rightarrow s = \log_e 2$$

Q. 3. $\frac{d^2s}{dt^2} = \frac{1}{2}\left(\frac{ds}{dt}\right)$

$$\Rightarrow \frac{dv}{dt} = \frac{1}{2}v$$

where $v = \frac{ds}{dt}$

$$\Rightarrow \int_3^v \frac{dv}{v} = \int_0^t \frac{1}{2}\, dt$$

$$\Rightarrow \log_e v\Big|_3^v = \frac{1}{2}t\Big|_0^t$$

$$\Rightarrow \log_e v - \log_e 3 = \frac{1}{2}t$$

$$\Rightarrow \log_e \frac{v}{3} = \frac{1}{2}t$$
$$\Rightarrow \frac{v}{3} = e^{\frac{1}{2}t}$$
$$\Rightarrow v = 3e^{\frac{1}{2}t}$$
$$\Rightarrow \frac{ds}{dt} = 3e^{\frac{1}{2}t}$$
$$\Rightarrow \int_0^s ds = \int_4^5 3e^{\frac{1}{2}t}\, dt$$
(since the 5th second is from $t = 4$ to $t = 5$)
$$\Rightarrow s\Big|_0^s = 6e^{\frac{1}{2}t}\Big|_4^5$$
$$\Rightarrow s = 6\left(e^{\frac{5}{2}} - e^2\right) \text{ m}$$

Q. 4. $\frac{d^2s}{dt^2} = k\left(\frac{ds}{dt}\right)^2 \qquad \Rightarrow \frac{dv}{dt} = -kv^2$

when $t = 0, s = 0, v = 20$;

when $s = 100, v = 10$.

$$\Rightarrow \int_{20}^{v} v^{-2}\, dv = \int_0^t -k\, dt$$
$$\Rightarrow -\frac{1}{v}\Big|_{20}^{v} = -kt\Big|_0^t$$
$$\Rightarrow \frac{1}{v}\Big|_{20}^{v} = kt\Big|_0^t$$
$$\Rightarrow \frac{1}{v} - \frac{1}{20} = kt$$
$$\Rightarrow \frac{1}{v} = kt + \frac{1}{20}$$
$$\Rightarrow \frac{1}{v} = \frac{20kt + 1}{20}$$
$$\Rightarrow v = \frac{20}{20kt + 1}$$
$$\Rightarrow \frac{ds}{dt} = \frac{20}{20kt + 1}$$
$$\Rightarrow \int_0^{100} ds = \int_0^t \frac{20}{20kt + 1}$$
$$\Rightarrow s\Big|_0^{100} = \frac{1}{k}\log_e (20kt + 1)\Big|_0^t$$
$$\Rightarrow \frac{1}{k}[\log_e (20kt + 1) - \log_e 1] = 100 - 0 \qquad \ldots \log_e 1 = 0$$
$$\Rightarrow \underbrace{\frac{1}{k}\log_e (20kt + 1) = 100}_{\text{Equation 1}}$$
$$\ldots\text{when } \underbrace{\frac{20}{20kt + 1}}_{\text{Equation 2}} = 10$$
$$\Rightarrow \underbrace{20kt + 1 = 2}_{\text{Equation 3}}$$

$$\Rightarrow \frac{1}{k}\log_e 2 = 100$$
$$\Rightarrow k = \frac{1}{100}\log_e 2$$
Putting this into equation 3 gives
$$\frac{1}{5}(\log_e 2)t + 1 = 2$$
$$\Rightarrow t = \frac{5}{\log_e 2} = 7.2 \text{ s}$$

Q. 5. $a = -\frac{v^3}{25} \quad \ldots①$

$$\Rightarrow v\frac{dv}{dx} = -\frac{v^3}{25}$$
$$\Rightarrow \int_{25}^{v} \frac{dv}{v^2} = -\frac{1}{25}\int_0^x dx$$
$$\Rightarrow -\left[\frac{1}{v}\right]_{25}^{v} = -\frac{1}{25}[x]_0^x$$
$$\Rightarrow \frac{1}{v} - \frac{1}{25} = \frac{x}{25}$$
$$\Rightarrow \frac{1}{v} = \frac{x + 1}{25}$$

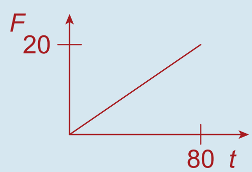
(i) $\Rightarrow v = \frac{25}{x + 1}$

$$\Rightarrow v\,\Big|_{x = 99} = \frac{1}{4} \text{ m/s}$$

(ii) $\frac{dv}{dt} = -\frac{v^3}{25}$ from ①

$$\Rightarrow \int_0^5 v^{-3}\, dv = -\frac{1}{25}\int_0^t dt$$
$$\Rightarrow -\frac{1}{2}\left[\frac{1}{v^2}\right]_{10}^{5} = -\frac{1}{25}t$$
$$\Rightarrow t = \frac{25}{2}\left[\frac{1}{25} - \frac{1}{100}\right]$$
$$\Rightarrow t = \frac{3}{8} \text{ s}$$

Q. 6.



$$F = \frac{20}{80}t + 0$$
$$\Rightarrow F = \frac{t}{4}$$

$\boxed{2} \rightarrow \frac{t}{4}$

NZL: $\Sigma F = ma$

$\overset{+}{\rightarrow} \frac{t}{4} = 2a$

(i) $\Rightarrow a = \frac{t}{8}$ **QED**

(ii) $\frac{dv}{dt} = \frac{t}{8}$

$$\Rightarrow \int_0^v dv = \frac{1}{8}\int_0^t t\, dt$$
$$\Rightarrow v = \frac{t^2}{16} \quad \ldots①$$

$$\Rightarrow \frac{dx}{dt} = \frac{t^2}{16}$$

$$\Rightarrow \int_0^x dx = \frac{1}{16}\int_0^t t^2\,dt$$

$$\Rightarrow x = \frac{t^3}{48} \quad \ldots (2)$$

From (1) $t = 4v^{\frac{1}{2}}$

From (2) $x = \left(\frac{4v^{\frac{1}{2}}}{48}\right)^3$

$$\Rightarrow x = \frac{64v^{\frac{3}{2}}}{48}$$

$$\Rightarrow x = \frac{4v^{\frac{3}{2}}}{3}$$

$$\Rightarrow 3x = 4v^{\frac{3}{2}}v$$

$$9x^2 = 16\,v^3$$

Exercise 12J

Q. 1.

$$P = Fv$$

$$\Rightarrow F = \frac{P}{v}$$

NZL: $\Sigma F = ma$

$$\overset{+}{\rightarrow}\ \frac{25m}{v} = m\frac{dv}{dt}$$

$$\Rightarrow \int_1^3 v\,dv = 25\int_0^t dt$$

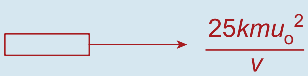
$$\Rightarrow 25t = \frac{1}{2}\left[v^2\right]_1^3$$

$$\Rightarrow t = \frac{1}{50}[9 - 1]$$

$$\Rightarrow t = \frac{4}{25}$$

$$\Rightarrow t = 0.16$$

Q. 2. $F = \frac{P}{v} = \frac{25kmu_0^2}{v}$



NZL: $\Sigma F = ma$

$$\overset{+}{\rightarrow}\ \frac{25\,kmu_0^2}{v} = mv\,\frac{dv}{ds}$$

(i) $\Rightarrow v^2\,dv = 25ku_0^2\,ds$ **QED**

(ii) $\int_{u_0}^{4u_0} v^2\,dv = 25ku_0^2\int_0^s ds$

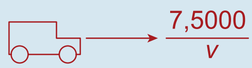
$$\Rightarrow \frac{1}{3}\left[v^3\right]_{u_0}^{4u_0} = 25k\,u_0^2 s$$

$$\Rightarrow \frac{1}{3}[64u_0^3 - u_0^3] = 25ku_0^2 s$$

$$\Rightarrow 21u_0^3 = 25ku_0^2 s$$

$$\Rightarrow s = \frac{21u_0}{25k} \quad \textbf{QED}$$

Q. 3. $F = \frac{P}{v} = \frac{75{,}000}{v}$



NZL: $\Sigma F = ma$

$$\rightarrow \frac{75{,}000}{v} = 1{,}000a$$

(i) $\Rightarrow a = \frac{75}{v} \quad \ldots (1)$

$$\Rightarrow \frac{dv}{dt} = \frac{75}{v}$$

(ii) $\Rightarrow \int_5^{25} v\,dv = 75\int_0^t dt$

$$\Rightarrow \frac{1}{2}\left[v^2\right]_5^{25} = 75t$$

$$\Rightarrow \frac{1}{2}[625 - 25] = 75\,t$$

$$\Rightarrow t = \frac{600}{150}$$

$$\Rightarrow t = 4\text{ s}$$

(iii) From (1)

$$v\frac{dv}{dx} = \frac{75}{v}$$

$$\Rightarrow \int_5^{25} v^2\,dv = 75\int_0^x dx$$

$$\Rightarrow \frac{1}{3}\left[v^3\right]_5^{25} = 75\,x$$

$$\Rightarrow x = \frac{1}{225}[25^3 - 5^3]$$

$$\Rightarrow x = \frac{620}{9}$$

$$\Rightarrow x = 68.89\text{ m}$$