Chapter 4 Exercise 4A

Q. 1. (i) $\vec{v_{BA}} = 30\vec{i} - 25\vec{i}$

$= 5\vec{i}$ m/s

(ii) $\frac{1{,}000}{5} = 200$ s

Q. 2. (i) $\vec{v_A} = 4\vec{i}$

$\vec{v_B} = 7\vec{i}$

$\vec{v_{BA}} = \vec{v_B} - \vec{v_A} = 7\vec{i} - 4\vec{i} = 3\vec{i}$ m/s

(ii) Relative distance = relative speed × time = 3 × 60 = 180 m

(iii) Time $= \frac{\text{relative distance}}{\text{relative speed}}$

$= \frac{600}{3}$

$= 200$ s

Q. 3. (i) $\vec{v_c} = 10\vec{i}$ m/s

(ii) $\vec{v_L} = -15\vec{i}$ m/s

$\therefore \vec{v_{CL}} = 10\vec{i} - (-15\vec{i}) = 25\vec{i}$ m/s

(iii) $\frac{500}{25} = 20$ s

Q. 4. (i) $\vec{v_g} = 1.2\vec{i}$ m/s

(ii) $\vec{v_b} = -1.3\vec{i}$ m/s

$\vec{v_{gb}} = \vec{v_g} - \vec{v_b}$

$= 1.2\vec{i} - (-1.3\vec{i})$

$= 1.2\vec{i} + 1.3\vec{i} = 2.5\vec{i}$ m/s

(iii) Time $= \frac{\text{relative distance}}{\text{relative speed}}$

$= \frac{250}{2.5}$

$= 100$ s

Q. 5. (i) $\vec{v_{pq}} = \vec{v_p} - \vec{v_q}$

$= (5\vec{i} + 2\vec{j}) - (2\vec{i} - 2\vec{j})$

$= 3\vec{i} + 4\vec{j}$ km/h

(ii) $|\vec{v_{pq}}| = \sqrt{3^2 + 4^2}$

$= 5$ km/h

(iii) $\frac{20}{5} = 4$ hours

Q. 6. (i) $\vec{v_{AB}} = (4\vec{i} - 3\vec{j}) - (6\vec{i} - \vec{j})$

$= -2\vec{i} - 2\vec{j}$ m/s

$|\vec{v_{AB}}| = \sqrt{4 + 4} = \sqrt{8}$ m/s

Direction = SW

(ii) $\vec{v_{CB}} = 8\vec{i} - (6\vec{i} - \vec{j})$

$= 2\vec{i} + \vec{j}$ m/s

$|\vec{v_{CB}}| = \sqrt{4 + 1} = \sqrt{5}$ m/s

$\tan\theta = \frac{1}{2} \Rightarrow \theta = 26° 34'$

Direction: E 26° 34′N

Q. 7. $\vec{r_{BA}} = (-3\vec{i} + 6\vec{j}) - (4\vec{i} + 2\vec{j}) = -7\vec{i} + 4\vec{j}$

$\vec{r_{CA}} = (-4\vec{i} + 2\vec{j}) - (4\vec{i} + 2\vec{j}) = -8\vec{i}$

$|\vec{r_{BA}}| = \sqrt{49 + 16} = \sqrt{65}$

$|\vec{r_{CA}}| = \sqrt{64} = 8$

Since $|\vec{r_{BA}}| > |\vec{r_{CA}}|$, B is farther

Q. 8. (i) $\vec{r_{QP}} = (-4\vec{i} + \vec{j}) - (\vec{i} - 2\vec{j})$

$= -5\vec{i} + 3\vec{j}$

(ii) Let $\vec{r_T} = a\vec{i} + b\vec{j}$

$\vec{r_{TS}} = \vec{r_{QP}}$

$(a + 3)\vec{i} + (b - 5)\vec{j} = -5\vec{i} + 3\vec{j}$

$a + 3 = -5$ and $b - 5 = 3$

$a = -8$ and $b = 8$

$\therefore \vec{r_T} = -8\vec{i} + 8\vec{j}$

Q. 9. $\vec{v_{CT}} = \vec{v_C} - \vec{v_T} = 10\vec{i} + 6\vec{j} - 30\vec{j}$

$= 10\vec{i} - 24\vec{j}$

$|\vec{v_{CT}}| = \sqrt{100 + 576} = 26$ m/s

$\tan\theta = \frac{24}{10} = 2.4 \Rightarrow \theta = 67° 23'$

Direction: E 67° 23′S

Q. 10. $\vec{v_{QP}} = (-4\vec{i} + 2\vec{j}) - (6\vec{i} + 2\vec{j}) = 10\vec{i}$ m/s

Time $= \frac{100}{10} = 10$ s

Q. 11. (i) $\vec{v_A} = 4\vec{i} + 3\vec{j}$

$\vec{v_B} = -\vec{i} + 3\vec{j}$

$\vec{v_{BA}} = \vec{v_B} - \vec{v_A} = -\vec{i} + 3\vec{j} - (4\vec{i} + 3\vec{j})$

$= -5\vec{i}$ km/h

(ii) The position of B relative to A is $\vec{r_{BA}} = 40\vec{i}$ km

$\Rightarrow \vec{v_{BA}} = -\frac{1}{8}(\vec{r_{BA}})$

Since $\vec{v_{BA}} = -k(\vec{r_{BA}})$ where k is a positive constant, they must be on a collision course.

(iii) The time of the collision is given by

$$t = \frac{\text{relative distance}}{\text{relative speed}}$$

$$= \frac{40}{5}$$

$$= 8 \text{ hours later}$$

Q. 12. (i) $\vec{v_A} = 12\vec{i} + 4\vec{j}$

$\vec{v_B} = 4\vec{j}$

$\overrightarrow{v_{BA}} = \vec{v_B} - \vec{v_A} = 4\vec{j} - (12\vec{i} + 4\vec{j})$

$= -12\vec{i}$

(ii) The position of B relative to A is

$\overrightarrow{r_{BA}} = 60\vec{i}$ km

$\Rightarrow \overrightarrow{v_{BA}} = -\frac{1}{5}(\overrightarrow{r_{BA}})$

Since $\overrightarrow{v_{BA}} = -k(\overrightarrow{r_{BA}})$ where k is a positive constant, they must be on a collision course.

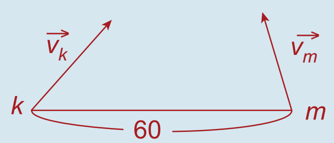
(iii) The time of the collision is given by

$$t = \frac{\text{relative distance}}{\text{relative speed}}$$

$$= \frac{60}{12} = 5 \text{ hours later}$$

Q. 13. (a) $\sqrt{t^2 + 9} = 5 \Rightarrow t = 4$

(b) (i)



$\vec{v_m} = -2\vec{i} + 3\vec{j}$

$\vec{v_k} = t\vec{i} + 3\vec{j}$

But $|t\vec{i} + 3\vec{j}| = 5$

$t = 4$, as before

$\therefore \vec{v_k} = 4\vec{i} + 3\vec{j}$

(ii) $\overrightarrow{v_{mk}} = (-2\vec{i} + 3\vec{j}) - (4\vec{i} + 3\vec{j})$

$= -6\vec{i}$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

$$= \frac{60}{6} = 10 \text{ hours}$$

Q. 14. (i)

$\vec{v_P} = 5\vec{i} + 5\vec{j}$ m/s

(ii) $\vec{v_Q} = K\vec{i} + 5\vec{j}$

But $|K\vec{i} + 5\vec{j}| = 13$

$\Rightarrow K = -12$ (it must be negative so that Q approaches P)

$\therefore \vec{v_Q} = -12\vec{i} + 5\vec{j}$

(iii) $\overrightarrow{v_{QP}} = (-12\vec{i} + 5\vec{j}) - (5\vec{i} + 5\vec{j})$

$= -17\vec{j}$

$\therefore |\overrightarrow{v_{QP}}| = 17$ km/h

$$\text{Time} = \frac{3{,}400}{17} = 200 \text{ s}$$

Q. 15. (i) $\vec{v_K} = 12\vec{i} + 6\vec{j}$

For collision to occur, $\vec{j}$-velocities must match. Therefore, the minimum velocity at which H must travel in order for a collision to occur is $6\vec{j}$ m/s i.e. a minimum speed of 6 m/s due north.

(ii) Let $\vec{v_H} = a\vec{i} + 6\vec{j}$ m/s, $a \in R$

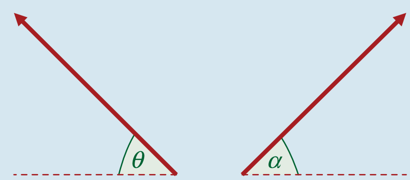
$\sqrt{a^2 + 6^2} = 10$ … we are told that the speed of H is 10 m/s

$\Rightarrow a^2 + 36 = 100$

$\Rightarrow a^2 = 64 \Rightarrow a = \pm 8$

$\Rightarrow$ Two possibilities for $\vec{v_H}$ are

$\vec{v_H} = -8\vec{i} + 6\vec{j}$ and $\vec{v_H} = 8\vec{i} + 6\vec{j}$



$\tan\theta = \tan\alpha = \frac{6}{8} = \frac{3}{4}$

$\Rightarrow \theta = \alpha = 36.87°$

$\Rightarrow$ Possible directions for H are 36.87° N of W and 36.87° N of E.

Using $\vec{v_H} = -8\vec{i} + 6\vec{j}$ gives

$\overrightarrow{v_{KH}} = \vec{v_K} - \vec{v_H} = 20\vec{i}$ m/s

$$\text{Time of Interception} = \frac{\text{relative distance}}{\text{relative speed}}$$

$$= \frac{3{,}000}{20}$$

$$= 150 \text{ s}$$

Using $\vec{v_H} = 8\vec{i} + 6\vec{j}$ gives

$\overrightarrow{v_{KH}} = \vec{v_K} - \vec{v_H} = 4\vec{i}$ m/s

$$\text{Time of Interception} = \frac{\text{relative distance}}{\text{relative speed}}$$

$$= \frac{3{,}000}{4}$$

$$= 750 \text{ s}$$

Q. 16. (i) $\vec{v_Y} = 10\vec{j}$ km/h

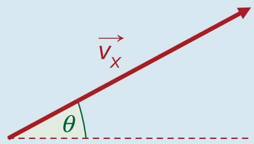
$\vec{v_X} = a\vec{i} + 10\vec{j},\ a \in R,$

where $\sqrt{a^2 + 10^2} = 20$

$\Rightarrow a^2 + 100 = 400$

$\Rightarrow a = \sqrt{300} = 10\sqrt{3}$

$\Rightarrow \vec{v_X} = 10\sqrt{3}\,\vec{i} + 10\vec{j}$



$\tan\theta = \dfrac{10}{10\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \theta = \tan^{-1}\dfrac{1}{\sqrt{3}} = 30°$

$\Rightarrow$ Captain must steer in a direction 30° N of E.

(ii) $\vec{v_{XY}} = \vec{v_X} - \vec{v_Y}$

$= 10\sqrt{3}\,\vec{i} + 10\vec{j} - 10\vec{j} = 10\sqrt{3}\,\vec{i}$

Time to interception $= \dfrac{\text{relative distance}}{\text{relative speed}}$

$= \dfrac{40}{10\sqrt{3}}$

$\simeq 2.309$ hours

$\simeq 2$ hours 19 mins

Q. 17. (i) $\vec{r_{BA}} = \vec{r_B} - \vec{r_A}$

$= 37\vec{i} + 25\vec{j} - (2\vec{i} - 3\vec{j})$

$= 35\vec{i} + 28\vec{j}$

$\vec{v_{BA}} = \vec{v_B} - \vec{v_A}$

$= -2\vec{i} - 3\vec{j} - (3\vec{i} + \vec{j})$

$= -5\vec{i} - 4\vec{j}$

$\Rightarrow \vec{v_{BA}} = -\frac{1}{7}(\vec{r_{BA}})$

Since $\vec{v_{BA}} = -k(\vec{r_{BA}})$ where k is a positive constant, they must be on a collision course.

(ii) Time to collision $= \dfrac{\text{relative distance}}{\text{relative speed}}$

$= \dfrac{\sqrt{35^2 + 28^2}}{\sqrt{(-5)^2 + (-4)^2}}$

$= \dfrac{7\sqrt{41}}{\sqrt{41}} = 7$ hours

$\Rightarrow$ Collision occurs at 17.00 hours.

Q. 18. (i) $\vec{r_A} = -8\vec{i} + 4\vec{j}$

$\vec{r_B} = 24\vec{i} - 12\vec{j}$

$\vec{r_{BA}} = \vec{r_B} - \vec{r_A}$

$= 24\vec{i} - 12\vec{j} - (-8\vec{i} + 4\vec{j})$

$= 32\vec{i} - 16\vec{j}$

$\vec{v_A} = 3\vec{i} + \vec{j}$

$\vec{v_B} = \vec{i} + 2\vec{j}$

$\vec{v_{BA}} = \vec{v_B} - \vec{v_A} = \vec{i} + 2\vec{j} - (3\vec{i} + \vec{j})$

$= -2\vec{i} + \vec{j}$

$\vec{v_{BA}} = -\frac{1}{8}\left(\vec{r_{BA}}\right)$

Since $\vec{v_{BA}} = -k\left(\vec{r_{BA}}\right)$ where k is a positive constant, they must be on a collision course.

(ii) Time to collision $= \dfrac{\text{relative distance}}{\text{relative speed}}$

$= \dfrac{\sqrt{32^2 + (-16)^2}}{\sqrt{(-2)^2 + 1^2}}$

$= \dfrac{16\sqrt{5}}{\sqrt{5}} = 16$ hours

$\Rightarrow$ Collision will occur at 16.00 hours.

Q. 19. (i) $\vec{r_X} = 10\vec{i} - 4\vec{j}$

$\vec{r_Y} = 37\vec{i} + k\vec{j}$

$\vec{r_{YX}} = \vec{r_Y} - \vec{r_X} = 27\vec{i} + (k + 4)\vec{j}$

$\vec{v_X} = 3\vec{i} + \vec{j}$

$\vec{v_Y} = -\vec{j}$

$\vec{v_{YX}} = \vec{v_Y} - \vec{v_X} = -3\vec{i} - 2\vec{j}$

$\dfrac{27}{-3} = \dfrac{k+4}{-2}$ ...collision course $\Rightarrow \vec{v_{YX}}$ is a scalar multiple of $\vec{r_{YX}}$

$\Rightarrow 3k + 12 = 54$

$\Rightarrow 3k = 42$

$\Rightarrow k = 14$

(ii) $\vec{r_{YX}} = 27\vec{i} + 18\vec{j}$

$\vec{v_{YX}} = -3\vec{i} - 2\vec{j}$

Time to collision $= \dfrac{\text{relative distance}}{\text{relative speed}}$

$= \dfrac{\sqrt{27^2 + 18^2}}{\sqrt{(-3)^2 + (-2)^2}}$

$= \dfrac{9\sqrt{13}}{\sqrt{13}}$

$= 9$ hours

Collision occurs at 10.00 hours.

Q. 20. (i) $\vec{r_P} = -11\vec{i} + \vec{j}$

$\vec{r_Q} = 4\vec{i} - 13\vec{j}$

$\overrightarrow{r_{QP}} = \vec{r_Q} - \vec{r_P} = 15\vec{i} - 14\vec{j}$

$\vec{v_P} = 3\vec{i}$

$\vec{v_Q} = x\vec{j}$

$\overrightarrow{v_{QP}} = \vec{v_Q} - \vec{v_P} = -3\vec{i} + x\vec{j}$

$\frac{15}{-3} = \frac{-14}{x}$ ...collision course $\Rightarrow \overrightarrow{v_{QP}}$ is a scalar multiple of $\overrightarrow{r_{QP}}$

$\Rightarrow 15x = 42$

$\Rightarrow x = \frac{14}{5}$

(ii) $\overrightarrow{r_{QP}} = 15\vec{i} - 14\vec{j}$

$\overrightarrow{v_{QP}} = -3\vec{i} + \frac{14}{5}\vec{j}$

Time to collision $= \frac{\text{relative distance}}{\text{relative speed}}$

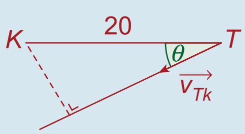
$= \frac{\sqrt{15^2 + (-14)^2}}{\sqrt{(-3)^2 + \left(\frac{14}{5}\right)^2}}$

$= 5$ hours

$\Rightarrow$ Collision occurs at 17.00 hours.

Exercise 4B

Q. 1.



$\overrightarrow{v_{KT}} = (\vec{i} + 2\vec{j}) - (-2\vec{i} - 2\vec{j})$

$= 3\vec{i} + 4\vec{j}$ m/s

$\tan\theta = \frac{4}{3}$

$\Rightarrow \sin\theta = \frac{4}{5}$

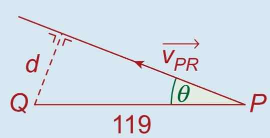
$d = 20\sin\theta$

$= 20\left(\frac{4}{5}\right) = 16$ m

Q. 2. (i) $\overrightarrow{v_{PQ}} = (-8\vec{i} + 12\vec{j}) - (7\vec{i} + 4\vec{j})$

$= -15\vec{i} + 8\vec{j}$

(ii)



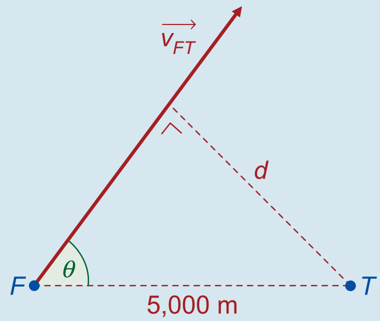
(iii) $\tan\theta = \frac{8}{15}$

$\Rightarrow \sin\theta = \frac{8}{17}$

$d = 119\sin\theta$

$= 119\left(\frac{8}{17}\right) = 56$ units

Q. 3.



(i) $\vec{v_F} = 2\vec{i} + 5\vec{j}$

$\vec{v_T} = -4\vec{i} - 3\vec{j}$

$\overrightarrow{v_{FT}} = \vec{v_F} - \vec{v_T}$

$= 2\vec{i} + 5\vec{j} + 4\vec{i} + 3\vec{j}$

$= 6\vec{i} + 8\vec{j}$

(ii) $\tan\theta = \frac{8}{6}$

$= \frac{4}{3}$

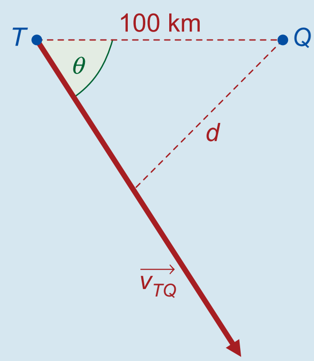
$\Rightarrow \sin\theta = \frac{4}{5}$

But, $\sin\theta = \frac{d}{5{,}000}$

$\Rightarrow \frac{d}{5{,}000} = \frac{4}{5}$

$\Rightarrow d = 4000$ m ... shortest distance between P and Q in subsequent motion.

Q. 4.



(i) $\vec{v_T} = 10\cos 30°\vec{i} - 10\sin 30°\vec{j}$

$= 10\left(\frac{\sqrt{3}}{2}\right)\vec{i} - 10\left(\frac{1}{2}\right)\vec{j}$

$= 5\sqrt{3}\,\vec{i} - 5\vec{j}$

$\overrightarrow{v_Q} = -20\cos 45°\vec{i} + 20\sin 45°\vec{j}$

$= -20\left(\frac{1}{\sqrt{2}}\right)\vec{i} + 20\left(\frac{1}{\sqrt{2}}\right)\vec{j}$

$= -10\sqrt{2}\,\vec{i} + 10\sqrt{2}\,\vec{j}$

$\overrightarrow{v_{TQ}} = \overrightarrow{v_T} - \overrightarrow{v_Q}$

$= (5\sqrt{3} + 10\sqrt{2})\vec{i} - (5 + 10\sqrt{2})\vec{j}$

$= 22.8\vec{i} - 19.14\vec{j}$ km/h

(ii) $\left|\overrightarrow{v_{TQ}}\right| = \sqrt{22.8^2 + 19.14^2}$

$= 29.77$ m/s

$\tan\theta = \frac{19.14}{22.8}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{19.14}{22.8}\right)$

$= 40°$

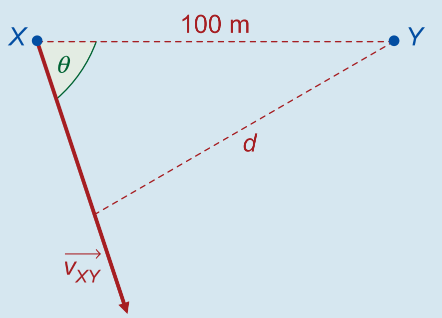
$\Rightarrow 40°$ S of E

(iii) $\sin 40° = \frac{d}{100}$

$\Rightarrow d = 100\sin 40°$

$\Rightarrow d = 64.3$ km

Q. 5.



(i) $\overrightarrow{v_X} = 7\vec{i}$

$\overrightarrow{v_Y} = 24\vec{j}$

$\overrightarrow{v_{XY}} = \overrightarrow{v_X} - \overrightarrow{v_Y}$

$= 7\vec{i} - 24\vec{j}$ m/s

(ii) $\tan\theta = \frac{24}{7}$

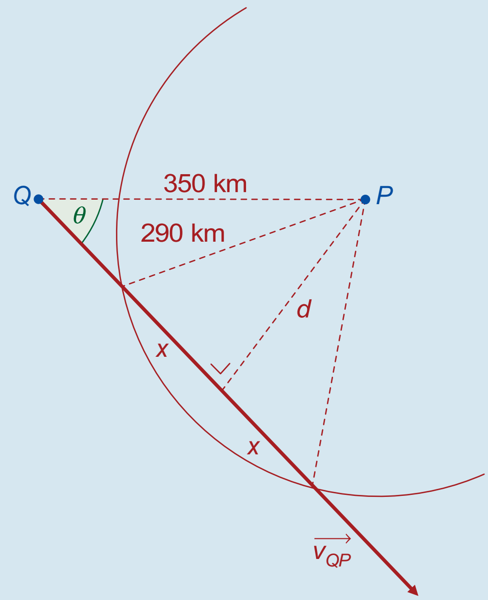
$\Rightarrow \sin\theta = \frac{24}{25}$

But, $\sin\theta = \frac{d}{100}$

$\Rightarrow \frac{d}{100} = \frac{24}{25}$

$\Rightarrow d = 96$ m

Q. 6.



(i) $\overrightarrow{v_P} = -\vec{i} + \vec{j}$

$\overrightarrow{v_Q} = 3\vec{i} - 2\vec{j}$

$\overrightarrow{v_{QP}} = \overrightarrow{v_Q} - \overrightarrow{v_P}$

$= 4\vec{i} - 3\vec{j}$

(ii) $\left|\overrightarrow{v_{QP}}\right| = \sqrt{4^2 + 3^2}$

$= 5$ km/h

$\tan\theta = \frac{3}{4}$

$\Rightarrow \theta = \tan^{-1}\frac{3}{4}$

$= 36.87°$

$\Rightarrow 36°52'$ S of E

(iii) $\tan\theta = \frac{3}{4}$

$\Rightarrow \sin\theta = \frac{3}{5}$

But, $\sin\theta = \frac{d}{350}$

$\Rightarrow \frac{d}{350} = \frac{3}{5}$

$\Rightarrow d = 210$ km

(iv) Insert circle with centre P and radius 290 km. As long as the relative path, $\overrightarrow{v_{QP}}$, is within this circle, P and Q will be able to exchange signals.

From the diagram, they will be within range for a relative distance of $2x$.

$x^2 + d^2 = 290^2$ … but $d = 210$

$\Rightarrow x = \sqrt{290^2 - 210^2}$

$= 200$ km

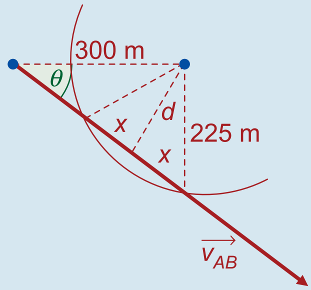
⇒ P and Q are within range for a relative distance of 400 km.

$$\text{Time} = \frac{\text{relative distance}}{\text{relative speed}}$$

$$= \frac{400}{5}$$

$$= 80 \text{ hours}$$

Q. 7.



(i) $\overrightarrow{v_A} = 2\vec{i} - \vec{j}$

$\overrightarrow{v_B} = -2\vec{i} + 2\vec{j}$

$\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B}$

$= 4\vec{i} - 3\vec{j}$ m/s

(ii) $\tan\theta = \frac{3}{4}$

$\Rightarrow \sin\theta = \frac{3}{5}$

But, $\sin\theta = \frac{d}{300}$

$\Rightarrow \frac{d}{300} = \frac{3}{5}$

$\Rightarrow 5d = 900$

$\Rightarrow d = 180$ m

(iii) Draw a circle of radius 225 metres with centre at B.

A and B will be able to exchange signals as long as the relative path, $\overrightarrow{v_{AB}}$, is inside this circle. This will be for a relative distance of $2x$.

$x^2 + d^2 = 225^2$ … but $d = 180$

$\Rightarrow x = \sqrt{225^2 - 180^2} = 135$

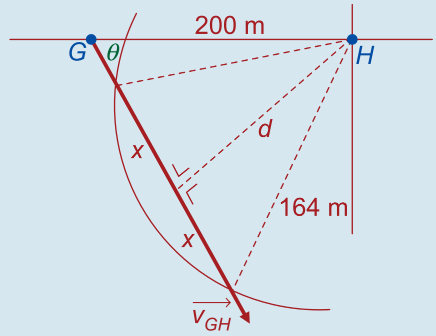
⇒ A and B will be able to exchange signals for a relative distance of 270 m.

$$\text{Time} = \frac{\text{relative distance}}{\text{relative speed}}$$

$$= \frac{270}{\sqrt{4^2 + (-3)^2}}$$

$$= 54 \text{ s}$$

Q. 8.



(i) $\overrightarrow{v_G} = 6\vec{i}$

$\overrightarrow{v_H} = 8\vec{j}$

$\overrightarrow{v_{GH}} = \overrightarrow{v_G} - \overrightarrow{v_H} = 6\vec{i} - 8\vec{j}$

$\tan\theta = \frac{8}{6} = \frac{4}{3} \Rightarrow \sin\theta = \frac{4}{5}$

But, $\sin\theta = \frac{d}{200}$

$\Rightarrow \frac{d}{200} = \frac{4}{5} \quad \Rightarrow d = 160$ m

(ii) Draw a circle with radius 164 m with centre H. As long as the relative path, $\overrightarrow{v_{GH}}$, is inside this circle, the cars will be no more than 164 m apart. This will be for a distance of $2x$.

$x^2 + d^2 = 164^2$ … but $d = 160$

$\Rightarrow x = \sqrt{164^2 - 160^2} = 36$

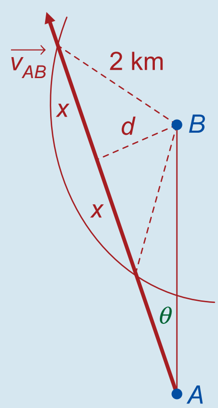
⇒ Less than or equal to 164 m apart for a relative distance of 72 m.

$$\text{Time} = \frac{\text{relative distance}}{\text{relative speed}}$$

$$= \frac{72}{\sqrt{6^2 + (-8)^2}}$$

$$= 7.2 \text{ s}$$

Q. 9.



(i) $\overrightarrow{v_A} = 16\cos 45°\vec{i} + 16\sin 45°\vec{j}$

$= 16\left(\frac{1}{\sqrt{2}}\right)\vec{i} + 16\left(\frac{1}{\sqrt{2}}\right)\vec{j}$

$= 8\sqrt{2}\,\vec{i} + 8\sqrt{2}\,\vec{j}$

$$\vec{v_B} = 20\cos 45°\vec{i} - 20\sin 45°\vec{j}$$
$$= 20\left(\frac{1}{\sqrt{2}}\right)\vec{i} - 20\left(\frac{1}{\sqrt{2}}\right)\vec{j}$$
$$= 10\sqrt{2}\,\vec{i} - 10\sqrt{2}\,\vec{j}$$
$$\vec{v_{AB}} = \vec{v_A} - \vec{v_B}$$
$$= -2\sqrt{2}\,\vec{i} + 18\sqrt{2}\,\vec{j} \text{ km/h}$$

(ii) $\tan\theta = \dfrac{2\sqrt{2}}{18\sqrt{2}} = \dfrac{1}{9}$

$\Rightarrow \sin\theta = \dfrac{1}{\sqrt{82}}$

But, $\sin\theta = \dfrac{d}{10}$

$\Rightarrow \dfrac{d}{10} = \dfrac{1}{\sqrt{82}}$

$\Rightarrow d = \dfrac{10}{\sqrt{82}} = 1.104 \text{ km} = 1{,}104 \text{ m}$

(iii) Draw a circle of radius 2 km with its centre at B.

As long as the relative path, $\vec{v_{AB}}$, is within this circle, the ships will be in visual contact.

This will be for a relative distance of $2x$.

$x^2 + d^2 = 2^2$ … but $d = \dfrac{10}{\sqrt{82}}$

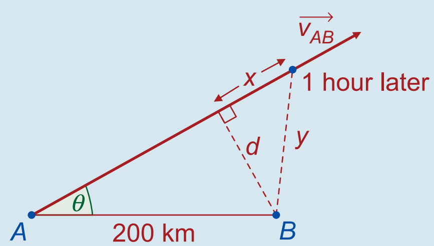
$\Rightarrow x = \sqrt{4 - \dfrac{100}{82}}$

$= 1.6675 \text{ km} = 1667.5 \text{ m}$

$\Rightarrow$ Ships will be in visual contact for a relative distance of $2(1.6675) = 3.335$ km

$$\text{Time} = \frac{\text{relative distance}}{\text{relative speed}}$$
$$= \frac{3.335}{\sqrt{(-2\sqrt{2})^2 + (18\sqrt{2})^2}}$$
$$= 0.13 \text{ h} = 7 \text{ min } 49 \text{ s}$$

Q. 10.



(i) $\vec{v_B} = 10\vec{j}$

$$\vec{v_A} = 20\cos 45°\vec{i} + 20\sin 45°\vec{j}$$
$$= 20\left(\frac{1}{\sqrt{2}}\right)\vec{i} + 20\left(\frac{1}{\sqrt{2}}\right)\vec{j}$$
$$= 10\sqrt{2}\,\vec{i} + 10\sqrt{2}\,\vec{j}$$

$$\vec{v_{AB}} = \vec{v_A} - \vec{v_B}$$
$$= 10\sqrt{2}\,\vec{i} + (10\sqrt{2} - 10)\vec{j}$$
$$= 14.14\vec{i} + 4.14\vec{j}$$

(ii) $\tan\theta = \dfrac{10(\sqrt{2} - 1)}{10\sqrt{2}} = \dfrac{\sqrt{2} - 1}{\sqrt{2}}$

$\Rightarrow \theta = 16.325°$

$\Rightarrow \sin\theta = 0.281$

But, $\sin\theta = \dfrac{d}{200}$

$\Rightarrow \dfrac{d}{200} = 0.281$

$\Rightarrow d = 56.2 \text{ km}$

(iii) One hour later:

relative distance = relative speed × time

$\Rightarrow x = \sqrt{(10\sqrt{2})^2 + (10\sqrt{2} - 10)^2} \times (1)$

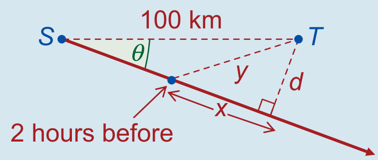
$= 14.736 \text{ km}$

$y^2 = x^2 + d^2$ … from diagram

$\Rightarrow y = \sqrt{14.736^2 + 56.2^2}$

$\Rightarrow y = 58.1 \text{ km}$

Q. 11.



(i) $\vec{v_T} = -8\vec{j}$

$$\vec{v_S} = 20\cos 30°\vec{i} - 20\sin 30°\vec{j}$$
$$= 20\left(\frac{\sqrt{3}}{2}\right)\vec{i} - 20\left(\frac{1}{2}\right)\vec{j}$$
$$= 10\sqrt{3}\,\vec{i} - 10\vec{j}$$

$$\vec{v_{ST}} = \vec{v_S} - \vec{v_T}$$
$$= 10\sqrt{3}\,\vec{i} - 2\vec{j}$$

$$|\vec{v_{ST}}| = \sqrt{(10\sqrt{3})^2 + (-2)^2}$$
$$= 4\sqrt{19} = 17.44 \text{ km/h}$$

$$\tan\theta = \frac{2}{10\sqrt{3}}$$
$$= \frac{1}{5\sqrt{3}}$$

$\Rightarrow \theta = 6.59°$

$\Rightarrow 6.59°$ S of E

(ii) $\tan\theta = \dfrac{1}{5\sqrt{3}}$

$\Rightarrow \sin\theta = \dfrac{1}{\sqrt{76}}$

But, $\sin\theta = \dfrac{d}{100}$

$\Rightarrow \dfrac{d}{100} = \dfrac{1}{\sqrt{76}}$

$\Rightarrow d = \dfrac{100}{\sqrt{76}}$

$= 11.47$ km

(iii) Two hours before:

relative distance = relative speed × time

$\Rightarrow x = 17.44 \times 2$

$= 34.88$ km

$y^2 = x^2 + d^2$

$\Rightarrow y = \sqrt{34.88^2 + 11.47^2}$

$= 36.72$ km

(iv) Draw a circle of radius 30 m with centre B.

As long as the relative path, $\overrightarrow{v_{AB}}$, is inside this circle, Adam and Barbara will be within 30 m of each other. This will be for a relative distance of 2x.

$x^2 + d^2 = 30^2$ … but $d = 22.5$

$\Rightarrow x = \sqrt{30^2 - 22.5^2}$

$= 19.843$ m

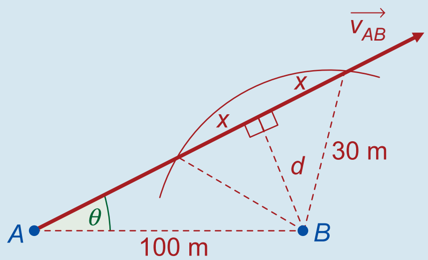
$\Rightarrow$ Adam and Barbara will be within 30 m of each other for a relative distance of 2(19.843) = 39.686 m

$\text{Time} = \dfrac{\text{relative distance}}{\text{relative speed}}$

$= \dfrac{39.686}{\sqrt{79}}$

$= 4.47$ s

Q. 12.



(i) $\overrightarrow{v_A} = 10\cos 30°\vec{i} + 10\sin 30°\vec{j}$

$= 10\left(\dfrac{\sqrt{3}}{2}\right)\vec{i} + 10\left(\dfrac{1}{2}\right)\vec{j}$

$= 5\sqrt{3}\,\vec{i} + 5\vec{j}$

$\overrightarrow{v_B} = 3\vec{j}$

$\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B} = 5\sqrt{3}\,\vec{i} + 2\vec{j}$

(ii) $|\overrightarrow{v_{AB}}| = \sqrt{(5\sqrt{3})^2 + 2^2} = \sqrt{79}$ m/s

$\tan\theta = \dfrac{2}{5\sqrt{3}}$

$\Rightarrow \theta = 13°$

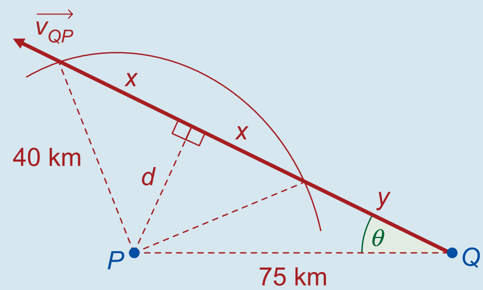
$\Rightarrow$ 13° N of E

(iii) $\dfrac{d}{100} = \sin 13°$

$\Rightarrow d = 100\sin 13°$

$= 22.5$ m

Q. 13.



(i) $\overrightarrow{v_P} = 50\cos 45°\vec{i} - 50\sin 45°\vec{j}$

$= 50\left(\dfrac{1}{\sqrt{2}}\right)\vec{i} - 50\left(\dfrac{1}{\sqrt{2}}\right)\vec{j}$

$= 25\sqrt{2}\,\vec{i} - 25\sqrt{2}\,\vec{j}$

$\overrightarrow{v_Q} = -30\vec{j}$

$\overrightarrow{v_{QP}} = \overrightarrow{v_Q} - \overrightarrow{v_P}$

$= -25\sqrt{2}\,\vec{i} + (25\sqrt{2} - 30)\vec{j}$

$= -35.36\vec{i} + 5.36\vec{j}$

$|\overrightarrow{v_{PQ}}| = \sqrt{(-35.36)^2 + (5.36)^2}$

$= 35.76$ m/s

$\tan\theta = \dfrac{5.36}{35.36}$

$\Rightarrow \theta = 8.62°$

$\Rightarrow$ 8.62° N of W

(ii) $\dfrac{d}{75} = \sin 8.62°$

$\Rightarrow d = 75\sin 8.62°$

$\Rightarrow d = 11.24$ km

(iii) Draw a circle of radius 40 km with centre at P.

Ships will be within range of each other while the relative path, $\overrightarrow{v_{QP}}$, is inside this circle.

This will be for a relative distance of $2x$.

$x^2 + d^2 = 40^2$ … but $d = 11.24$

$\Rightarrow x = \sqrt{40^2 - 11.24^2} = 38.39$ km

$\Rightarrow$ Ships will be within range of each other for a relative distance of $2(38.39) = 76.78$ km.

$$\text{Time} = \frac{\text{relative distance}}{\text{relative speed}} = \frac{76.78}{35.76} = 2.15 \text{ hours} = 2 \text{ hours } 9 \text{ mins}$$

From the diagram,

$(x + y)^2 + d^2 = 75^2$ … but $d = 11.24$

$\Rightarrow x + y = \sqrt{75^2 - 11.24^2}$

$= 74.15$ … but $x = 38.39$

$\Rightarrow y = 74.15 - 38.39$

$= 35.76$

Time before coming into range:

$$\frac{\text{relative distance}}{\text{relative speed}} = \frac{35.76}{35.76} = 1 \text{ hour}$$

$\Rightarrow$ Ships will come into range at 13.00 hours.

Ships stay within range for 2 hours and 9 minutes.

$\Rightarrow$ Ships will lose sight of each other at 15.09 hours.

Exercise 4C

Q. 1. (i) $\text{Time} = \frac{\text{distance}}{\text{speed}} = \frac{60}{12} = 5$ s

(ii) Distance travelled by Q = speed × time = 5 × 5 = 25 m

$\Rightarrow$ Distance from $O = 51 - 25 = 26$ m

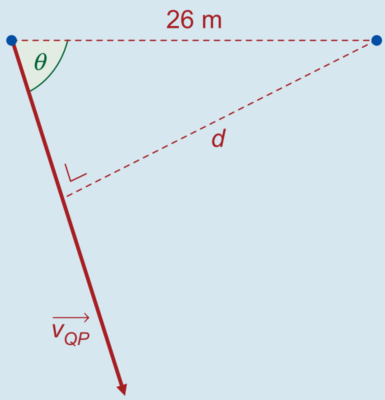
(iii) $\overrightarrow{v_P} = 12\vec{j}$

$\overrightarrow{v_Q} = 5\vec{i}$

$\overrightarrow{v_{QP}} = \overrightarrow{v_Q} - \overrightarrow{v_P}$

$= 5\vec{i} - 12\vec{j}$

(iv)



$\tan\theta = \frac{12}{5}$

$\Rightarrow \sin\theta = \frac{12}{13}$

But, $\sin\theta = \frac{d}{26}$

$\Rightarrow \frac{d}{26} = \frac{12}{13}$

$\Rightarrow d = 24$ m

Q. 2. (i) $\text{Time} = \frac{\text{distance}}{\text{speed}} = \frac{20}{8} = 2.5$ s

(ii) Distance travelled by M = speed × time = 6 × 2.5 = 15 m

$\Rightarrow$ Distance from $O = 65 - 15 = 50$ m

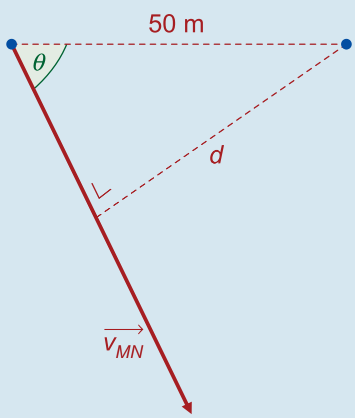
(iii) $\overrightarrow{v_M} = 6\vec{i}$

$\overrightarrow{v_N} = 8\vec{j}$

$\overrightarrow{v_{MN}} = \overrightarrow{v_M} - \overrightarrow{v_N}$

$= 6\vec{i} - 8\vec{j}$ m/s

(iv)



$\tan\theta = \frac{8}{6} = \frac{4}{3}$

$\Rightarrow \sin\theta = \frac{4}{5}$

But, $\sin\theta = \frac{d}{50}$

$\Rightarrow \frac{d}{50} = \frac{4}{5}$

$\Rightarrow d = 40$ m

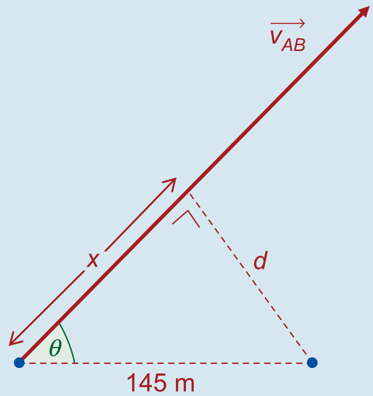
Q. 3. (i) $\vec{v_A} = 21\vec{i}$

$\vec{v_B} = -20\vec{j}$

$\overrightarrow{v_{AB}} = \vec{v_A} - \vec{v_B}$

$= 21\vec{i} + 20\vec{j}$ m/s

(ii)



Wait until B reaches the intersection.

$\text{Time} = \frac{\text{distance}}{\text{speed}} = \frac{100}{20} = 5$ s

Find how far A has travelled in this time.

Distance = speed × time

$= 21 \times 5 = 105$ m

⇒ Distance from $O = 250 - 105$

$= 145$ m

$\tan\theta = \frac{20}{21}$

$\Rightarrow \sin\theta = \frac{20}{29}$

But, $\sin\theta = \frac{d}{145}$

$\Rightarrow \frac{d}{145} = \frac{20}{29}$

$\Rightarrow d = 100$ m

(iii) $x^2 + d^2 = 145^2$ … but $d = 100$

$\Rightarrow x = \sqrt{145^2 - 100^2} = 105$

$\text{Time} = \frac{\text{relative distance}}{\text{relative speed}}$

$= \frac{105}{\sqrt{21^2 + 20^2}}$

$= 3.62$ s

Q. 4. (i) $\text{Time} = \frac{\text{distance}}{\text{speed}}$

$= \frac{100}{5}$

$= 20$ s

(ii) Distance travelled by B = speed × time = $8 \times 20 = 160$ m

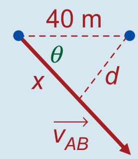
⇒ Distance from $O = 200 - 160 = 40$ m

⇒ Distance between A and B = 40 m

(iii) $\vec{v_A} = -5\cos\theta\vec{i} - 5\sin\theta\vec{j}$

$= -5\left(\frac{4}{5}\right)\vec{i} - 5\left(\frac{3}{5}\right)\vec{j}$

$= -4\vec{i} - 3\vec{j}$



$\vec{v_B} = -8\vec{i}$

$\overrightarrow{v_{AB}} = \vec{v_A} - \vec{v_B}$

$= 4\vec{i} - 3\vec{j}$

$\Rightarrow |\overrightarrow{v_{AB}}| = \sqrt{4^2 + (-3)^2}$

$= 5$ m/s

$\tan\theta = \frac{3}{4}$

$\Rightarrow \theta = 36.87°$

⇒ 36.87° S of E

(iv) $\tan\theta = \frac{3}{4}$

$\Rightarrow \sin\theta = \frac{3}{5}$

But, $\sin\theta = \frac{d}{40}$

$\Rightarrow \frac{d}{40} = \frac{3}{5}$

$\Rightarrow d = 24$ m

(v) $x^2 + d^2 = 40^2$ … but $d = 24$

$\Rightarrow x = \sqrt{40^2 - 24^2} = 32$ m

$\text{Time} = \frac{\text{relative distance}}{\text{relative speed}}$

$= \frac{32}{5} = 6.4$ s

But, A and B had already been travelling for 20 seconds.

⇒ Time = 26 s

(vi) A: Distance from intersection

$= 100 - 5t$

B: Distance from intersection

$= 200 - 8t$

⇒ Equidistant from O when

$100 - 5t = 200 - 8t$

$\Rightarrow 3t = 100$

$\Rightarrow t = \frac{100}{3}$

$= 33\frac{1}{3}$ s

Q. 5. $\vec{v_A} = -16 \cos \theta \vec{i} - 16 \sin \theta \vec{j}$

$= -16\left(\frac{3}{5}\right)\vec{i} - 16\left(\frac{4}{5}\right)\vec{j} = -9.6\vec{i} - 12.8\vec{j}$

$\vec{v_B} = -v\vec{i}$

$\vec{v_{AB}} = \vec{v_A} - \vec{v_B} = (v - 9.6)\vec{i} - 12.8\vec{j}$

$|\vec{v_{AB}}| = 16$

$\Rightarrow \sqrt{\left(v - \frac{48}{5}\right)^2 + \left(-\frac{64}{5}\right)^2} = 16$

$\Rightarrow v^2 - \frac{96}{5}v + \frac{2,304}{25} + \frac{4,096}{25} = 256$

$\Rightarrow 25v^2 - 480v + 6,400 = 6,400$

$\Rightarrow 25v^2 - 480v = 0$

$\Rightarrow 5v^2 - 96v = 0$

$\Rightarrow v(5v - 96) = 0$

$\Rightarrow v = \frac{96}{5}$

$= 19.2$ m/s

(i) Find out how long it takes for A to get to the junction.

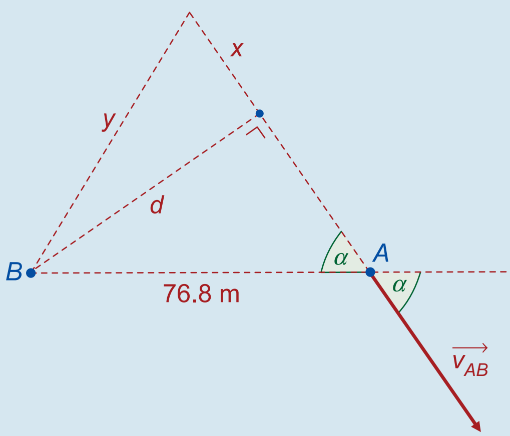
Time $= \frac{\text{distance}}{\text{speed}} = \frac{96}{16} = 6$ s

Find out how far B has travelled at this time.

Distance = speed × time

$= 19.2 \times 6 = 115.2$ m

Since B was 38.4 m from O at the beginning, B is now 76.8 m past O.



$\vec{v_{AB}} = 9.6\vec{i} - 12.8\vec{j}$

$\tan \alpha = \frac{12.8}{9.6} = \frac{4}{3} \quad \Rightarrow \quad \sin \alpha = \frac{4}{5}$

But, $\sin \alpha = \frac{d}{76.8}$

$\Rightarrow \frac{d}{76.8} = \frac{4}{5}$

$\Rightarrow d = 61.44$ m

(ii) 2 seconds before:

Relative distance $= x$

$=$ relative speed × time

$= 16 \times 2 = 32$ m

Actual distance $= y$

$y^2 = 32^2 + 61.44^2$

$\Rightarrow y = 69$ m

Q. 6. $\vec{v_A} = -10 \cos \theta \vec{i} - 10 \sin \theta \vec{j}$

$= -10\left(\frac{4}{5}\right)\vec{i} - 10\left(\frac{3}{5}\right)\vec{j} = -8\vec{i} - 6\vec{j}$

$\vec{v_B} = -20\vec{i}$

$\vec{v_{AB}} = \vec{v_A} - \vec{v_B} = 12\vec{i} - 6\vec{j}$

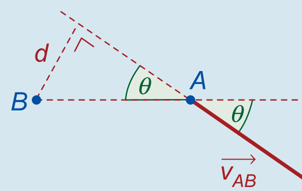
Find out how long it takes for A to get to the junction.

Time $= \frac{\text{distance}}{\text{speed}} = \frac{100}{10} = 10$ s

Find out how far B has travelled at this time.

Distance = speed × time $= 20 \times 10$

$= 200$ m



Since B was 100 m from O at the beginning, B is now 100 m past O.

$\tan \theta = \frac{6}{12} = \frac{1}{2} \quad \Rightarrow \quad \sin \theta = \frac{1}{\sqrt{5}}$

But, $\sin \theta = \frac{d}{100}$

$\Rightarrow \frac{d}{100} = \frac{1}{\sqrt{5}}$

$\Rightarrow d = \frac{100}{\sqrt{5}}$

$= 20\sqrt{5}$

$= 44.72$ m

Q. 7. $\vec{v_A} = -30 \cos 60°\vec{i} - 30 \sin 60°\vec{j}$

$= -30\left(\frac{1}{2}\right)\vec{i} - 30\left(\frac{\sqrt{3}}{2}\right)\vec{j} = -15\vec{i} - 15\sqrt{3}\vec{j}$

$\vec{v_B} = -40\vec{i}$

$\vec{v_{AB}} = \vec{v_A} - \vec{v_B}$

$= 25\vec{i} - 15\sqrt{3}\vec{j}$

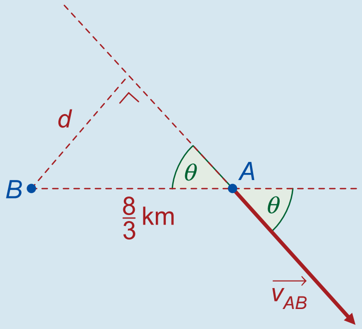
Time for A to get to junction $= \frac{\text{distance}}{\text{speed}}$

$= \frac{3.5}{30}$

$= \frac{7}{60}$ h

Distance travelled by B = speed × time

$= 40 \times \frac{7}{60} = \frac{14}{3}$ km



$\Rightarrow$ When A is at the junction, B is $\frac{8}{3}$ km past the junction.

$\tan\theta = \frac{15\sqrt{3}}{25}$

$= \frac{3\sqrt{3}}{5}$

$\Rightarrow \sin\theta = \frac{3\sqrt{3}}{2\sqrt{13}}$

But, $\sin\theta = \frac{d}{\frac{8}{3}} = \frac{3d}{8}$

$\Rightarrow \frac{3d}{8} = \frac{3\sqrt{3}}{2\sqrt{13}}$

$\Rightarrow d = \frac{4\sqrt{39}}{13}$

$= 1.92$ km

Q. 8. (a) $\overrightarrow{v_A} = -16\cos 60°\vec{i} - 16\sin 60°\vec{j}$

$= -16\left(\frac{1}{2}\right)\vec{i} - 16\left(\frac{\sqrt{3}}{2}\right)\vec{j}$

$= -8\vec{i} - 8\sqrt{3}\vec{j}$

$v_B = 20\vec{i}$

$\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B}$

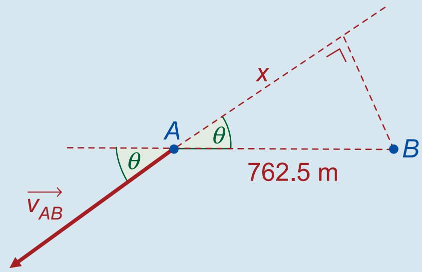
$= -28\vec{i} - 8\sqrt{3}\vec{j}$

(b) (i) Find out how long it takes for A to reach O:

Time $= \frac{\text{distance}}{\text{speed}}$

$= \frac{450}{16}$

$= 28.125$ s



Find out how far B has travelled in this time:

Distance = speed × time

$= 20 \times 28.125 = 562.5$ m

$\Rightarrow B$ is now 762.5 m from O.

$\tan\theta = \frac{8\sqrt{3}}{28} = \frac{2\sqrt{3}}{7}$

$\Rightarrow \cos\theta = \frac{7}{\sqrt{61}}$

But, $\cos\theta = \frac{x}{762.5}$

$\Rightarrow \frac{x}{762.5} = \frac{7}{\sqrt{61}}$

$\Rightarrow x = 683.4$ m

Time $= \frac{\text{relative distance}}{\text{relative speed}}$

$= \frac{683.4}{\sqrt{(-28)^2 + (-8\sqrt{3})^2}}$

$= 21.875$ s

$\Rightarrow$ Closest together 21.875 seconds before they were side by side.

$28.125 - 21.875 = 6.25$ s

(ii) Distance of A from $O = 450 - 16t$

Distance of B from $O = 200 + 20t$

Equidistant from O when
$450 - 16t = 200 + 20t$

$\Rightarrow 36t = 250$

$\Rightarrow t = 6.94$ s

Exercise 4D

Q. 1. (i) $\overrightarrow{v_B} = \vec{i} + 2\vec{j}$

Time across $= \frac{\text{distance across}}{\text{speed across}}$

$= \frac{40}{2}$

$= 20$ s

(ii) Distance downstream:
speed downstream × time

$= 1 \times 20$

$= 20$ m

Q. 2. $\vec{v_B} = 5\vec{i} + 12\vec{j}$

Time across $= \dfrac{\text{distance across}}{\text{speed across}}$

$= \dfrac{60}{12}$

$= 5$ s

Distance downstream:
speed downstream × time

$= 5 \times 5$

$= 25$ m

Q. 3. (i) Puts all effort into going across:

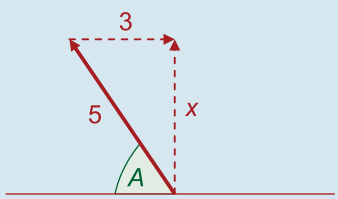
$\Rightarrow \vec{v_B} = 3\vec{i} + 5\vec{j}$

Time across $= \dfrac{\text{distance across}}{\text{speed across}}$

$= \dfrac{60}{5}$

$= 12$ s

(ii) Heads upstream at an angle A to the bank at full speed, 5 m/s.



$x^2 + 3^2 = 5^2$

$\Rightarrow x = 4$

$\Rightarrow$ Boat travels at 4 m/s straight across.

Time across $= \dfrac{\text{distance across}}{\text{speed across}}$

$= \dfrac{60}{4}$

$= 15$ s

Q. 4. (i) Puts all effort into going across:

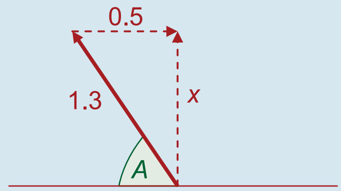
$\Rightarrow \vec{v_B} = 0.5\vec{i} + 1.3\vec{j}$

Time across $= \dfrac{\text{distance across}}{\text{speed across}}$

$= \dfrac{39}{1.3}$

$= 30$ s

(ii) Heads upstream at an angle A to the bank at full speed, 1.3 m/s.



$x^2 + 0.5^2 = 1.3^2$

$\Rightarrow x = 1.2$

$\Rightarrow$ Boat travels at 1.2 m/s straight across.

Time across $= \dfrac{\text{distance across}}{\text{speed across}}$

$= \dfrac{39}{1.2}$

$= 32.5$ s

Q. 5. (i) Puts all effort into going across:

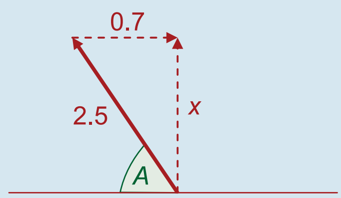
$\Rightarrow \vec{v_B} = 0.7\vec{i} + 2.5\vec{j}$

Time across $= \dfrac{\text{distance across}}{\text{speed across}}$

$= \dfrac{60}{2.5}$

$= 24$ s

(ii) Heads upstream at an angle A to the bank at full speed, 2.5 m/s.



$x^2 + 0.7^2 = 2.5^2$

$\Rightarrow x = 2.4$

$\Rightarrow$ Boat travels at 2.4 m/s straight across.

Time across $= \dfrac{\text{distance across}}{\text{speed across}}$

$= \dfrac{60}{2.4}$

$= 25$ s

Q. 6. (i) Puts all effort into going across:

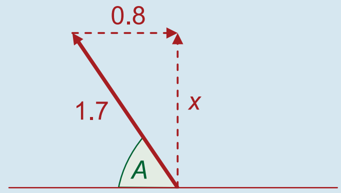
$\Rightarrow \vec{v_B} = 0.8\vec{i} + 1.7\vec{j}$

Time across $= \dfrac{\text{distance across}}{\text{speed across}}$

$= \dfrac{510}{1.7}$

$= 300$ s

(ii) Heads upstream at an angle A to the bank at full speed, 2.5 m/s.



$x^2 + 0.8^2 = 1.7^2$

$\Rightarrow x = 1.5$

$\Rightarrow$ Boat travels at 1.5 m/s straight across.

$$\text{Time across} = \frac{\text{distance across}}{\text{speed across}}$$
$$= \frac{510}{1.5}$$
$$= 340 \text{ s}$$

Q. 7. (i) Puts all effort into going across:

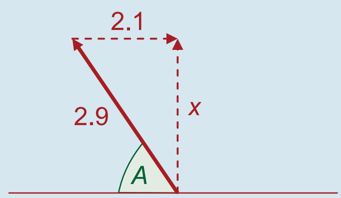
$\Rightarrow \vec{v_B} = 2.1\vec{i} + 2.9\vec{j}$

$$\text{Time across} = \frac{\text{distance across}}{\text{speed across}}$$
$$= \frac{58}{2.9}$$
$$= 20 \text{ s}$$

Distance downstream:
speed downstream × time

$$= 2.1 \times 20$$
$$= 42 \text{ m}$$

(ii) Heads upstream at an angle A to the bank at full speed, 2.9 m/s.



$x^2 + 2.1^2 = 2.9^2$

$\Rightarrow x = 2$

$\Rightarrow$ Boat travels at 2 m/s straight across.

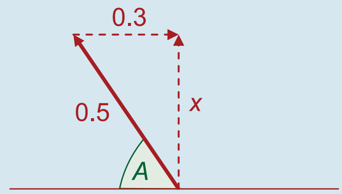
$$\text{Time across} = \frac{\text{distance across}}{\text{speed across}}$$
$$= \frac{58}{2}$$
$$= 29 \text{ s}$$

Q. 8. Quickest route: Puts all effort into going across:

$\Rightarrow \vec{v_B} = 0.3\vec{i} + 0.5\vec{j}$

$$\text{Time across} = \frac{\text{distance across}}{\text{speed across}}$$
$$= \frac{50}{0.5}$$
$$= 100 \text{ s}$$

Shortest route: Heads upstream at an angle A to the bank at full speed, 0.5 m/s.



$x^2 + 0.3^2 = 0.5^2$

$\Rightarrow x = 0.4$

$\Rightarrow$ Boat travels at 0.4 m/s straight across.

$$\text{Time across} = \frac{\text{distance across}}{\text{speed across}}$$
$$= \frac{50}{0.4}$$
$$= 125 \text{ s}$$

$\Rightarrow$ Crossing times differ by 25 seconds.

Q. 9. (i) He should head straight across.

(ii) $\vec{v_B} = \frac{5}{6}\vec{i} + \frac{5}{9}\vec{j}$

$$\text{Time across} = \frac{\text{distance across}}{\text{speed across}}$$
$$= \frac{50}{\frac{5}{9}}$$
$$= 50\left(\frac{9}{5}\right)$$
$$= 90 \text{ s}$$

(iii) Distance downstream:
speed downstream × time

$$= \frac{5}{6} \times 90$$
$$= 75 \text{ m}$$

Q. 10. $\overrightarrow{v_{PW}} = 100\vec{i}$

$\overrightarrow{v_W} = -10\vec{j}$

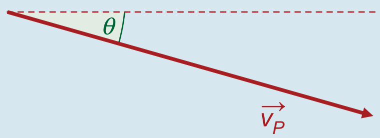
$\Rightarrow \overrightarrow{v_P} = 100\vec{i} - 10\vec{j}$

Speed $= |\overrightarrow{v_P}|$

$= \sqrt{100^2 + (-10)^2}$

$= 10\sqrt{101}$

$= 100.5$ m/s



$\tan\theta = \frac{10}{100}$

$= \frac{1}{10}$

$\Rightarrow \theta = 5.71°$

$= 5°43'$

$\Rightarrow 5°43'$ S of E

Q. 11. Upstream:

$\overrightarrow{v_C} = \overrightarrow{v_{CR}} + \overrightarrow{v_R}$

$= -5\vec{i} + (3\vec{i})$

$= -2\vec{i}$

Time $= \frac{80}{2}$

$= 40$ s

Downstream:

$\overrightarrow{v_C} = 5\vec{i} + 3\vec{i}$

$= 8\vec{i}$

Time $= \frac{80}{8}$

$= 10$ s

Total time $= 40 + 10 = 50$ s

Lake: Total time $= \frac{80}{5} + \frac{80}{5}$

$= 32$ s

which is 18 seconds less

Q. 12. Still water: Time $= \frac{\text{distance}}{\text{speed}}$

$= \frac{960}{8} = 120$ s

Current: A to B: Time $= \frac{\text{distance}}{\text{speed}}$

$= \frac{480}{10}$

$= 48$ s

Current: B to A: Time $= \frac{\text{distance}}{\text{speed}}$

$= \frac{480}{6}$

$= 80$ s

Total time $= 48 + 80$

$= 128$ s

$\Rightarrow$ It takes 8 seconds longer when there is a current of 2 m/s from A to B.

Q. 13. (i) $\overrightarrow{v_R} = 12\vec{i}$

$\overrightarrow{v_{BR}} = 5\vec{j}$

$\overrightarrow{v_B} = \overrightarrow{v_{BR}} + \overrightarrow{v_R}$

$= 12\vec{i} + 5\vec{j}$ m/s

Magnitude: $\left|\overrightarrow{v_B}\right| = \sqrt{12^2 + 5^2}$

$= 13$ m/s

(ii) Time $= \frac{\text{distance}}{\text{speed}}$

$= \frac{240}{5}$

$= 48$ s

Distance downstream:
speed downstream × time

$= 12 \times 48$

$= 576$ m

Q. 14. $v = \sqrt{15^2 + 8^2}$

$= 17$ m/s

Q. 15. (i) $\overrightarrow{v_R} = 7\vec{i}$

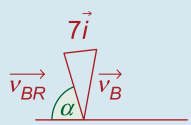
$\overrightarrow{v_{BR}} = -25\cos\alpha\,\vec{i} + 25\sin\alpha\,\vec{j}$

$\therefore \overrightarrow{v_B} = (7 - 25\cos\alpha)\vec{i} + 25\sin\alpha\,\vec{j}$

$7 - 25\cos\alpha = 0$

$\Rightarrow \cos\alpha = \frac{7}{25}$

$\Rightarrow \sin\alpha = \frac{24}{25}$



Since $\cos\alpha = \frac{7}{25}$

$= 0.28$

$\alpha = 73°44'$

(ii) $v_B = 0\vec{i} + 25\left(\frac{24}{25}\right)\vec{j}$

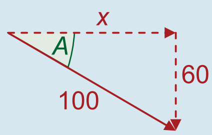
$= 24\vec{j}$

Time $= \frac{120}{24}$

$= 5$ s

Q. 16. $\vec{v}_W = 60\vec{j}$

$\Rightarrow$ Plane must head at an angle A as shown in order to counteract the wind.



$x^2 + 60^2 = 100^2$

$\Rightarrow x = 80$

$\Rightarrow$ Plane actually flies at 80 m/s due East.

$\text{Time} = \frac{\text{distance}}{\text{speed}}$

$= \frac{189}{80}$

$= 2.3625$ h

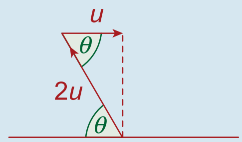
$= 2$ h 21 m 45 s

The time taken for the return journey is the same because the wind is blowing directly from the south. This means that the wind will have no effect on the $\vec{i}$-velocity of the plane. The plane will still fly at 80 km/h but in the opposite direction.

Q. 17. $\cos\theta = \frac{u}{2u}$

$= \frac{1}{2}$

$\Rightarrow \theta = 60°$



Q. 18. $\vec{v}_c = -\vec{j}$

$\vec{v}_s = 2\cos 45°\vec{i} - 2\sin 45°\vec{j}$

$= 1.414\vec{i} - 1.414\vec{j}$

$\vec{v}_{sc} = \vec{v}_s - \vec{v}_c = 1.414\vec{i} - 0.414\vec{j}$

$v_{sc} = \sqrt{(1.414)^2 + (-0.414)^2}$

$= 1.47$ m/s

Q. 19. $\vec{v}_A = -100\vec{i}$

$\vec{v}_W = -20\cos 30\vec{i} + 20\sin 30\vec{j}$

$= -17.32\vec{i} + 10\vec{j}$

$\vec{v}_{AW} = \vec{v}_A - \vec{v}_W$

$= -100\vec{i} - (17.32\vec{i} + 10\vec{j})$

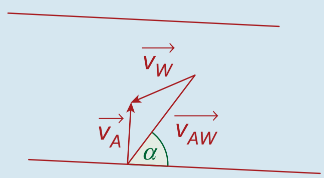
$= -82.68\vec{i} - 10\vec{j}$

$|\vec{v}_{AW}| = \sqrt{(-82.68)^2 + (-10)^2}$

$= 83.28$ km/h

Q. 20. $\vec{v}_w = -50\cos 45°\vec{i} - 50\sin 45°\vec{j}$

$= -35.355\vec{i} - 35.355\vec{j}$



$\vec{v}_{Aw} = 200\cos\alpha\vec{i} + 200\sin\alpha\vec{j}$

$\vec{v}_A = (200\cos\alpha - 35.355)\vec{i} + (200\sin\alpha - 35.355)\vec{j}$

But $200\cos\alpha - 35.355 = 0$

$\Rightarrow \cos\alpha = \frac{35.355}{200} = 0.1768$

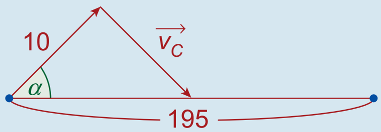
$\Rightarrow \alpha = 79°\ 49'$

$\therefore \vec{v}_A = 0\vec{i} + (200(0.9843) - 35.355)\vec{j}$

$= 161.505\vec{j}$

$= 161.5$ km/h

Q. 21.



$\overrightarrow{v_{SC}} = 10\cos\alpha\vec{i} + 10\sin\alpha\vec{j}$

$\vec{v_C} = 5\vec{i} - 6\vec{j}$

$\vec{v_S} = \overrightarrow{v_{SC}} + \vec{v_C}$

$= (10\cos\alpha + 5)\vec{i} + (10\sin\alpha - 6)\vec{j}$

j-component $= 0$

$10\sin\alpha - 6 = 0$

$\Rightarrow \sin\alpha = \frac{3}{5}$

$\Rightarrow \cos\alpha = \frac{4}{5}$

$\therefore \vec{v_S} = \left(10\left(\frac{4}{5}\right) + 5\right)\vec{i} + 0\vec{j} = 13\vec{i}$

Time $= \frac{195}{13}$

$= 15$ s

Returning is similar, giving the result

$\vec{v_S} = (-10\cos\alpha + 5)\vec{i} + (10\sin\alpha - 6)\vec{j}$

$10\sin\alpha - 6 = 0$

$\Rightarrow \sin\alpha = \frac{3}{5}$

$\Rightarrow \cos\alpha = \frac{4}{5}$

$\therefore \vec{v_S} = \left(-10\left(\frac{4}{5}\right) + 5\right)\vec{i}$

$= -3\vec{i}$

Time $= \frac{195}{3}$

$= 65$ s

$\therefore$ Total time $= 15 + 65$

$= 80$ s

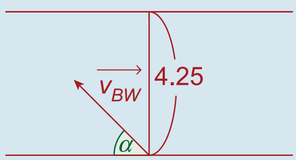
The difference between the outward and return speeds must be $2 \times 5 = 10$ m/s

(Since outward speed gains 5 m/s from the current, but return speed loses 5 m/s)

The outward speed $= \frac{195}{13} = 15$ m/s, the return speed will be $15 - 10 = 5$ m/s.

The time will be $\frac{195}{5} = 39$ s

Q. 22.



$\overrightarrow{v_{BW}} = -18\cos\alpha\vec{i} + 18\sin\alpha\vec{j}$

$\vec{v_W} = 8\sqrt{2}\vec{i} - 8\sqrt{2}\vec{j}$

$\therefore \vec{v_B} = (-18\cos\alpha + 8\sqrt{2})\vec{i} + (18\sin\alpha - 8\sqrt{2})\vec{j}$

The i-component is zero $\Rightarrow -18\cos\alpha + 8\sqrt{2} = 0$

$\Rightarrow \cos\alpha = \frac{8\sqrt{2}}{18} = \frac{4\sqrt{2}}{9}$

$\Rightarrow \sin\alpha = \frac{7}{9}$

$\therefore \vec{v_B} = 0\vec{i} + \left(18\left(\frac{7}{9}\right) - 8\sqrt{2}\right)\vec{j} = (14 - 8\sqrt{2})\vec{j}$

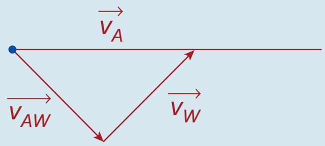
Time $= \frac{4.25}{14 - 8\sqrt{2}}$

Similarly, returning time $= \frac{4.25}{14 + 8\sqrt{2}}$

Total time $= \frac{4.25}{14 - 8\sqrt{2}} + \frac{4.25}{14 + 8\sqrt{2}}$

$= \frac{4.25(14 + 8\sqrt{2}) + 4.25(14 - 8\sqrt{2})}{(14 - 8\sqrt{2})(14 + 8\sqrt{2})} = \frac{7}{4}$ hours

Q. 23. (i)



$$\overrightarrow{v_W} = \frac{v}{\sqrt{2}}\vec{i} + \frac{v}{\sqrt{2}}\vec{j}$$

$$\overrightarrow{v_{AW}} = x\cos\alpha\vec{i} - x\sin\alpha\vec{j}$$

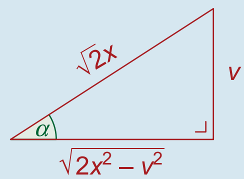
$$\therefore \overrightarrow{v_A} = \left(\frac{v}{\sqrt{2}} + x\cos\alpha\right)\vec{i} + \left(\frac{v}{\sqrt{2}} - x\sin\alpha\right)\vec{j}$$

j-component $= 0$

$$\Rightarrow \frac{\vec{v}}{2} - x\sin\alpha = 0$$

$$\Rightarrow \sin\alpha = \frac{v}{\sqrt{2}}x$$

$$\therefore \cos\alpha = \frac{\sqrt{2x^2 - v^2}}{\sqrt{2}x}$$



$$\therefore \overrightarrow{v_A} = \left(\frac{v}{2} + \frac{x\sqrt{2x^2 - v^2}}{\sqrt{2}x}\right)\vec{i}$$

$$= \left(\frac{v}{\sqrt{2}} + \frac{\sqrt{2x^2 - v^2}}{\sqrt{2}}\right)\vec{i}$$

$$\therefore \left|\overrightarrow{v_A}\right| = \frac{v + \sqrt{2x^2 - v^2}}{\sqrt{2}} = U_1$$

Similarly $U_2 = \dfrac{\sqrt{2x^2 - v^2} - v}{\sqrt{2}}$

$$\therefore U_1 - U_2 = \frac{2v}{\sqrt{2}} = \sqrt{2}v \quad \textbf{QED}$$

(ii) $$U_1 U_2 = \left(\frac{\sqrt{2x^2 - v^2} + v}{\sqrt{2}}\right)\left(\frac{\sqrt{2x^2 - v^2} - v}{\sqrt{2}}\right)$$

$$= \frac{2x^2 - 2v^2}{2}$$

$$= x^2 - v^2 \quad \textbf{QED}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{d}{\frac{\sqrt{2x^2 - v^2} + v}{\sqrt{2}}} + \frac{d}{\frac{\left(\sqrt{2x^2 - v^2} - v\right)}{\sqrt{2}}}$$

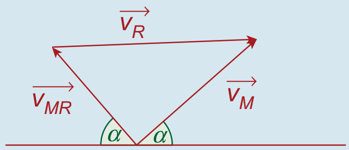
$$= \frac{\sqrt{2}d}{\sqrt{2x^2 - v^2} + v} + \frac{\sqrt{2}d}{\sqrt{2x^2 - v^2} - v}$$

$$= \frac{\sqrt{2}d\left(\sqrt{2x^2 - v^2} - v\right) + \sqrt{2}d\sqrt{2x^2 - v^2} + v}{\left(\sqrt{2x^2 - v^2} + v\right)\left(\sqrt{2x^2 - v^2} - v\right)}$$

$$= \frac{2\sqrt{4x^2 - 2v^2}d}{2x^2 - 2v^2}$$

$$= \frac{\sqrt{4x^2 - 2v^2}\,d}{x^2 - v^2}$$

Q. 24.



$\overrightarrow{v_{MR}} = -5\cos\alpha\vec{i} + 5\sin\alpha\vec{j}$

$\vec{v_R} = 13\vec{i}$

$\vec{v_M} = (13 - 5\cos\alpha)\vec{i} + 5\sin\alpha\vec{j}$

$\tan\theta = \dfrac{j\text{-component}}{i\text{-component}} = \dfrac{5\sin\alpha}{13 - 5\cos\alpha}$

$\dfrac{d(\tan\theta)}{d\alpha} = \dfrac{(13 - 5\cos\alpha)(5\cos\alpha) - 5\sin\alpha\,(5\sin\alpha)}{(13 - 5\cos\alpha)^2} = 0$

$\Rightarrow 65\cos\alpha - 25\cos^2\alpha - 25\sin^2\alpha = 0$

$\Rightarrow 65\cos\alpha - 25(\cos^2\alpha + \sin^2\alpha) = 0$

$\Rightarrow 65\cos\alpha - 25 = 0$

$\Rightarrow \cos\alpha = \dfrac{5}{13}$

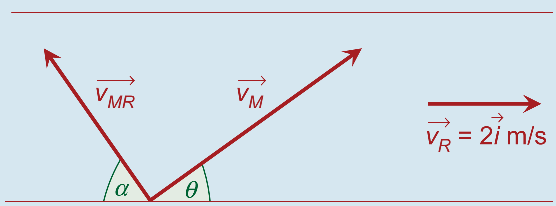
The shortest path is where θ is a maximum and therefore where $\tan\theta$ is a maximum, since $\tan\theta$ is an increasing function in θ. That is to say that the shortest path is where $\cos\alpha = \dfrac{5}{13}$, and hence $\sin\alpha = \dfrac{12}{13}$

In this case $\vec{v_M} = \left(13 - 5\left(\dfrac{5}{13}\right)\right)\vec{i} + 5\left(\dfrac{12}{13}\right)\vec{j}$

$= \dfrac{144}{13}\vec{i} + \dfrac{60}{13}\vec{j}$

Crossing time $= \dfrac{60}{\frac{60}{13}} = 13$ s

Q. 25.



$\overrightarrow{v_{MR}} = -\cos\alpha\vec{i} + \sin\alpha\vec{j}$

$\vec{v_R} = 2\vec{i}$

$\overrightarrow{v_{MR}} = \vec{v_M} - \vec{v_R}$

$\Rightarrow \vec{v_M} = \overrightarrow{v_{MR}} + \vec{v_R}$

$= (2 - \cos\alpha)\vec{i} + \sin\alpha\vec{j}$

$\Rightarrow \tan\theta = \dfrac{\sin\alpha}{2 - \cos\alpha}$

$\tan\theta$ will have a maximum value when $\dfrac{d}{d\alpha}(\tan\theta) = 0$

$\dfrac{d}{d\alpha}(\tan\theta) = \dfrac{(2 - \cos\alpha)(\cos\alpha) - (\sin\alpha)(\sin\alpha)}{(2 - \cos\alpha)^2}$ … using the Quotient Rule

$$\Rightarrow \frac{d}{d\alpha}(\tan\theta) = \frac{2\cos\alpha - \cos^2\alpha - \sin^2\alpha}{(2-\cos\alpha)^2}$$

$$\Rightarrow \frac{d}{d\alpha}(\tan\theta) = \frac{2\cos\alpha - (\cos^2\alpha + \sin^2\alpha)}{(2-\cos\alpha)^2} \quad \ldots \cos^2\alpha + \sin^2\alpha = 1$$

$$\Rightarrow \frac{d}{d\alpha}(\tan\theta) = \frac{2\cos\alpha - 1}{(2-\cos\alpha)^2}$$

Putting $\frac{d}{d\alpha}(\tan\theta) = 0$ gives

$$\frac{2\cos\alpha - 1}{(2-\cos\alpha)^2} = 0$$

$$\Rightarrow 2\cos\alpha - 1 = 0$$

$$\Rightarrow \cos\alpha = \frac{1}{2}$$

$$\Rightarrow \alpha = 60^\circ$$

Shortest path will occur when $\alpha = 60^\circ$

$$\Rightarrow \vec{v_M} = (2 - \cos 60^\circ)\vec{i} + \sin 60^\circ\vec{j}$$

$$= \left(2 - \frac{1}{2}\right)\vec{i} + \frac{\sqrt{3}}{2}\vec{j} = \frac{3}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$$

$$\text{Time across} = \frac{\text{distance across}}{\text{speed across}}$$

$$= \frac{36}{\frac{\sqrt{3}}{2}}$$

$$= \frac{72}{\sqrt{3}}$$

$$= 24\sqrt{3}\text{ s}$$

Q. 26. (a) $\cos A = \sqrt{1 - \sin^2 A}$

(b) $\vec{v_{BC}} = 5\vec{i} - 2\vec{j}$

$\vec{v_C} = 5\cos\alpha\vec{i} + 5\sin\alpha\vec{j}$

$\vec{v_B} = (5 + 5\cos\alpha)\vec{i} + (-2 + 5\sin\alpha)\vec{j}$

This is in a N.E. direction

$$\therefore \frac{-2 + 5\sin\alpha}{5 + 5\cos\alpha} = \tan 45^\circ = 1$$

$$\Rightarrow -2 + 5\sin\alpha = 5 + 5\cos\alpha$$

$$\Rightarrow -2 + 5\sin\alpha = 5 + 5\sqrt{1 - \sin^2\alpha}$$

$$\Rightarrow -7 + 5\sin\alpha = 5\sqrt{1 - \sin^2\alpha}$$

$$\Rightarrow 49 - 70\sin\alpha + 25\sin^2\alpha = 25(1 - \sin^2\alpha)$$

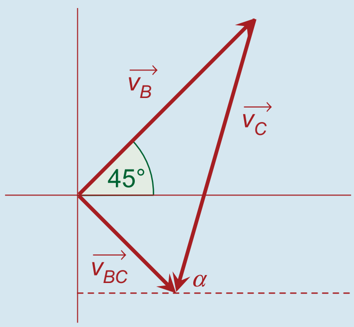
$$\Rightarrow 50\sin^2\alpha - 70\sin\alpha + 24 = 0$$

$$\Rightarrow 25\sin^2\alpha - 35\sin\alpha + 12 = 0$$

$$\Rightarrow (5\sin\alpha - 3)(5\sin\alpha - 4) = 0$$

$$\Rightarrow \sin\alpha = \frac{3}{5} \quad \textbf{OR} \quad \sin\alpha = \frac{4}{5}$$

$$\Rightarrow \cos\alpha = \pm\frac{4}{5} \quad \textbf{OR} \quad \cos\alpha = \pm\frac{3}{5}$$



Possibility 1: $\sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5} \quad \Rightarrow \vec{v_B} = 9\vec{i} + 3\vec{j}$. Reject

Possibility 2: $\sin\alpha = \frac{3}{5}, \cos\alpha = -\frac{4}{5} \quad \Rightarrow \vec{v_B} = \vec{i} + \vec{j}$. Correct

Possibility 3: $\sin\alpha = \frac{4}{5}, \cos\alpha = \frac{3}{5} \quad \Rightarrow \vec{v_B} = 8\vec{i} + 2\vec{j}$. Reject

Possibility 4: $\sin\alpha = \frac{4}{5}, \cos\alpha = -\frac{3}{5} \quad \Rightarrow \vec{v_B} = 2\vec{i} + 2\vec{j}$. Correct

(i) $\vec{v_C} = -4\vec{i} + 3\vec{j}$ **OR** $-3\vec{i} + 4\vec{j}$ m/s

(ii) $\vec{v_B} = \vec{i} + \vec{j}$ **OR** $2\vec{i} + 2\vec{j}$ m/s

Exercise 4E

Q. 1. (i) **Case 1:** $\vec{v_M} = 4\vec{i}$

$\vec{v_W} = x\vec{i} + y\vec{j}$

$\Rightarrow \vec{v_{WM}} = \vec{v_W} - \vec{v_M}$

$= (x-4)\vec{i} + y\vec{j}$

$\vec{v_{WM}}$ from the north

$\Rightarrow x - 4 = 0$

$\Rightarrow x = 4$

Case 2: Let $\vec{v_L}$ = velocity of the woman

$\vec{v_L} = -\vec{j}$

$\vec{v_W} = x\vec{i} + y\vec{j}$

$\Rightarrow \vec{v_{WL}} = \vec{v_W} - \vec{v_L}$

$= x\vec{i} + (y+1)\vec{j}$

$\vec{v_{WL}}$ from the north-west

$\Rightarrow x = -(y+1)$ … but $x = 4$

$\Rightarrow 4 = -y - 1$

$\Rightarrow y = -5$

$\Rightarrow \vec{v_W} = 4\vec{i} - 5\vec{j}$ m/s

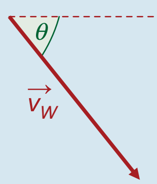
(ii) Speed $= |\vec{v_W}| = \sqrt{4^2 + (-5)^2}$

$= \sqrt{41}$ m/s

$\Rightarrow \tan\theta = \frac{5}{4}$

$\Rightarrow \theta = 51.34°$

51.34° S of E



(iii) $\vec{v_M} = 4\vec{j}$

$\vec{v_W} = 4\vec{i} - 5\vec{j}$

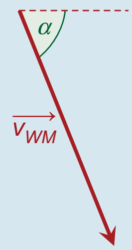
$\vec{v_{WM}} = \vec{v_W} - \vec{v_M}$

$= 4\vec{i} - 9\vec{j}$

$\tan\alpha = \frac{9}{4}$

$\Rightarrow \alpha = 66°$

$\Rightarrow$ 66° S of E



Q. 2. **Case 1:** Walking South

$\vec{v_M} = -\vec{j}$

$\vec{v_W} = x\vec{i} + y\vec{j}$

$\vec{v_{WM}} = \vec{v_W} - \vec{v_M}$

$= x\vec{i} + (y+1)\vec{j}$

$\vec{v_{WM}}$ from South-West

$\Rightarrow x = y + 1$

$\Rightarrow x - y = 1$

Case 2: Walking North

$\vec{v_M} = 3\vec{j}$

$\vec{v_W} = x\vec{i} + y\vec{j}$

$\vec{v_{WM}} = \vec{v_W} - \vec{v_M} = x\vec{i} + (y-3)\vec{j}$

$\vec{v_{WM}}$ from North-West

$\Rightarrow x = -(y-3)$

$\Rightarrow x + y = 3$

But, $\underline{x - y = 1}$ … add

$\Rightarrow 2x = 4$

$\Rightarrow x = 2$

$\Rightarrow y = 1$

$\Rightarrow \vec{v_W} = 2\vec{i} + \vec{j}$ m/s

Q. 3. (i) Let the velocity of the woman be $\vec{v_L}$, and the velocity of the wind, $\vec{v_W}$

Case 1: $\vec{v_L} = -2\vec{j}$

$\overrightarrow{v_W} = x\vec{i} + y\vec{j}$

$\overrightarrow{v_{WL}} = \overrightarrow{v_W} - \vec{v_L}$

$= x\vec{i} + (y + 2)\vec{j}$

$\overrightarrow{v_{WL}}$ from North-West

$\Rightarrow x = -(y + 2)$

$\Rightarrow x + y = -2$

Case 2: $\vec{v_L} = -14\vec{j}$

$\overrightarrow{v_W} = x\vec{i} + y\vec{j}$

$\overrightarrow{v_{WL}} = \overrightarrow{v_W} - \vec{v_L}$

$= x\vec{i} + (y + 14)\vec{j}$

$\overrightarrow{v_{WL}}$ towards North-East

$\Rightarrow x = y + 14$

$\Rightarrow x - y = 14$

But, $\underline{x + y = -2}$ ... add

$\Rightarrow 2x = 12$

$\Rightarrow x = 6$

$\Rightarrow y = -8$

$\Rightarrow \overrightarrow{v_W} = 6\vec{i} - 8\vec{j}$ m/s

(ii) Speed $= |\overrightarrow{v_W}| = \sqrt{6^2 + (-8)^2}$

$= 10$ m/s

Q. 4. **Case 1:** $\vec{v_C} = 7\vec{j}$

$\overrightarrow{v_W} = x\vec{i} + y\vec{j}$

$\overrightarrow{v_{WC}} = \overrightarrow{v_W} - \vec{v_C}$

$= x\vec{i} + (y - 7)\vec{j}$

$\overrightarrow{v_{WC}}$ from North-West

$\Rightarrow x = -(y - 7)$

$\Rightarrow x + y = 7$

Case 2: $\vec{v_P} = -\vec{i}$

$\overrightarrow{v_W} = x\vec{i} + y\vec{j}$

$\overrightarrow{v_{WP}} = \overrightarrow{v_W} - \vec{v_P} = (x + 1)\vec{i} + y\vec{j}$

$\overrightarrow{v_{WP}}$ from South-West

$\Rightarrow x + 1 = y$

$\Rightarrow x - y = -1$

But, $\underline{x + y = 7}$ ... add

$\Rightarrow 2x = 6$

$\Rightarrow x = 3$

$\Rightarrow y = 4$

$\Rightarrow \overrightarrow{v_W} = 3\vec{i} + 4\vec{j}$

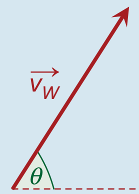
$|\overrightarrow{v_W}| = \sqrt{3^2 + 4^2}$

$= 5$ m/s

$\tan\theta = \frac{4}{3}$

$\Rightarrow \theta = 53.13°$

$\Rightarrow 53.13°$ N of E



Q. 5. **Case 1:** $\vec{v_B} = \vec{j}$

$\overrightarrow{v_W} = x\vec{i} + y\vec{j}$

$\overrightarrow{v_{WB}} = \overrightarrow{v_W} - \vec{v_B}$

$= x\vec{i} + (y - 1)\vec{j}$

$\overrightarrow{v_{WB}}$ from South-West

$\Rightarrow x = y - 1$

$\Rightarrow x - y = -1$

Case 2: $\vec{v_B} = 5\vec{j}$

$\overrightarrow{v_W} = x\vec{i} + y\vec{j}$

$\overrightarrow{v_{WB}} = \overrightarrow{v_W} - \vec{v_B} = x\vec{i} + (y - 5)\vec{j}$

$\overrightarrow{v_{WB}}$ from North-West

$\Rightarrow x = -(y - 5)$

$\Rightarrow x + y = 5$

But, $\underline{x - y = -1}$ ... add

$\Rightarrow 2x = 4$

$\Rightarrow x = 2$

$\Rightarrow y = 3$

$\Rightarrow \overrightarrow{v_W} = 2\vec{i} + 3\vec{j}$ m/s

Q. 6. **Case 1:** $\vec{v_C} = 3\vec{i} + 2\vec{j}$

$\overrightarrow{v_W} = x\vec{i} + y\vec{j}$

$\overrightarrow{v_{WC}} = \overrightarrow{v_W} - \vec{v_C}$

$= (x - 3)\vec{i} + (y - 2)\vec{j}$

$\overrightarrow{v_{WC}}$ from North-West

$\Rightarrow x - 3 = -(y - 2)$

$\Rightarrow x + y = 5$

Case 2: $\vec{v_C} = 7\vec{i}$

$\overrightarrow{v_W} = x\vec{i} + y\vec{j}$

$\overrightarrow{v_{WC}} = \overrightarrow{v_W} - \vec{v_C} = (x - 7)\vec{i} + y\vec{j}$

$\overrightarrow{v_{WC}}$ from North

$\Rightarrow x - 7 = 0 \quad \Rightarrow \quad x = 7$

$\Rightarrow \quad y = -2$

$\Rightarrow \overrightarrow{v_W} = 7\vec{i} - 2\vec{j}$ m/s

Q. 7. (i) **Case 1:** $\vec{v_C} = 3\vec{j}$

$\vec{v_W} = x\vec{i} + y\vec{j}$

$\vec{v_{WC}} = \vec{v_W} - \vec{v_C}$

$= x\vec{i} + (y - 3)\vec{j}$

$\vec{v_{WC}}$ from South-West

$\Rightarrow x = y - 3$

$\Rightarrow x - y = -3$

Case 2: $\vec{v_C} = 9\vec{j}$

$\vec{v_W} = x\vec{i} + y\vec{j}$

$\vec{v_{WC}} = \vec{v_W} - \vec{v_C}$

$= x\vec{i} + (y - 9)\vec{j}$

$\vec{v_{WC}}$ from North-West

$\Rightarrow x = -(y - 9)$

$\Rightarrow x + y = 9$

But, $\underline{x - y = -3}$ ... add

$\Rightarrow 2x = 6$

$\Rightarrow x = 3$

$\Rightarrow y = 6$

$\Rightarrow \vec{v_W} = 3\vec{i} + 6\vec{j}$ m/s

(ii) $\vec{v_C} = p\vec{j}$

$\vec{v_W} = 3\vec{i} + 6\vec{j}$

$\vec{v_{WC}} = \vec{v_W} - \vec{v_C} = 3\vec{i} + (6 - p)\vec{j}$

$\vec{v_{WC}}$ from West

$\Rightarrow 6 - p = 0 \quad \Rightarrow \quad p = 6$

$\Rightarrow$ She must cycle at 6 m/s North.

Q. 8. (i) $\vec{v_M} = -2\vec{j}$

$\vec{v_W} = x\vec{i} + y\vec{j}$

$\vec{v_{WM}} = \vec{v_W} - \vec{v_M} = x\vec{i} + (y + 2)\vec{j}$

$\vec{v_{WM}}$ from North-West

$\Rightarrow x = -(y + 2)$

Also, $\sqrt{x^2 + y^2} = 10$

$\Rightarrow x^2 + y^2 = 100$ ... let $x = -(y + 2)$

$\Rightarrow (y + 2)^2 + y^2 = 100$

$\Rightarrow y^2 + 4y + 4 + y^2 = 100$

$\Rightarrow 2y^2 + 4y - 96 = 0$

$\Rightarrow y^2 + 2y - 48 = 0$

$\Rightarrow (y + 8)(y - 6) = 0$

$\Rightarrow y = -8$, ~~$y = 6$~~

$\Rightarrow x = 6$

$\Rightarrow \vec{v_W} = 6\vec{i} - 8\vec{j}$ m/s

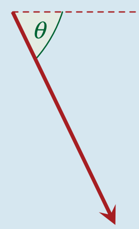
Note: The $y = 6$ solution is excluded because this would mean the man is cycling into the wind while travelling south. The wind could not therefore appear to be coming from the North-West.

(ii) $\vec{v_M} = 2\vec{j}$

$\vec{v_W} = 6\vec{i} - 8\vec{j}$ m/s

$\vec{v_{WM}} = \vec{v_W} - \vec{v_M}$

$= 6\vec{i} - 10\vec{j}$



$\tan\theta = \frac{10}{6} = \frac{5}{3}$

$\Rightarrow \theta = 59°$ N of W

Q. 9. (i) **Case 1:** $\vec{v_G} = 4\vec{i}$

$\vec{v_W} = x\vec{i} + y\vec{j}$

$\vec{v_{WG}} = \vec{v_W} - \vec{v_G}$

$= (x - 4)\vec{i} + y\vec{j}$

$\vec{v_{WG}}$ from the North-West

$\Rightarrow x - 4 = -y \quad \Rightarrow \quad x = 4 - y$

Case 2: $\vec{v_G} = 6\vec{j}$

$\vec{v_W} = x\vec{i} + y\vec{j}$

$\vec{v_{WG}} = \vec{v_W} - \vec{v_G}$

$= x\vec{i} + (y - 6)\vec{j}$

$|\vec{v_{WG}}| = 10$

$\Rightarrow \sqrt{x^2 + (y - 6)^2} = 10$ ... but $x = 4 - y$

$\Rightarrow (4 - y)^2 + (y - 6)^2 = 100$

$\Rightarrow 16 - 8y + y^2 + y^2 - 12y + 36 - 100 = 0$

$\Rightarrow 2y^2 - 20y - 48 = 0$

$\Rightarrow y^2 - 10y - 24 = 0$

$\Rightarrow (y - 12)(y + 2) = 0$

$\Rightarrow y = 12, y = -2$

$\Rightarrow x = -8, x = 6$

Let $x = -8$ and $y = 12$

$\Rightarrow \vec{v_W} = -8\vec{i} + 12\vec{j}$

$\vec{v_{WC}} = -12\vec{i} + 12\vec{j}$

Let $x = 6$ and $y = -2$

$\Rightarrow \vec{v_W} = 6\vec{i} - 2\vec{j}$

$\vec{v_{WC}} = 2\vec{i} - 2\vec{j}$

$\vec{v_{WC}} = -12\vec{i} + 12\vec{j}$ is not from the North-West. It is, in fact, towards the North-West. We therefore exclude $x = -8$ and $y = 12$

$\vec{v_{WC}} = 2\vec{i} - 2\vec{j}$ is from the North-West as required.

$\Rightarrow \vec{v_W} = 6\vec{i} - 2\vec{j}$ is the actual velocity of the wind.

(ii) $\vec{v_G} = -p\vec{i}, \quad p > 0$

$\vec{v_W} = 6\vec{i} - 2\vec{j}$

$\vec{v_{WG}} = \vec{v_W} - \vec{v_G} = (6 + p)\vec{i} - 2\vec{j}$

$|\vec{v_{WG}}| = 8$

$\Rightarrow \sqrt{(6 + p)^2 + (-2)^2} = 8$

$\Rightarrow 36 + 12p + p^2 + 4 = 64$

$\Rightarrow p^2 + 12p - 24 = 0$

$$\Rightarrow p = \frac{-12 \pm \sqrt{(12)^2 - 4(1)(-24)}}{2}$$

$$= \frac{-12 \pm \sqrt{240}}{2}$$

$p > 0$

$$\Rightarrow p = \frac{-12 + \sqrt{240}}{2} = 1.75$$

$\Rightarrow$ Girl should cycle at 1.75 m/s due west.

Q. 10. $\vec{v_T} = 4\vec{i}$

$\vec{v_S} = x\vec{i} + y\vec{j}$

$\vec{v_{ST}} = \vec{v_S} - \vec{v_T} = (x - 4)\vec{i} + y\vec{j}$

$\vec{v_{ST}}$ towards south-east

$\Rightarrow x - 4 = -y \quad \Rightarrow \quad x = 4 - y$

Also, $|\vec{v_S}| = 20$

$\Rightarrow \sqrt{x^2 + y^2} = 20$ ... but $x = 4 - y$

$\Rightarrow (4 - y)^2 + y^2 = 400$

$\Rightarrow 16 - 8y + y^2 + y^2 = 400$

$\Rightarrow 2y^2 - 8y - 384 = 0$

$\Rightarrow y^2 - 4y - 192 = 0$

$\Rightarrow (y - 16)(y + 12) = 0$

$\Rightarrow y = 16, y = -12$

$\Rightarrow x = -12, x = 16$

Taking $x = -12$ and $y = 16$ gives $\vec{v_{ST}} = -16\vec{i} + 16\vec{j}$. This is not towards the south-east. It is, in fact, from the south-east. These values are therefore excluded.

Taking $x = 16$ and $y = -12$ gives $\overrightarrow{v_{ST}} = 12\vec{i} - 12\vec{j}$. This is towards the south-east.

$\Rightarrow \vec{v_S} = 16\vec{i} - 12\vec{j}$ m/s.

Q. 11. (i) $\vec{v_H} = 17\vec{i}$

$\vec{v_K} = x\vec{i} + y\vec{j}$

$\overrightarrow{v_{KH}} = \vec{v_K} - \vec{v_H} = (x - 17)\vec{i} + y\vec{j}$

$\overrightarrow{v_{KH}}$ north-east

$\Rightarrow x - 17 = y \quad \Rightarrow \quad x = y + 17$

Also, $|\vec{v_K}| = 25$

$\Rightarrow \sqrt{x^2 + y^2} = 25 \quad \ldots$ but $x = y + 17$

$\Rightarrow (y + 17)^2 + y^2 = 625$

$\Rightarrow y^2 + 34y + 289 + y^2 = 625$

$\Rightarrow 2y^2 + 34y - 336 = 0$

$\Rightarrow y^2 + 17y - 168 = 0$

$\Rightarrow (y + 24)(y - 7) = 0$

$\Rightarrow y = -24, y = 7$

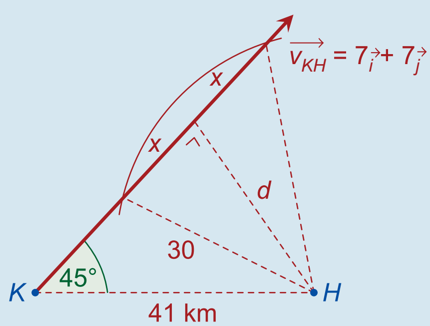
$\Rightarrow x = -7, x = 24$

Taking $x = -7$ and $y = -24$ gives $\overrightarrow{v_{KH}} = -24\vec{i} - 24\vec{j}$. This is not towards the north-east. It is, in fact, from the north-east. These values of x and y are therefore excluded.

Taking $x = 24$ and $y = 7$ gives $\overrightarrow{v_{KH}} = 7\vec{i} + 7\vec{j}$. This is towards the north-east.

$\Rightarrow \vec{v_K} = 24\vec{i} + 7\vec{j}$ km/h

(ii)



$\sin 45° = \frac{d}{41}$

$\Rightarrow d = 41 \sin 45°$

$= 29$ km

(iii) Draw a circle of radius 30 km with centre at H.

As long as the relative path, $\overrightarrow{v_{KH}}$ is inside this circle, K and H will be within 30 km of each other. This will be for a relative distance of $2x$.

$x^2 + d^2 = 30^2 \quad \ldots$ but $d = 29$

$\Rightarrow x = \sqrt{30^2 - 29^2} = \sqrt{59}$

$\Rightarrow 2x = 2\sqrt{59}$

$$\text{Time} = \frac{\text{relative distance}}{\text{relative speed}}$$

$$= \frac{2\sqrt{59}}{\sqrt{7^2 + 7^2}} = 1.55 \text{ h}$$

$= 93$ mins.

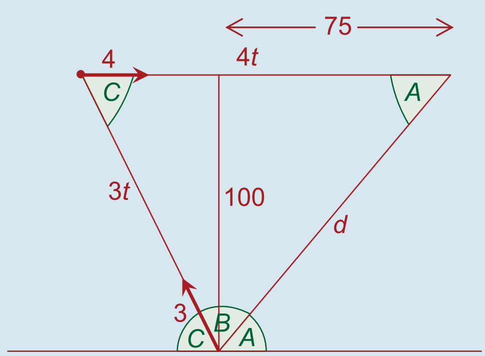
Exercise 4F

Q. 1. Let t = the time taken to cross the river.

The boat will head upstream at 3 m/s, and would travel a distance of $3t$.

Meanwhile, the river carries the boat downstream a distance $4t$.

The boat lands 75 m downstream.



$\tan A = \frac{100}{75} = \frac{4}{3}$

$\Rightarrow A = 53.13°$

$d^2 = 75^2 + 100^2$

$\Rightarrow d = 125$ m

Using the Sine Rule:

$\frac{3t}{\sin A} = \frac{4t}{\sin B} \quad \ldots$ but $\sin A = \frac{4}{5}$

$\Rightarrow 3t\left(\frac{5}{4}\right) = \frac{4t}{\sin B}$

$\Rightarrow \frac{15t}{4} = \frac{4t}{\sin B}$

$\Rightarrow \sin B = \frac{16}{15}$

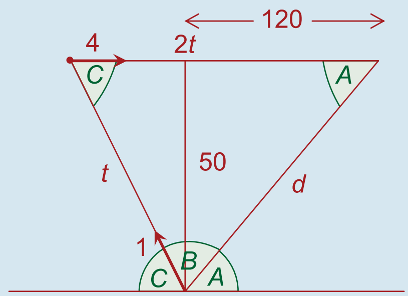
Note: This question can be solved by replacing 75 with 105.

Q. 2. (i) Let t = the time taken to cross the river.

The boat will head upstream at 1 m/s, and would travel a distance of t.

Meanwhile, the river carries the boat downstream a distance $2t$.

The boat lands 120 m downstream.



$\tan A = \frac{50}{120}$

$= \frac{5}{12}$

$\Rightarrow A = 22.62°$

$d^2 = 120^2 + 50^2$

$\Rightarrow d = 130$ m

Using the Sine Rule:

$\frac{t}{\sin A} = \frac{2t}{\sin B}$ … but $\sin A = \frac{5}{13}$

$\Rightarrow \frac{13}{5} = \frac{2}{\sin B}$

$\Rightarrow \sin B = \frac{10}{13}$

$\Rightarrow B = 50.28°$ **OR** $B = 129.72°$

Case 1: $B = 50.28°$

$C = 180° - 50.28° - 22.62°$

$\Rightarrow C = 107.1°$

$\Rightarrow 72.9°$ to the downstream direction

Case 2: $B = 129.72°$

$C = 180° - 129.72° - 22.62°$

$\Rightarrow C = 27.66°$

$\Rightarrow 27.66°$ to the upstream direction.

Using the Sine rule:

$\frac{t}{\sin A} = \frac{d}{\sin C}$

$\Rightarrow \frac{13t}{5} = \frac{130}{\sin 107.1°}$

$\Rightarrow t = \frac{50}{\sin 107.1°} = 52$ s

$\Rightarrow \frac{13t}{5} = \frac{130}{\sin 27.66°}$

$\Rightarrow t = \frac{50}{\sin 27.66°} = 108$ s

Q. 3. (i) $\vec{v_R} = q\vec{i}$

$\vec{v_{GR}} = p\vec{j}$ … tries to go straight across.

$\vec{v_G} = \vec{v_{GR}} + \vec{v_R}$

$= q\vec{i} + p\vec{j}$

Time across $= \frac{\text{distance across}}{\text{speed across}}$

$= \frac{60}{p}$

$\Rightarrow \frac{60}{p} = 100 \Rightarrow p = 0.6$

distance downstream =
speed downstream × time $= q \times 100$

$= 100q$

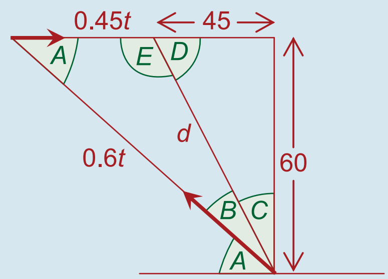
$\Rightarrow 100q = 45 \Rightarrow q = 0.45$

(ii) Let t = the time taken to cross the river.

The girl will head upstream at 0.6 m/s, and would travel a distance $0.6t$

Meanwhile the river carries her downstream a distance $0.45t$.

She lands 45 m upstream.



$d^2 = 45^2 + 60^2 \Rightarrow d = 75$ m

$\tan D = \frac{60}{45} = \frac{4}{3}$

$\Rightarrow D = 53.13°$

$\Rightarrow E = 126.87°$

Using the Sine Rule:

$\frac{0.6t}{\sin 126.87°} = \frac{0.45t}{\sin B}$

$\Rightarrow \sin B = \frac{0.45 \sin 126.87°}{0.6}$

$= 0.6$

$\Rightarrow B = 36.87°$ **OR** $B = 143.13°$

Case 1: $B = 36.87°$

$A = 180° - 36.87° - 126.87°$

$\Rightarrow A = 16.26°$

Using the Sine Rule:

$\frac{75}{\sin 16.26} = \frac{0.6t}{\sin 126.87°}$

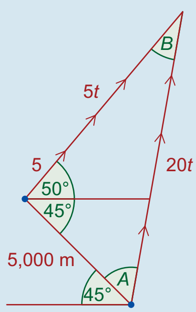
$\Rightarrow t = 357$ s

Case 2: $B = 143.13$

$A = 180° - 143.13° - 126.87°$

$\Rightarrow A = -90°$ … not possible

Q. 4.



$\frac{20t}{\sin 95°} = \frac{5t}{\sin A}$

$\therefore 20 \sin A = 5 \sin 95°$

$\therefore A = 14.42°$ **OR** ~~165.578°~~

$\therefore B = 180° - 95° - 14.42° = 70.58°$

$\therefore$ Speedboat must travel
$(45 + 14.42) = 59.42°$ North of West

$\frac{5{,}000}{\sin 70.58°} = \frac{20\,t}{\sin 95°}$

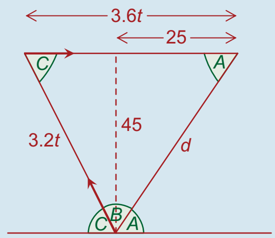
$\therefore t = \frac{250 \sin 95°}{\sin 70.58°} = 264$ s

Q. 5. (i) Let t = the time taken to cross the river.

The boat will head upstream at 3.2 m/s, and would travel a distance $3.2t$.

Meanwhile the river carries the boat downstream a distance $3.6t$.

The boat lands 25 m downstream.



$\tan A = \frac{45}{25} = \frac{9}{5}$

$\Rightarrow A = 60.945°$

$d^2 = 25^2 + 45^2$

$\Rightarrow d = 51.478$ m

Using the Sine Rule:

$\frac{3.2t}{\sin 60.945} = \frac{3.6t}{\sin B}$

$\Rightarrow B = \sin^{-1}\left[\frac{3.6 \sin 60.945°}{3.2}\right]$

$\Rightarrow B = 79.553°$ **OR** $B = 100.447°$

(ii) **Case 1:** $B = 79.553°$

$C = 180° - 79.553° - 60.945°$

$\Rightarrow C = 39.502°$

Case 2: $B = 100.447°$

$C = 180° - 100.447° - 60.945°$

$\Rightarrow C = 18.608°$

Using the Sine rule:

$\frac{3.2t}{\sin 60.945°} = \frac{51.478}{\sin 39.502°}$

$\Rightarrow t = \frac{51.478 \sin 60.945°}{3.2 \sin 39.502°}$

$\Rightarrow t = 22$ s

$\frac{3.2t}{\sin 60.945°} = \frac{51.478}{\sin 18.608°}$

$\Rightarrow t = \frac{51.478 \sin 60.945°}{3.2 \sin 18.608°}$

$\Rightarrow t = 44$ s