MarkingScheme

SeguencesSeriesH

Question 1 (2017)

(a)	$r = \frac{42{\cdot}75}{95} = \frac{9}{20} \qquad T_n = ar^{n-1} < 0{\cdot}01$ $95\left(\frac{9}{20}\right)^{n-1} < 0{\cdot}01$ $\left(\frac{9}{20}\right)^{n-1} < \frac{0{\cdot}01}{95}$ $(n-1)\log\left(\frac{9}{20}\right) < \log\left(\frac{0.01}{95}\right)$ $(n-1) > \frac{\log\left(\frac{0.01}{95}\right)}{\log\left(\frac{9}{20}\right)}$ (since $\log\left(\frac{9}{20}\right)$ is negative) $n-1 > 11{\cdot}47$ $n > 12{\cdot}47$ 12th day	**Scale 15D (0, 5, 8, 12, 15)** *Low Partial Credit:* • r found • T_n of a GP with some substitution *Mid Partial Credit:* • Inequality in n written *High Partial Credit:* • Inequality in n simplified (log handled) *Full Credit:* • Accept $n = 12{\cdot}47$

Question 2 (2017)

	$4(2) + 4\sqrt{2} + 4 + \cdots\cdots$ $a = 8 \quad r = \frac{1}{\sqrt{2}}$ $S_\infty = \frac{a}{1-r}$ $S_\infty = \frac{8}{1-\frac{1}{\sqrt{2}}}$ $S_\infty = \frac{8}{1-\frac{1}{\sqrt{2}}} \cdot \frac{1+\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$ $S_\infty = \frac{8\left(1+\frac{1}{\sqrt{2}}\right)}{\frac{1}{2}}$ $S_\infty = 16 + 8\sqrt{2}$	**Scale 10C (0, 5, 8, 10)** *Low Partial Credit:* • length of one side of new square *High Partial Credit:* • S_∞ fully substituted • Correct work with one side only

Q9	Model Solution – 55 Marks	Marking Notes
(a)(i)	$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ $S_n = \frac{a(1-r^n)}{1-r}$ $S_n = \frac{4\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 7{\cdot}9375$ $-\frac{1}{2^n} = -\frac{1}{128}$ $n = 7$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • some listing of terms • S_n formula *High Partial Credit* • listing of exactly 7 correct terms • formula fully substituted
(a) (ii)	$S_\infty = \frac{a}{1-r}$ $S_\infty = \frac{4}{1-\frac{1}{2}} = 8$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • S_∞ formula *High Partial Credit* • formula fully substituted

	1	2	3	4	5	6	7	8	9
Chg x	+4	0	-1	0	$\frac{1}{4}$	0	$-\frac{1}{16}$	0	$\frac{1}{64}$
Chg y	0	2	0	$-\frac{1}{2}$	0	$\frac{1}{8}$	0	$-\frac{1}{32}$	0

(a) (iii)	$S_\infty = \frac{4}{1-\left(-\frac{1}{4}\right)} = 3{\cdot}2 = \frac{16}{5}$ $S_\infty = \frac{2}{1-\left(-\frac{1}{4}\right)} = 1{\cdot}6 = \frac{8}{5}$ $\left(\frac{16}{5}, \frac{8}{5}\right)$ or $(3{\cdot}2, 1{\cdot}6)$	Scale 15D (0, 4, 7, 11, 15) *Low Partial Credit* • 2 extra entries correct in either row *Mid Partial Credit* • either row fully correct *High Partial Credit* • one co-ordinate correct **Notes:** - *need to see S_∞ correctly used to move beyond Mid Partial Credit* - *no S_∞ merits Mid Partial Credit at most*

(a) Complete the table below to show the maximum height, **in fraction form**, reached by the ball on each of the first four bounces.

Bounce	0	1	2	3	4
Height (m)	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{27}{32}$	$\frac{81}{128}$

(b) Find, in metres, the total vertical distance (up and down) the ball had travelled when it hit the ground for the 5th time. Give your answer in fraction form.

$$2+2\left(\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\frac{81}{128}\right)=2+2\left(\frac{525}{128}\right)=\frac{653}{64}=10\tfrac{13}{64}\text{ m}$$

or

$$2+2\left(\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\frac{81}{128}\right)=2+2S_4$$

$$=2+2\left(\frac{\frac{3}{2}\left(1-\left(\frac{3}{4}\right)^4\right)}{1-\frac{3}{4}}\right)$$

$$=2+\frac{525}{64}=\frac{653}{64}=10\tfrac{13}{64}\text{ m}$$

(c) If the ball were to continue to bounce indefinitely, find, in metres, the total vertical distance it would travel.

$$2+2\left(\frac{3}{2}+\frac{9}{8}+\dots\right)=2+2\left(\frac{a}{1-r}\right)$$

$$=2+2\left(\frac{\frac{3}{2}}{1-\frac{3}{4}}\right)$$

$$=2+12=14\text{ m}$$

Question 5 (2015)

$$S = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}$$
$$a = 1, \quad r = \omega$$
$$S = \frac{1(1-\omega^n)}{1-\omega} = \frac{1(1-1)}{1-\omega} = 0$$

Question 6 (2015)

(a) Find the probability that Michael is successful (S) with all three of his first three free throws in a game.

$$P(S, S, S) = 0 \cdot 7 \times 0 \cdot 8 \times 0 \cdot 8 = 0 \cdot 448$$

(b) Find the probability that Michael is unsuccessful (U) with his first two free throws and successful with the third.

$$P(U, U, S) = 0 \cdot 3 \times 0 \cdot 4 \times 0 \cdot 6 = 0 \cdot 072$$

(c) List all the ways that Michael could be successful with his third free throw in a game and hence find the probability that Michael is successful with his third free throw.

S, S, S   U, U, S   S, U, S   U, S, S

$$P(S, S, S) = 0 \cdot 7 \times 0 \cdot 8 \times 0 \cdot 8 = 0 \cdot 448$$
$$P(U, U, S) = 0 \cdot 3 \times 0 \cdot 4 \times 0 \cdot 6 = 0 \cdot 072$$
$$P(S, U, S) = 0 \cdot 7 \times 0 \cdot 2 \times 0 \cdot 6 = 0 \cdot 084$$
$$P(U, S, S) = 0 \cdot 3 \times 0 \cdot 6 \times 0 \cdot 8 = 0 \cdot 144$$

$$P = 0 \cdot 448 + 0 \cdot 072 + 0 \cdot 084 + 0 \cdot 144 = 0 \cdot 748$$

(d) (i) Let p_n be the probability that Michael is successful with his n^{th} free throw in the game (and hence $(1-p_n)$ is the probability that Michael is unsuccessful with his n^{th} free throw). Show that $p_{n+1} = 0{\cdot}6 + 0{\cdot}2p_n$.

$$\begin{aligned} p_{n+1} &= \mathrm{P}(\mathrm{S,S}) + \mathrm{P}(\mathrm{U,S}) \\ &= p_n \times 0{\cdot}8 + (1-p_n)0{\cdot}6 \\ &= 0{\cdot}6 + 0{\cdot}2p_n \end{aligned}$$

(ii) Assume that p is Michael's success rate in the long run; that is, for large values of n, we have $p_{n+1} \approx p_n \approx p$.
Using the result from part **(d) (i)** above, or otherwise, show that $p = 0{\cdot}75$.

$$p \approx p_n \approx p_{n+1} = 0{\cdot}6 + 0{\cdot}2p_n$$
$$\Rightarrow 0{\cdot}8p_n = 0{\cdot}6$$
$$\Rightarrow p_n = \frac{0{\cdot}6}{0{\cdot}8} = 0{\cdot}75 = p$$

(e) For all positive integers n, let $a_n = p - p_n$, where $p = 0{\cdot}75$ as above.

(i) Use the ratio $\frac{a_{n+1}}{a_n}$ to show that a_n is a geometric sequence with common ratio $\frac{1}{5}$.

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{p - p_{n+1}}{p - p_n} \\ &= \frac{0{\cdot}75 - (0{\cdot}6 + 0{\cdot}2p_n)}{0{\cdot}75 - p_n} \\ &= \frac{0{\cdot}15 - 0{\cdot}2p_n}{5(0{\cdot}15 - 0{\cdot}2p_n)} = \frac{1}{5} \end{aligned}$$

(ii) Find the smallest value of n for which $p - p_n < 0{\cdot}00001$.

$$a_n = p - p_n$$
$$a_1 = p - p_1 = 0\cdot75 - 0\cdot7 = 0\cdot05$$

$$ar^{n-1} = 0\cdot05(0\cdot2)^{n-1} < 0\cdot00001$$
$$(n-1)\ln 0\cdot2 < \ln 0\cdot0002$$
$$\Rightarrow n-1 > \frac{\ln 0\cdot0002}{\ln 0\cdot2} = 5\cdot29$$
$$\Rightarrow n > 6\cdot29$$

$$n = 7$$

(f) You arrive at a game in which Michael is playing. You know that he has already taken many free throws, but you do not know what pattern of success he has had.

(i) Based on this knowledge, what is your estimate of the probability that Michael will be successful with his next free throw in the game?

Answer: $0{\cdot}75$ **or** p

(ii) Why would it **not** be appropriate to consider Michael's subsequent free throws in the game as a sequence of Bernoulli trials?

Events not independent

(a) **(i)** Show that T_1, T_2, and T_3 are in arithmetic sequence.

$T_1 = \ln a$, $\quad T_2 = \ln a^2 = 2\ln a$, $\quad T_3 = \ln a^3 = 3\ln a$.

$T_2 - T_1 = 2\ln a - \ln a = \ln a$

$T_3 - T_2 = 3\ln a - 2\ln a = \ln a$

$T_3 - T_2 = T_2 - T_1$. Hence, terms are in arithmetic sequence.

(ii) Prove that the sequence is arithmetic and find the common difference.

$T_n = \ln a^n = n\ln a,$

$T_{n-1} = \ln a^{n-1} = (n-1)\ln a.$

$T_n - T_{n-1} = n\ln a - (n-1)\ln a = \ln a$, (a constant).

Hence, the sequence is arithmetic.

Common difference: $T_n - T_{n-1} = \ln a$

(b) Find the value of a for which $T_1 + T_2 + T_3 + \cdots + T_{98} + T_{99} + T_{100} = 10\,100$.

$T_1 + T_2 + T_3 + \cdots + T_{98} + T_{99} + T_{100} = 10\,100$

$\Rightarrow \ln a + 2\ln a + 3\ln a + \cdots + 100\ln a = 10\,100$

$\Rightarrow \dfrac{100}{2}\left[2\ln a + (100-1)\ln a\right] = 10\,100$

$\Rightarrow 50\left[101\ln a\right] = 10\,100$

$\Rightarrow 5050\ln a = 10\,100$

$\Rightarrow \ln a = 2$

$\Rightarrow a = e^2 = 7\cdot389$

(c) Verify that, for all values of a,

$$\left(T_1+T_2+T_3+\cdots+T_{10}\right)+100d=\left(T_{11}+T_{12}+T_{13}+\cdots+T_{20}\right),$$

where d is the common difference of the sequence.

$$\begin{aligned}\left(T_1+T_2+T_3+\cdots+T_{10}\right)+100d &= (T_1+10d)+(T_2+10d)+(T_3+10d)+\cdots+(T_{10}+10d)\\ &= T_{11}+T_{12}+T_{13}+\cdots+T_{20}.\end{aligned}$$

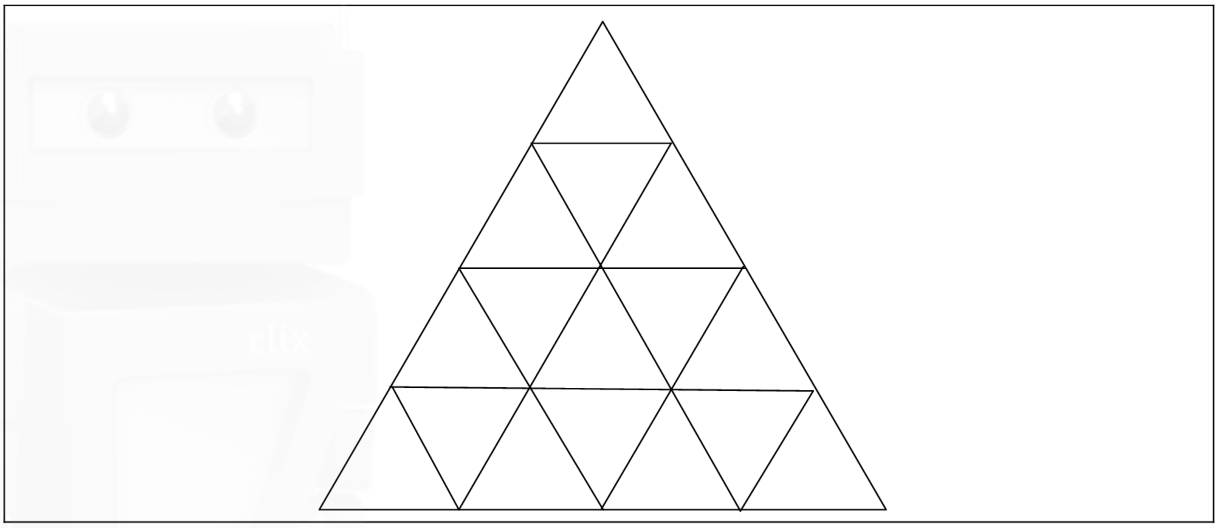
OR

$$\begin{aligned}\left(T_1+T_2+T_3+\cdots+T_{10}\right)+100d &= (\ln a+2\ln a+3\ln a+\cdots+10\ln a)+100\ln a\\ &= \tfrac{10}{2}\left(2\ln a+(10-1)\ln a\right)+100\ln a\\ &= 5(11\ln a)+100\ln a\\ &= 155\ln a\end{aligned}$$

$$\begin{aligned}\left(T_{11}+T_{12}+T_{13}+\cdots+T_{20}\right) &= 11\ln a+12\ln a+13\ln a+\cdots+20\ln a\\ &= \tfrac{10}{2}\left(22\ln a+(10-1)\ln a\right)\\ &= 5\left(31\ln a\right)\\ &= 155\ln a\end{aligned}$$

Hence, L.H.S = R.H.S

(a) **(i)** Draw the 4th pattern in the sequence.



(ii) The table below shows the number of small triangles in each pattern and the number of matchsticks needed to create each pattern. Complete the table.

Pattern	1st	2nd	3rd	4th
Number of small triangles	1	4	9	16
Number of matchsticks	3	9	18	30

(b) Write an expression in n for the number of triangles in the n^{th} pattern in the sequence.

n^2

OR

1   4   9   16   25

1^{st} Diff: 3   5   7   9

2^{nd} Diff: 2   2   2 $\leftarrow$ Constant $\Rightarrow$ quadratic

$$T_n = an^2 + bn + c$$
$$2a = 2 \Rightarrow a = 1$$

$$T_n = n^2 + bn + c$$
$$T_1 = 1 + b + c = 1$$
$$T_2 = 4 + 2b + c = 4$$
$$\Rightarrow b = 0$$
$$c = 0$$

$$\Rightarrow T_n = n^2$$

(c) Find an expression, in n, for the number of matchsticks needed to turn the $(n-1)^{\text{th}}$ pattern into the n^{th} pattern.

$3n$

OR

3   9   18   30

1^{st} Diff: 6   9   12

2^{nd} Diff: 3   3

Second difference constant $\Rightarrow$ quadratic pattern

$$T_n = an^2 + bn + c$$
$$2a = 3 \Rightarrow a = \frac{3}{2}$$

$$T_n = \frac{3}{2}n^2 + bn + c$$
$$T_1 = \frac{3}{2}(1)^2 + b(1) + c = 3 \quad \Rightarrow b + c = \frac{3}{2}$$
$$T_2 = \frac{3}{2}(2)^2 + b(2) + c = 9 \quad \Rightarrow 2b + c = 3$$

$$b + c = \frac{3}{2}$$
$$\underline{2b + c = 3}$$
$$b = \frac{3}{2}$$
$$c = 0$$

$$T_n = \frac{3}{2}n^2 + \frac{3}{2}n$$
$$T_{n-1} = \frac{3}{2}(n-1)^2 + \frac{3}{2}(n-1)$$
$$T_n - T_{n-1} = -\frac{3}{2}(n^2 - 2n + 1) - \frac{3}{2}(n-1) + \frac{3}{2}n^2 + \frac{3}{2}n$$
$$= -\frac{3}{2}n^2 + 3n - \frac{3}{2} - \frac{3}{2}n + \frac{3}{2} + \frac{3}{2}n^2 + \frac{3}{2}n$$
$$= 3n$$

Question 9 (2013)

Initial Exploration:

2 × 2: In this case there are 4 jigsaw pieces and all of those are edge pieces.
3 × 2: In this case there are 6 jigsaw pieces and all of those are edge pieces too. In fact any jigsaw which is $m \times 2$ or $2 \times n$ consists entirely of edge pieces and no interior ones.
3 × 3: There are 9 jigsaw pieces and 8 of those are edge pieces, 1 is an interior piece.
4 × 3: There are 12 jigsaw pieces and 10 of those are edge pieces, 2 are interior pieces.
4 × 4: There are 16 jigsaw pieces and 12 of those are edge pieces, 4 are interior pieces.

(a) How do the number of edge pieces and the number of interior pieces compare in cases where either $m \leq 4$ or $n \leq 4$?

For an $m \times n$ jigsaw the edge pieces can be counted by looking at the perimeter (be careful not to count the corner pieces twice).

The left side of the jigsaw has m pieces, as does the right side.

The top side has n pieces but we've already counted the 2 corner pieces which means there are $n-2$ pieces left on the top to count (the same is true on the bottom of the jigsaw). So

$$\begin{aligned}\text{number of edge pieces} &= m+m+(n-2)+(n-2)\\ &= 2m+2n-4\end{aligned}$$

To calculate the interior pieces we can take the number of edge pieces, $2m+2n-4$ away from the total, mn to get

$$\begin{aligned}\text{number of interior pieces} &= mn-(2m+2n-4)\\ &= mn-2m-2n+4\end{aligned}$$

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(b) Show that if the number of edge pieces is equal to the number of interior pieces, then

$$m = 4+\frac{8}{n-4}$$

First let's simplify the equation we're given by writing the right-hand side as one fraction

$$\begin{aligned}m &= 4+\frac{8}{n-4}\\ m &= \frac{4(n-4)}{n-4}+\frac{8}{n-4}\\ m &= \frac{4n-16+8}{n-4}\\ m &= \frac{4n-8}{n-4}\end{aligned}$$

From part **(a)** we get

$$\begin{aligned}\text{number of edge pieces} &= \text{number of interior pieces}\\ mn-2m-2n+4 &= 2m+2n-4\\ mn-4m &= 4n-8\\ m(n-4) &= 4n-8\\ m &= \frac{4n-8}{n-4}\end{aligned}$$

Therefore, from above:

$$m = 4+\frac{8}{n-4}$$

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altitude (km)	0	1	2	3	4	5
pressure (kPa)	101·3	89·4	79·0	69·7	61·6	54·4
		–0·6%	–0·6%	–0·6%	0%	+0·7%

(i) Taking any **one** value other than 0 for the altitude, verify that the pressure given by Thomas's model and the pressure given by Hannah's model differ by less than 0·01 kPa.

Altitude 1: $101{\cdot}3e^{-0{\cdot}1244} - 101{\cdot}3(0{\cdot}883) \approx 0{\cdot}0027 < 0{\cdot}01$

Altitude 2: $101{\cdot}3e^{-0{\cdot}1244\times2} - 101{\cdot}3(0{\cdot}883)^2 \approx 0{\cdot}0048 < 0{\cdot}01$

Altitude 3: $101{\cdot}3e^{-0{\cdot}1244\times3} - 101{\cdot}3(0{\cdot}883)^3 \approx 0{\cdot}0063 < 0{\cdot}01$

Altitude 4: $101{\cdot}3e^{-0{\cdot}1244\times4} - 101{\cdot}3(0{\cdot}883)^4 \approx 0{\cdot}0074 < 0{\cdot}01$

Altitude 5: $101{\cdot}3e^{-0{\cdot}1244\times5} - 101{\cdot}3(0{\cdot}883)^5 \approx 0{\cdot}0081 < 0{\cdot}01$

(ii) Explain how Thomas might have arrived at the value of the constant 0·1244 in his model.

He might have assumed it was of the form $101{\cdot}3e^{-kt}$ and then used one of the observations to find *k*.

He might have put various values of *t* into $p(t) = 101{\cdot}3e^{-kt}$, and found an average of the resulting values of *k*.

He might have got the natural log of the ratio of consecutive terms.

He might have plotted the log of the pressure against the altitude and used the slope of the best-fit line to find *k*.

(c) Hannah's model is *discrete*, while Thomas's is *continuous*.

(i) Explain what this means.

Hannah's model gives values for the pressure at separate (whole number) values for the altitude.

Thomas's model gives a value for the pressure at any real value of the altitude, whether it's a whole number or not.

(ii) State one advantage of a continuous model over a discrete one.

You are not restricted to the specific discrete values of the independent variable; you can also work with values between any two given values – any value you like.

(d) Use Thomas's model to estimate the atmospheric pressure at the altitude of the top of Mount Everest: 8848 metres.

$$p(8{\cdot}848) = 101{\cdot}3e^{-0{\cdot}1244\times8{\cdot}848} \approx 33{\cdot}7 \text{ kPa}.$$

(e) Using Thomas's model, find an estimate for the altitude at which the atmospheric pressure is half of its value at sea level (altitude 0 km).

$$p(h) = \tfrac{1}{2}\,p(0)$$
$$101{\cdot}3e^{-0{\cdot}1244h} = \tfrac{1}{2}(101.3)$$
$$e^{0{\cdot}1244h} = 2$$
$$0{\cdot}1244h = \ln 2$$
$$h \approx 5{\cdot}57 \text{ km}$$

(f) People sometimes experience a sensation in their ears when the pressure changes. This can happen when travelling in a fast lift in a tall building. Experiments indicate that many people feel such a sensation if the pressure changes rapidly by 1 kilopascal or more. Suppose that such a person steps into a lift that is close to sea level. Taking a suitable approximation for the distance between two floors, estimate the number of floors that the person would need to travel in order to feel this sensation.

$$p(h) = 100{\cdot}3$$
$$101{\cdot}3e^{-0{\cdot}1244h} = 100{\cdot}3$$
$$e^{-0{\cdot}1244h} = \frac{100{\cdot}3}{101{\cdot}3}$$
$$-0{\cdot}1244h = \ln\left(\frac{100{\cdot}3}{101{\cdot}3}\right)$$
$$h = \frac{1}{0{\cdot}1244}\left(\ln 101{\cdot}3 - \ln 100{\cdot}3\right)$$
$$h \approx 0{\cdot}0797 \text{ km} \approx 80 \text{ m}$$

Allowing 3 m per floor, it's about 27 floors.