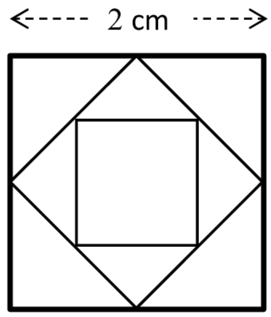
Question 1

Question 4 **(25 marks)**

(a) The amount of a substance remaining in a solution reduces exponentially over time. An experiment measures the percentage of the substance remaining in the solution. The percentage is measured at the same time each day. The data collected over the first 4 days are given in the table below. Based on the data in the table, estimate which is the first day on which the percentage of the substance in the solution will be less than 0·01%.

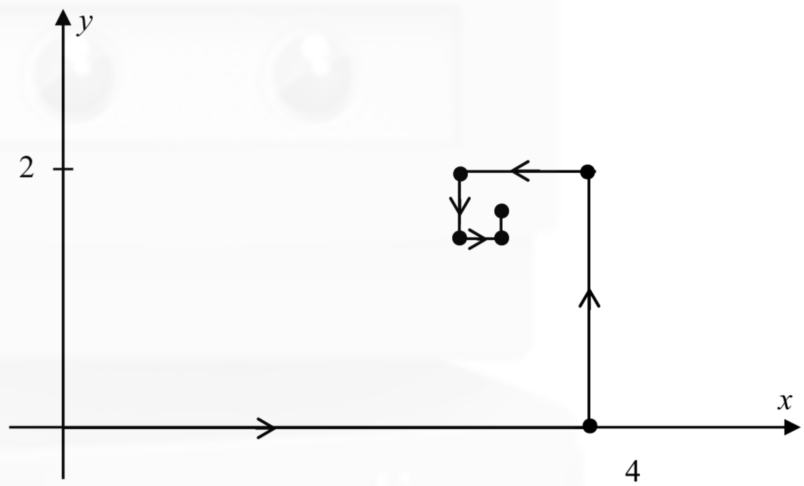
Day	**1**	**2**	**3**	**4**
Percentage of substance (%)	95	42·75	19·2375	8·6569

(b) A square has sides of length 2 cm. The midpoints of the sides of this square are joined to form another square. This process is continued.
The first three squares in the process are shown below.
Find the sum of the perimeters of the squares if this process is continued indefinitely.
Give your answer in the form $a + b\sqrt{c}$ cm, where a, b, and $c \in \mathbb{N}$.



Question 3

(a) At the first stage of a pattern, a point moves 4 units from the origin in the positive direction along the x-axis. For the second stage, it turns left and moves 2 units parallel to the y-axis. For the third stage, it turns left and moves 1 unit parallel to the x-axis.
At each stage, after the first one, the point turns left and moves half the distance of the previous stage, as shown.



(i) How many stages has the point completed when the total distance it has travelled, along its path, is 7·9375 units?

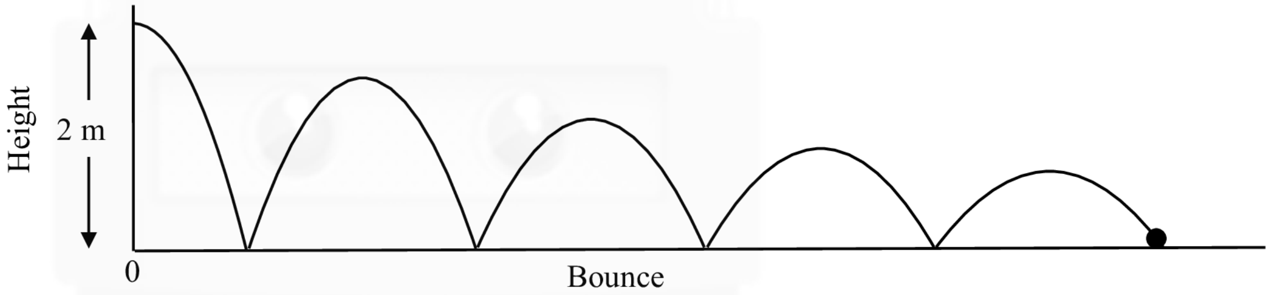
(ii) Find the maximum distance the point can move, along its path, if it continues in this pattern indefinitely.

(iii) Complete the second row of the table below showing the changes to the *x* co-ordinate, the first nine times the point moves to a new position. Hence, or otherwise, find the *x* co-ordinate and the *y* co-ordinate of the final position that the point is approaching, if it continues indefinitely in this pattern.

Stage	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**	**8th**	**9th**
Change in *x*	+4	0	–1						
Change in *y*									

Question 1 **(25 marks)**

Mary threw a ball onto level ground from a height of 2 m. Each time the ball hit the ground it bounced back up to $\frac{3}{4}$ of the height of the previous bounce, as shown.



(a) Complete the table below to show the maximum height, **in fraction form**, reached by the ball on each of the first four bounces.

Bounce	0	1	2	3	4
Height (m)	$\frac{2}{1}$				

(b) Find, in metres, the total vertical distance (up and down) the ball had travelled when it hit the ground for the 5^{th} time. Give your answer in fraction form.

(c) If the ball were to continue to bounce indefinitely, find, in metres, the total vertical distance it would travel.

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(b) Let ω be a complex number such that $\omega^n = 1$, $\omega \neq 1$, and $S = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}$. Use the formula for the sum of a finite geometric series to write the value of S in its simplest form.

Question 8 **(65 marks)**

In basketball, players often have to take free throws. When Michael takes his first free throw in any game, the probability that he is successful is 0·7.
For all subsequent free throws in the game, the probability that he is successful is:

- 0·8 if he has been successful on the previous throw
- 0·6 if he has been unsuccessful on the previous throw.

(a) Find the probability that Michael is successful (S) with all three of his first three free throws in a game.

P(S, S, S) =

(b) Find the probability that Michael is unsuccessful (U) with his first two free throws and successful with the third.

P(U, U, S) =

(c) List all the ways that Michael could be successful with his third free throw in a game and hence find the probability that Michael is successful with his third free throw.

(d) **(i)** Let p_n be the probability that Michael is successful with his n^{th} free throw in the game (and hence $(1-p_n)$ is the probability that Michael is unsuccessful with his n^{th} free throw). Show that $p_{n+1} = 0{\cdot}6 + 0{\cdot}2p_n$.

(ii) Assume that p is Michael's success rate in the long run; that is, for large values of n, we have $p_{n+1} \approx p_n \approx p$.
Using the result from part **(d) (i)** above, or otherwise, show that $p = 0{\cdot}75$.

(e) For all positive integers n, let $a_n = p - p_n$, where $p = 0{\cdot}75$ as above.

(i) Use the ratio $\dfrac{a_{n+1}}{a_n}$ to show that a_n is a geometric sequence with common ratio $\dfrac{1}{5}$.

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(ii) Find the smallest value of n for which $p - p_n < 0{\cdot}00001$.

(f) You arrive at a game in which Michael is playing. You know that he has already taken many free throws, but you do not know what pattern of success he has had.

(i) Based on this knowledge, what is your estimate of the probability that Michael will be successful with his next free throw in the game?

Answer: _______________

(ii) Why would it **not** be appropriate to consider Michael's subsequent free throws in the game as a sequence of Bernoulli trials?

Question 7

The n^{th} term of a sequence is $T_n = \ln a^n$, where $a > 0$ and a is a constant.

(a) **(i)** Show that T_1, T_2, and T_3 are in arithmetic sequence.

(ii) Prove that the sequence is arithmetic and find the common difference.

(b) Find the value of a for which $T_1 + T_2 + T_3 + \cdots + T_{98} + T_{99} + T_{100} = 10\,100$.

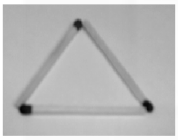
(c) Verify that, for all values of a,

$$(T_1 + T_2 + T_3 + \cdots + T_{10}) + 100\,d = (T_{11} + T_{12} + T_{13} + \cdots + T_{20}),$$

where d is the common difference of the sequence.

Shapes in the form of small equilateral triangles can be made using matchsticks of equal length. These shapes can be put together into patterns. The beginning of a sequence of these patterns is shown below.

1



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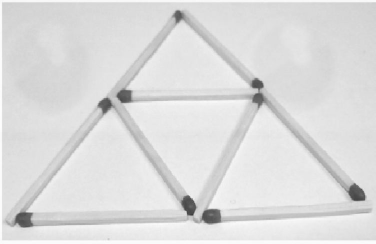

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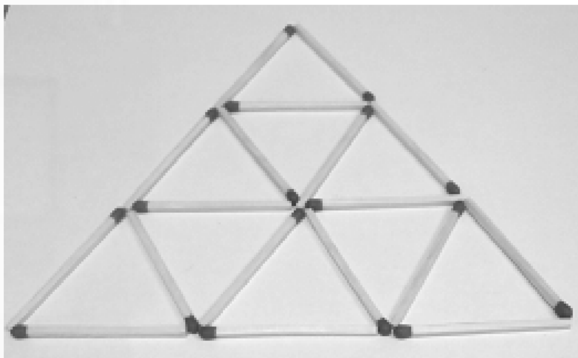


(a) **(i)** Draw the fourth pattern in the sequence.

(ii) The table below shows the number of small triangles in each pattern and the number of matchsticks needed to create each pattern. Complete the table.

Pattern	1^{st}	2^{nd}	3^{rd}	4^{th}
Number of small triangles	1		9	
Number of matchsticks	3	9		

(b) Write an expression in n for the number of triangles in the n^{th} pattern in the sequence.

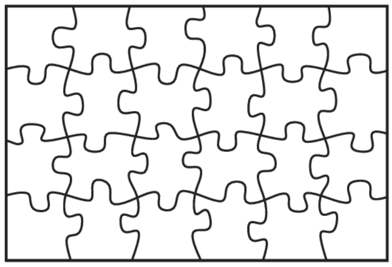
(c) Find an expression, in n, for the number of matchsticks needed to turn the $(n-1)^{\text{th}}$ pattern into the n^{th} pattern.

(d) The number of matchsticks in the n^{th} pattern in the sequence can be represented by the function $u_n = an^2 + bn$ where $a,\ b \in \mathbb{Q}$ and $n \in \mathbb{N}$. Find the value of a and the value of b.

(e) One of the patterns in the sequence has 4134 matchsticks. How many small triangles are in that pattern?

Question 6 **(50 marks)**

A rectangular jigsaw puzzle has pieces arranged in rows. Each row has the same number of pieces. For example, the picture on the right shows a 4×6 jigsaw puzzle – there are four rows with 6 pieces in each row.



Every piece of the puzzle is either an *edge piece* or an *interior piece*. The puzzle shown has 16 edge pieces and 8 interior pieces.

Investigate the number of edge pieces and the number of interior pieces in an $m \times n$ jigsaw puzzle, for different values of m and n. Start by exploring some particular cases, and then attempt to answer the questions that follow, with justification.

Initial exploration:

(a) How do the number of edge pieces and the number of interior pieces compare in cases where either $m \leq 4$ or $n \leq 4$?

(b) Show that if the number of edge pieces is equal to the number of interior pieces, then

$$m = 4 + \frac{8}{n-4}.$$

(c) Find all cases in which number of edge pieces is equal to the number of interior pieces.

(d) Determine the circumstances in which there are *fewer* interior pieces than edge pieces. Describe fully all such cases.

Question 9 **(50 marks)**

The *atmospheric pressure* is the pressure exerted by the air in the earth's atmosphere. It can be measured in kilopascals (kPa). The average atmospheric pressure varies with altitude: the higher up you go, the lower the pressure is.

Some students are investigating this variation in pressure, using some data that they found on the internet. They have information about the average pressure at various altitudes.

Six of the entries in the data set are as shown in the table below:

altitude (km)	0	1	2	3	4	5
pressure (kPa)	101·3	89·9	79·5	70·1	61·6	54·0

By looking at the pattern, the students are trying to find a suitable model to match the data.

(a) Hannah suggests that this is approximately a geometric sequence. She says she can match the data fairly well by taking the first term as 101·3 and the common ratio as 0·883.

(i) Complete the table below to show the values given by Hannah's model, correct to one decimal place.

altitude (km)	0	1	2	3	4	5
pressure (kPa)	101·3					

(ii) By considering the percentage errors in the above values, insert an appropriate number to complete the statement below.

"Hannah's model is accurate to within ________%."

(b) Thomas suggests modelling the data with the following exponential function:

$$p = 101{\cdot}3 \times e^{-0{\cdot}1244h}$$

where p is the pressure in kilopascals, and h is the altitude in kilometres.

(i) Taking any **one** value other than 0 for the altitude, verify that the pressure given by Thomas's model and the pressure given by Hannah's model differ by less than 0·01 kPa.

(ii) Explain how Thomas might have arrived at the value of the constant 0·1244 in his model.

(c) Hannah's model is *discrete*, while Thomas's is *continuous*.

(i) Explain what this means.

(ii) State one advantage of a continuous model over a discrete one.

(d) Use Thomas's model to estimate the atmospheric pressure at the altitude of the top of Mount Everest: 8848 metres.

(e) Using Thomas's model, find an estimate for the altitude at which the atmospheric pressure is half of its value at sea level (altitude 0 km).

(f) People sometimes experience a sensation in their ears when the pressure changes. This can happen when travelling in a fast lift in a tall building. Experiments indicate that many people feel such a sensation if the pressure changes rapidly by 1 kilopascal or more. Suppose that such a person steps into a lift that is close to sea level. Taking a suitable approximation for the distance between two floors, estimate the number of floors that the person would need to travel in order to feel this sensation.