SOLUTIONS TO PRISM PROBLEMS

Senior Level 2014

1. (C)
Since $14 = 2 \times 7,$ we see that the least common multiple of $3, 7$ and 14 is $3 \times 7 \times 2 = 42.$

2. (E)
We factor $x^2 + x - 12$ as $(x+3)(x-4)$ so the equation $x^2 + x - 12 = 0$ if and only if $(x+3)(x-4) = 0.$ Hence $x = 3$ or -4.
Alternatively, we can of course proceed in a more tedious manner by using the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the roots of a quadratic equation $ax^2 + bx + c = 0.$ Here $a = 1, b = 1$ and $= b - 12$ so the roots are

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-12)}}{2(1)} = \frac{-1 \pm \sqrt{49}}{2} = \frac{-1 \pm 7}{2} = 3, -4,$$ as above.

3. (A)
Many students will solve this problem rapidly by drawing a number line and positioning each of the five given choices on this line to easily see that $-3a$ is the largest among them.
If we proceed in a formal manner, we can algebraically compare $-3a$ with each of the other possible answers in turn. Notice first that $-3a > 6a$ because this inequality is the same as $-9a > 0$ which is true if (dividing across by -9) $a < 0,$ and since indeed $a < 0$ (given) the number $-3a$ indeed exceeds the number $6a.$ [Here and below it is important for students to note that when dealing with inequalities, we change the sign of the inequality when we multiply or divide across by a negative number.]
Next $-3a$ is greater than $-a + 1$ because the inequality $-3a > -a + 1$ is the same as $-2a > 1$ which is equivalent to $a < -\frac{1}{2},$ and this is true since we are given that a is a negative integer.
Next, observe that $-3a$ is greater than $-a$ because the inequality $-3\dot{a} > -a$ is the same as $-2a > 0$ or $a < 0,$ which is given.
Finally, we have $-3a > -a - 2$ because this inequality is equivalent to $-2a > -2$ which is true if and only if $a < \frac{-2}{-2} = 1,$ and this is true since we are given that a is negative

4. (C)
Let the marks obtained by the 5 students be x_1, x_2, x_3, x_4 and x_5 and let $\bar{x} = \frac{x_1 + x_2 + \ldots + x_5}{5} = 60$ be their average mark. If 4 of the students have their marks raised by 6 and the remaining student has his mark lowered by 1, then the new average will be $\bar{x}_{NEW} = \frac{x_1 + x_2 + \ldots + x_n + 6 + 6 + 6 + 6 - 4}{5}.$
Notice that this is $\bar{x}_{NEW} = \bar{x} + \frac{20}{5} = 60 + 4 = 64$

5. (E)

A short way of solving this problem is to note that M and E are two of the 5 equally likely ways in which the random arrangement of the letters of MATHS can commence Hence the required probability is $\frac{2}{5}$.

A more tedious but far more general method of proceeding is as follows.

With no restrictions, the letters of the word MATHS can be arranged in $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ (Students unfamiliar with this are encouraged to read the note below.)

Now if an arrangement is to start with an M or an E, we have two choices for the first letter and given the letter chosen here, we have $4! = 4 \times 3 \times 2 \times 1 = 24$ ways if arranging the remaining four letters. The total number of arrangements of that start with M or E is then (see the Fundamental Counting Rule below if this isn't clear) $2 \times 4!$

The required probability is then $\frac{2\times4!}{5!} = \frac{2\times4\times3\times2\times1}{5\times4\times3\times2\times1}$

Note [About the fundamental counting rule and the n-factorial formula.].

Students will find it helpful to recall the *fundamental theorem of counting*. One version of this says that if we have a ways of doing one experiment, then b ways of doing a second experiment, then c ways of doing a third experiment, and so on, there are $a \times b \times c \times \ldots$ ways of performing the combined experiment.

In the above problem, with no restrictions on the position of any letter in the word MATHS, there were 5 choices of letters for the first position in the arrangement, then 4 choices for the second position, 3 for the third, 2 four the fourth and 1 for the last. ways of choosing a person for the second position, then 2 choices for the third position and finally 1 choice for the last position. By the above rule, the total number of ways of arranging the letters of the word MATHS is then $5 \times 4 \times 3 \times 2 \times 1 = 120$.

Note that the fundamental counting rule shows that there are $n(n-1)(n-2) \times \ldots \times (3)(2)(1)$ ways of arranging n distinct objects in a row, because any one of n objects can be placed in the first position, then any one of the remaining $n-1$ objects can be put in the second position, and so on to 1 way of placing the last object. This number $n(n-1)(n-2) \times \ldots \times (3)(2)(1)$ is denoted $n!$ [read "*n-factorial*"]

Now if an arrangement must commence with the letter M or E then we have two choices for the first letter of the arrangement and for any such choice we have $4!$ ways of arranging the four letters than can occupy positions 2,3,4 and 5. By the above Fundamental Counting Rule, the total number of ways of arranging the letters of the word MATHS if an arrangement must commence in an M or an E is then $\frac{2\times4!}{5!}2 \times 4!$.

Finally, as given above, the required probability is then $\frac{2\times4!}{5!} = \frac{2}{5}$

6. (A)
Recall that the Remainder Theorem says that when a polynomial is divided by $x-a$, the remainder is $f(a)$.
Thus with $f(x) = (x^2 - x - 1)^{40}$, the given information implies that the remainder is $f(2)$ when $f(x)$ is divided by $x-2$. The answer is therefore
$f(2) = ((2)^2 - (2) - 1)^{40} = 1^{40} = 1$.
Note: The direct method of dividing $(x^2 - x - 1)^{40}$ by $x-2$ is not recommended as it is far too tedious and time-consuming.

7. (B)
One of the Laws of indices says that $x^{ab} = (x^a)^b$. Hence $9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27$. Also, $4^{\frac{5}{2}} = (4^{\frac{1}{2}})^5 = 2^5 = 32$. Since $27 < 30 < 32$, the relationship between $9^{\frac{3}{2}}, 4^{\frac{5}{2}}$ and 30 is then $9^{\frac{3}{2}} < 30 < 4^{\frac{5}{2}}$

8. (B) Since John has no brothers or sisters, his words “my father’s son” must be the same as saying “me”. Then John’s statement “that man’s father is my father’s son” is the same as saying “that man’s father is me, John”. Accordingly we see that the man in the picture is John’s son.

9. (A)
$$\frac{(3i)^{10} + 3^{10}}{3^{10}} = \frac{3^{10}i^{10} + 3^{10}}{3^{10}} = \frac{3^{10}(i^{2})^5 + 3^{10}}{3^{10}} = \frac{3^{10}(-1)^5 + 3^{10}}{3^{10}} = \frac{-3^{10} + 3^{10}}{3^{10}} = 0$$

10. (A)
We simply replace x by $x-2$ in each of the solution $x = 3$ and $x = 5$. This gives $x - 2 = 3$ and $x - 2 = 5$, so the solutions to $f(x-2) = 0$ are $x = 2 + 3 = 5$ and $c = 2 + 5 = 7$

11. (D)
We can proceed by the following enumeration.
Since each of the four children must get at least one of the six apples, one child must get 3 apples and each of the others get one apple each, or else each of two children gets 2 apples each and the remaining two children get 1 apple each.
Now the number of ways of giving one child 3 apples and each of the others 1 apple each is 4 (since the child who gets 3 apples could be any one of four children). Also the number of ways of choosing two children to get 2 apples each and the remaining two children to get 1 apple each is 6 (if the four children are denoted A, B, C and D, we could give two apples to each of A and B, or each of A and C or each of A and D or each of B and C or each of B and D or each of C and D).
The total number of ways of distributing the 6 apples so each child gets at least one apple is then $4 + 6 = 10$.

Note: There is a well-known combinatorial formula for solving the above problem but students would likely not know it. The number of ways of distributing r identical items to n people so that each person gets at least one item is $\binom{r-1}{n-1}$. In the present case, we obtain $\binom{6-1}{4-1} = \binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5\times4\times3\times2\times1}{3\times2\times1\times2\times1} = \frac{5\times4}{2\times1} = 10$.

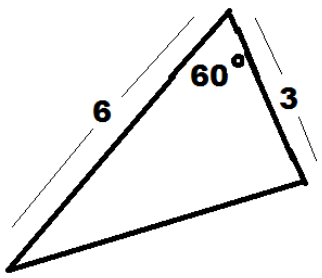
12. (D)
The information in the question tells us that among the 100 people in the school, exactly 12 of them are both females and teach mathematics. Therefore 12% if the entire set of 100 teachers are female teachers of mathematics. (Note that the 60% refers to a percentage of the females only; that is, *among* the females, 60% are teachers, but that is not what the report said!)

13. (C)
From the man's statement, there are three equally likely possibilities for the genders of his two children: (Boy, Boy), (Boy, Girl) and (Girl, Boy). *Therefore the probability that both are boys is 1/3.*

Note: Students who think the answer is 1/2 ignore the fact that the events (Boy, Girl) and (Girl, Boy) are different; if the man had said "I have two children A and B; A is a boy", then the probability that both are boys would be the probability that B is a boy, which would be 1/2.

14. (D)
The proportion of even numbers could be $\frac{2}{5}$ if e.g. Asia wrote down $3, 4, 5, 6, 7,$ for we would then have two even numbers (2,4) among the 5 numbers chosen. Similarly, the proportion of even numbers could be $\frac{1}{2}$ if e.g. Asia's list was $1, 2, 3, 4.$ The proportion could also be $\frac{4}{7}$ if e.g. Asia wrote down $2, 3, 4, 5, 6, 7, 8$ (for here we have 4 even numbers among the 7 chosen). Similarly the proportion could be $\frac{3}{5}$ if Asia's choices were e.g. $2, 3, 4, 5, 6.$ But $\frac{5}{8}$ is not a possible fraction (indeed it is obvious that the number of even digits chosen must equal, be one less than or be one more than the number of odd digits chosen). Putting everything together we see that four of the five fractions are possible for the proportion of even digits that Asia could choose.

15. (C)
The figure shown below illustrates the information given in the question.



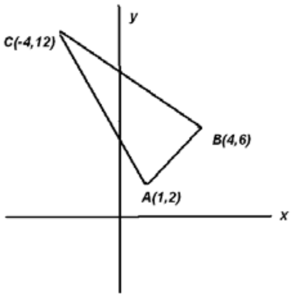
Students who know the cosine rule can immediately get the length c of the third side. We have
$c^2 = 6^2 + 3^2 - 2 \times 6 \times 3 \times \cos(60) = 36 + 9 - 2 \times 18 \times \frac{1}{2} = 27.$ Hence $c = \sqrt{27} = 6.$
Students who could note recall the cosine rule could proceed in other ways, or by enlightened guesswork from the choices given (although we deliberately inserted a 'none of these' option!

16. (B)

***One way*:**

One way of proceeding is to first calculate the slopes of the sides of the triangle. Recall that the slope of a line joining two points $D(x_1,y_1)$ and $E(x_2.y_2)$ is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

We use thus slope formula three times to find the slopes of the lines AB, AC and BC in the given triangle, which is shown (not quite to scale!) below.



The slope of the line AB is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{4-1} = \frac{4}{3}$

The slope of the line AC is $\frac{12-2}{-4-1} = \frac{10}{-5} = -2$

The slope of the line BC is $\frac{12-6}{-4-4} = \frac{6}{-8} = -\frac{3}{4}$

Notice that the product of the slopes of the lines AB and BC is $\frac{4}{3} \times \left(-\frac{3}{4}\right) = -1$, which means that these two lines are perpendicular, and so the angle $\angle ABC$ is $90°$.

The answer to the current problem is then (B), i.e. ABC is a right-angled triangle with the right angle at B.

***Another way*:**

A second way of proceeding is by first calculating the lengths of the three sides of the triangle. Recall that the length of a line segment connecting point (x_1,y_1) to (x_2,y_2) is $\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

The length of AC is $|AC| = \sqrt{(12-2)^2 + (-4-1))^2} = \sqrt{125}$.

Similarly we see that $|AB| = \sqrt{25} = 5$ and $|BC| = \sqrt{100} = 10$.

Now notice that $|AC|^2 = |AB|^2 + |BC|^2$ and we thus see that ABC is a right-angle triangle. Since AC is the longest side, we have that the right-angle is opposite this side, i.e. at B.

17. (D)

As explained in the note at the end of the solution above to Question 5, the expression $n(n-1)(n-2)\times\ldots\times(3)(2)(1)$, denoted $n!$, represents the number of ways of arranging n distinct objects in a row. In particular the number of ways of arranging the digits 1,2,3,4 is $4! = 4\times3\times2\times1 = 24$. Now to add the numbers in all of these 25 arrangements, it would be very tedious to first write down all of the 24 four-digit numbers. Observe instead that if we imagine writing down the 24 numbers, each of the four digits $1, 2, 3$ and 4 will appear exactly six times in each of the four columns. Hence when we add the 24 numbers in each column we will have $6\times1+6\times2+6\times3+6\times4 = 60$. However when carrying out the addition, we must be careful that a 6 will be carried from the units (ones) position to the tens column, a 6 will carry from the tens to the hundreds column and a 6 will also carry from the hundreds' column to the thousands' column. The sum of the 24 four-digit numbers is then 66660

18. (B)

Let x be the boat's speed in still water, in miles per day, and let y be the speed of the current. Travelling downstream, the boat's speed will be $x+y$ while travelling upstream against the current, its speed will be $x-y$. Since speed $=\frac{\text{distance}}{\text{time}}$, we see that the distance covered by the boat in travelling downstream is $3(x+y)$, while upstream it travels a distance of $4(x-y)$.

Since the distance travelled downstream equals the distance travelled upstream we have $3(x+y) = 4(x-y)$. Thus $3x+3y = 4x-4y$ and so $x = 7y$. Accordingly the distance between A and B is $3(x+y) = 3(7y+y) = 24y$.

Since the log travels at the same speed y as the current, the log will take 24 days to cover the distance $24y$.

19. (C)

A fraction $\left(\frac{1}{10}+\frac{1}{20}-\frac{1}{18000}\right)=\frac{2699}{18000}$ of the tank will be filled in one minute. Hence the entire tank will be filled in $\frac{18000}{2699}=6.669$ minutes, and to the nearest minute, this is 7 minutes.

Note: It may be useful to note that this question involves *approximation* of some quantity (the time for the tank to be filled). In approximating the answer to a problem, it often simplifies calculations greatly if we omit or round quantities that will have only a negligible effect on the answer. In the present problem, alert students may have noticed that since the rate at which the tank is emptied by pump C is very small in comparison with the rates at which A and B fill the tank, one could obtain a very good approximation to the answer by completely omitting pump C from our calculation. In fact, notice that $\frac{1}{10}+\frac{1}{20}-\frac{1}{18000}$ is very close to $\frac{1}{10}+\frac{1}{20}$ and then $\frac{1}{\frac{1}{10}+\frac{1}{20}-\frac{1}{18000}}=\frac{18000}{2699}=6.669$ is very close to $\frac{1}{\frac{1}{10}+\frac{1}{20}}=\frac{200}{30}=6.667,$ which is the time it takes for the tank to be filled if only pumps A and B are in operation. Incidentally, students might further note that the quantity $\frac{1}{\frac{1}{10}+\frac{1}{20}}$ is proportional to the so-called harmonic mean of the numbers 10 and 20.

Note: The harmonic mean mean of two numbers a and b is defined as $\frac{2}{\frac{1}{a}+\frac{1}{b}}$ which can be written as $\frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}}$, i.e. the reciprocal of the arithmetic mean of the reciprocals of the two numbers a and b

20. (D)

Exactly one zero will appear at the end of a number if 10 is a factor of the number, two zeros will appear if 100 is a factor, three zeros if 1,000 is a factor, and so on. Now in the number

$$30! = \underline{30}\times 29\times 28\times 27\times 26\times \underline{\underline{25}}\times 24\times 23\times 22\times 21\times \underline{20}\times 18\times 18\times 17\times 16\times \underline{15}\times 14\times$$
$$\times 11\times \underline{10}\times 9\times 8\times 7\times 6\times \underline{5}\times 4\times 3\times 2\times 1$$

we see that a factor of 10 will appear whenever a factor of 5 appears. This is because whenever a 5 appears, there will also be a 2 since there are in fact more appearances of factors 2 than of 5 in this number $30!$. Notice also that $25=5\times 5$ has two factors of 5.

The total number of factors of 10 is then:

5 (determined by the appearances of 30, 20,15 and 10 and 5 [underlined above], each of which will give a factor of 5) PLUS 2 (determined by the appearance of 25 [double-underlined above] which gives 2 factors of 5, since $25=5\times 5$) FOR A TOTAL OF 7.

Note: For purely note that $30! = 265,252,859,812,191,058,636,308,480,000,000$
