SOLUTIONS TO PRISM PROBLEMS

Junior Level 2014

1. (B)
Since 50% of 50 is $\frac{1}{2} \times 50 = 25$ and 50% of 40 is $\frac{1}{2} \times 40 = 20$, the first exceeds the second by $25 - 20 = 5$.

2. (A) One way of comparing the magnitudes of the numbers $\frac{3}{4}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}$ and 0.7 is to express them all as decimals. Since
$\frac{3}{4} = 0.75, \frac{3}{5} = 0.6. \frac{2}{3} \approx 0.667, \frac{5}{7} \approx 0.714,$ the largest of the five numbers is $\frac{3}{4}$.
Alternatively but far more tediously in the present problem, we could express all numbers as fractions with a common denominator. In the order given, the numbers are $\frac{315}{420}, \frac{252}{420}, \frac{280}{420}, \frac{300}{420}$ and $\frac{294}{420}$. and we again see that the largest is the first-appearing fraction which is $\frac{315}{420} = \frac{3}{4}$

3. (C)
The numbers between 1 and 40 that are multiples of 7 are $7, 14, 21, 28$ and 35 . The numbers between 1 and 40 that end in a 7 are $7, 17, 27$ and 37. Note that the number 7 appears on both lists above. The numbers between 1 and 40 that are either a multiple of 7 or end in a 7 are then $7, 14, 21, 28, 35, 17, 27,$ and 37. There are thus 8 such numbers.

4. (C)
Halfway between 8:04 and 18:38 is 13.21.

5. (D)
Let L, E and C be the price or value of a loaf of bread, an egg and a chocolate bar, respectively. We are given that $L = 4E$ and $C = 2E$. We use these two equations to get a relationship between L and C. If we multiply the equation $C = 2E$ across by 2, we get $2C = 4E$ (that is 2 chocolate bars cost the same as 4 eggs). But the first equation says that a loaf costs the same as four eggs. Hence we have that 2 chocolate bars are equal to one loaf, that is $2C = L$. Multiplying across by 3 we see that $6C = 3L$, i.e. 6 chocolate bars are worth 3 loaves.

6. (C)
If Peter ate only, say 6 sweets, there would be $18 - 6 = 12$ sweets to share among the other two boys so one or both of them could receive 6, which would contradict the fact that Peter ate more than each of the other two boys. This contradictions means that Peter did not eat 6 sweets. A similar argument shows that Peter cannot have ate less than 6 sweets. It is however possible for Peter to eat 7 sweets, as then there would be 11 left for the other two boys and the most each of these could receive (while ensuring that Peter gets more than each of them) is 6 and 5. Thus 7 is the minimum number of sweets that Peter could have received.

7. (D)
The first position in the row could be occupied by any one of 4 students, the second position then by any of three, the third position by one of the remaining two, and finally we then have just one choice of person for the fourth position. The total number of ways of arranging the four people is then $4 \times 3 \times 2 \times 1 = 24$.
Students who are unclear on why we multiple the four numbers $4, 3, 2,$ and 1 are requested to read the following note which will prove very useful in their Leaving Certificate course!

Note [About the fundamental counting rule and the n-factorial formula.].

Students will find it helpful to recall the *fundamental theorem of counting*. One version of this says that if we have a ways of doing one experiment, then b ways of doing a second experiment, then c ways of doing a third experiment, and so on, there are $a \times b \times c \times \ldots$ ways of performing the combined experiment.
In the above problem, there were 4 ways of occupying the first position, then 3 ways of choosing a person for the second position, then 2 choices for the third position and finally 1 choice for the last position. By the above rule, the total number of ways of seating the four people is then $4 \times 3 \times 2 \times 1 = 24$.
Note that the fundamental counting rule shows that there are $n(n-1)(n-2) \times \ldots \times (3)(2)(1)$ ways of arranging n distinct objects in a row, because any one of n objects can be placed in the first position, then any one of the remaining $n-1$ objects can be put in the second position, and so on to 1 way of placing the last object. This number $n(n-1)(n-2) \times \ldots \times (3)(2)(1)$ is denoted $n!$ [read "n-*factorial*"]

8. (A)
For each team to play one game, 5 games will be played (because each team will play some other team). Similarly, for each team to play their second game, another 5 games will be played. Finally when each team has played its third game, another 5 games will have been played. The total number of games played is then $5 + 5 + 5 = 15$.

9. (C)
One way of proceeding is to write each fraction as a decimal. We find that $\frac{2}{5} = 0.4$, $\frac{4}{10} = 0.4$, $\frac{11}{25} = 0.44$, $\frac{23}{50} = 0.46$ and $\frac{35}{75} = 0.4667$. We thus see that the largest is $\frac{35}{75}$.
Alternatively (and perhaps easier), students might compare the fractions by representing them all with the same common denominator. Clearly the denominators of the five given numbers all divide into 150 (which in fact is the least common multiple of the denominators). We can write
$\frac{2}{5} = \frac{2 \times 30}{5 \times 30} = \frac{60}{150}$, $\frac{4}{10} = \frac{60}{150}$, $\frac{11}{25} = \frac{66}{150}$, $\frac{23}{50} = \frac{69}{150}$ and $\frac{35}{75} = \frac{70}{150}$.
The largest numerator is now 70 so $\frac{35}{75} = \frac{70}{150}$ is the greatest of the five fractions.

10. (D)

One way of proceeding is by long division, from which we see that $\frac{1}{62500}$ is exactly equal to 0.000016, so there are six digits after the decimal point. Slightly shorter would be to factorize 62500 as $62500 = 625 \times 100 = 25 \times 25 \times 10 \times 10$. Then since $\frac{1}{25} = \frac{4}{100} = 0.04$, we have that $\frac{1}{62500} = \frac{1}{25\times25\times10\times10} = 0.04 \times 0.04 \times 0.1 \times 0.1 = 0.000016$, so again there are 6 digits that follow the decimal point.

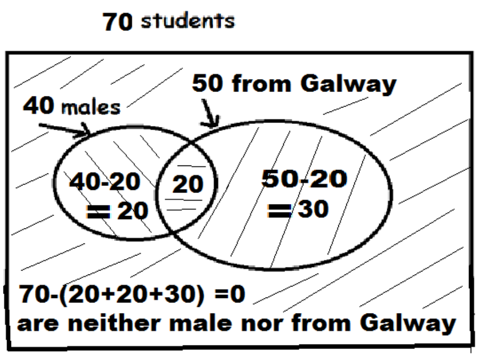
11. (B)

We solve the simultaneous equations $4x + 3y = 4$ and $4x + 2y = 4$. The coefficient of x is the same in both equations so we can immediately eliminate x by subtracting the second equation from he first. This immediately gives $y = 0$, uniquely. The corresponding value of x is got by inserting $y = 0$ into one of the equations, say the first. We get $4x + 0(y) = 4$ so 4x=4 and $x = 1$. The equations thus have exactly one solution $x = 1, y = 0$.

12. (D)

Let x be the price of a sweater and let y be the price of a shirt. We are given that $12x + 8y = 8x + 12y + 20$. Hence $4x = 4y + 20$, so $x = y + 5$. That is, a sweater costs €5 more than a shirt.

13. (A)



In the Venn diagram above, the entire rectangle represents all 70 students. The smaller ‘egg’ represents the set of 40 male students and the larger ‘egg’ is the set of 50 Galway students.
The intersection of these two ‘eggs’ is the set of male Galway students and we are given that the number of students in this set is 20. Hence the number of students who are male and not from Galway is $40 - 20 = 20$ and the number of students who are from Galway but not male is $50 - 20 - 30$.
Finally then the number of students who are neither male nor from Galway is the set of students within the rectangle who do not belong to either ‘egg’, and this number is $70 - (20 + 20 + 30) = 70 - 70 = 0$.

14. (D)

Let s be the distance from the bottom to the top of the hill so s is also the distance from the top to the bottom. Also let t_1 be the time taken to get from the bottom to the top of the hill, and let t_2 be the time taken to go from the top to the bottom. Since speed is distance divided by time, we have $30 = s/t_1$ and $45 = s/t_2$. Hence $s = 30t_1 = 45t_2$. Accordingly, $t_1 = \frac{45}{30}t_2 = \frac{3}{2}t_2$.

Now the total distance is the distance to top + distance to the bottom $= 30t_1 + 45t_2$. The average speed is the total distance divided by the total time, i.e. twice the distance to the top divided by the total time, i.e. $\frac{2\times 45t_2}{t_1+t_2}$. Substituting $t_2 = t_1/3$, we find that the average speed is $\frac{90t_2}{\frac{3}{2}t_2+t_2} = \frac{90}{\frac{3}{2}+1} = \frac{90}{\frac{5}{2}} = 36$.

Note: A study of the argument above shows that in computing average speed we do not use the arithmetic average of the speeds up and down the mountains, but rather the harmonic mean, $\dfrac{2}{\frac{1}{\text{speed up}} + \frac{1}{\text{speed down}}}$ which can be written as

$\dfrac{1}{\dfrac{\frac{1}{\text{speed up}} + \frac{1}{\text{speed down}}}{2}}$, i.e. the reciprocal of the arithmetic mean of the reciprocals of the two speeds!

15.(A)

The total amount, T, placed on the board is $T = 1 + 2 + 3 + \ldots + 63 + 64$.

It would be very tedious to add these numbers term-wise. However there is a well-known trick! We write the sum down with the numbers reversed, that is, we write

$T = 1 + 2 + 3 + \ldots + 63 + 64$ as

$T = 64 + 63 + \ldots + 3 + 2 + 1$

We now add these last two equations to get

$T + T = (1 + 64) + (2 + 63) + (3 + 62) + \ldots + (62 + 3) + (63 + 2) + (64 + 1)$.

That is, $2T = \underbrace{65 + 65 + 65 + \ldots + 65 + 65 + 65}_{\text{64 terms}}$

Thus $2T = 64 \times 65$ and so $T = \frac{64\times 65}{2} = 32 \times 65 = 2080$.

Note to students: The above method is only one way of adding any number of consecutive integers and was developed by a young precocious child named Gauss. If you google the word “Gauss” you will get a great many hits about this famous century mathematician who was born in 1777 and is sometimes referred to as the “Prince of Mathematicians” and the “greatest mathematician since antiquity”! The general formula for the sum of the first n integers $1, 2, 3, \ldots, n$ is $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ and this formula which is well worth memorizing is a special case of the formula for the sum of an arithmetic progression.

16. (C)
When Andrew crosses the finish line, Brian will have covered 800 metres. Thus the ratio of Brian's speed to Andrew's speed is $\frac{800}{1000} = 0.8.$ That is, Brian travelled at 90% of Andrew's speed. Similarly, Charlie travelled at 80% of Brian's speed, so Charlie travelled at $(80\%$ of $80\%) = 64\%$ of Andrew's speed. Accordingly, when Andrew has covered the 1000 metres, Charlie will have covered $0.64 \times 1000 = 640$ metres. Hence Andrew beat Charlie by $1000 - 640 = 360$ metres.

17. (B)
We are given that $ab = 6, ac = 18$ and $bc = 27.$ We require $abc.$ Many students will proceed by rial and error. A rigorous approach would note that since $ab = 6,$ we have $abc = 6c,$ so we could easily solve the problem if we can evaluate $c.$
From $ab = 6, ac = 18,$ we have $a = \frac{6}{b}$ and $a = \frac{18}{c}.$ Thus $\frac{6}{b} = \frac{18}{c}$ so $c = 3b.$ But $bc = 27$ so $b = \frac{27}{c}.$ Substituting $b = \frac{27}{c}$ into $c = 3b$ gives $c = 3 \times \frac{27}{c},$ or $c^2 = 81$ and (since $c > 0$) we have $c = 9.$ Thus $abc = (ab)c = 6 \times 9 = 54$

18. (D)
First note that a square with sides of length 6 has perimeter 5+5+5+5=20.
Next by Pythagoras' Theorem if the length of the side of the square is x then $x^2 + x^2 = (\sqrt{50})^2.$
Thus $2x^2 = 50,$ or $x^2 = 25$ or $x = 5.$ Thus the square in B) has again perimeter $4 \times 5 = 20.$
For C) note that a square with area 25 cm^2 has side length $\sqrt{25} = 5,$ so again the perimeter of this square is 25.
The circle in D) has radius $r = 3.5$ so its perimeter is $2\pi r = 2\pi(3.5) = 7.5\pi$ and since π is about 3.14, this perimeter is somewhat greater than $7.5(3) = 22.5$
Finally, if r denotes the radius of the circle in E), we are given that the area is $\pi r^2 = 9\pi.$
Hence $r^2 = 9.$ so $r = 3.$ The perimeter of this circle is $2\pi(3) = 6\pi$ which is certainly less than all of the answers in A), B), C) and D).
Putting everything together, we see that the figure with the largest perimeter is the circle in D).

19. (A)

Let x be the length of a side of the square. By Pythagoras' Theorem, the square of the diagonal is $x^2 + x^2 = 2x^2$. We are given that $2x^2 = 4 \times (4x)$. Thus $x = 8$. Accordingly the area of the square is $8 \times 8 = 64$ cm^2

20. (B)

Let the marks obtained by the n students be $x_1, x_2, \ldots, x_n$ and let $\bar{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$ be their average mark. Note that this average is obtained by adding up the marks of all of the students and dividing the result by the number of students.

When m of the students have their marks raised by 3 and the remaining $n - m$ have their marks raised by 1, the new average will be

$$\bar{x}_{NEW} = \frac{x_1 + x_2 + \ldots + x_n + \underbrace{3 + 3 + \ldots + 3}_{m \text{ times}} + \underbrace{1 + 1 + \ldots + 1}_{(n-m) \text{ times}}}{n}$$

Notice that this can be written as $\bar{x}_{NEW} = \dfrac{x_1 + x_2 + \ldots + x_n + 3m + n - m}{n} = \dfrac{x_1 + x_2 + \ldots + x_n + 2m + n}{n} = \dfrac{x_1 + x_2 + \ldots + x_n}{n} + \dfrac{2m + n}{n} = \bar{x} + \dfrac{2m + n}{n}$.

The difference $\bar{x}_{NEW} - \bar{x}$ is then $\dfrac{2m + n}{n}$ and we are given that this equals 1.5

Writing $\dfrac{2m + n}{n} = 1.5$ as $2\frac{m}{n} + 1 = 1.5$, we have $2\frac{m}{n} = 0.5$ so $\frac{m}{n} = \frac{1}{4}$