Question 1 (2017)

(a)	$Se^{\cdot 1(0)} \times 10^6 = 1100000$ $S = 1{\cdot}1$	**Scale 10B (0, 4, 10)** *Partial Credit* • equation in S with substitution
(b)	$p(5) = 1{\cdot}1e^{0\cdot 1(5)} \times 10^6$ $= 1{\cdot}813593 \times 10^6$ $= 1813593$	**Scale 10B (0, 4, 10)** *Partial Credit* • substitution into formula for $p(5)$
(c)	$p(6) = 1{\cdot}1e^{0\cdot 6} \times 10^6$ $p(5) = 1{\cdot}1e^{0\cdot 5} \times 10^6$ $p(6) - p(5) = (1{\cdot}1e^{0\cdot 6} - 1{\cdot}1e^{0\cdot 5}) \times 10^6$ $= 0{\cdot}1907372 \times 10^6$ $= 190737$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • substitution into formula for $p(6)$ • use of $p(5)$ from previous part • $p(6) - p(5)$ written or implied *High partial Credit* • Formulates $p(6) - p(5)$ with some substitution

(d)	$q(t) = 3{\cdot}9e^{kt} \times 10^6$ $3709795 = 3{\cdot}9e^{k} \times 10^6$ $\frac{3{\cdot}709795}{3{\cdot}9} = e^k$ $\log_e \frac{3{\cdot}709795}{3{\cdot}9} = k$ $k = -0{\cdot}0499 = -0{\cdot}05$	**Scale 15C (0, 5, 10, 15)** *Low Partial Credit* • Either substitution into formula for k • Verifies k value only. *High Partial Credit* • relevant equation in k
(e)	$p(t) = q(t)$ $1{\cdot}1e^{0{\cdot}1t} \times 10^6 = 3{\cdot}9e^{-0{\cdot}05t} \times 10^6$ $1{\cdot}1e^{0{\cdot}1t} = 3{\cdot}9e^{-0{\cdot}05t}$ $\frac{e^{0{\cdot}1t}}{e^{-0{\cdot}05t}} = \frac{3{\cdot}9}{1{\cdot}1}$ $e^{0{\cdot}15t} = \frac{39}{11}$ $\ln\frac{39}{11} = 0{\cdot}15t$ $t = 8{\cdot}44$ years In 2018 both populations equal	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit* • $p(t) = q(t)$ written or implied *High Partial Credit* • relevant equation in t
(f)	$\frac{1}{15}\int_0^{15} 3{\cdot}9e^{-0{\cdot}05t} \times 10^6 dt$ $\frac{1}{15}\left[\frac{3{\cdot}9}{-0{\cdot}05}e^{-0{\cdot}05(15)} - \frac{3{\cdot}9}{-0{\cdot}05}e^{-0{\cdot}05(0)}\right]$ $\times 10^6$ $2{\cdot}743694 \times 10^6$ 2743694	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • integral formulated (with limits) *High Partial Credit:* • integration with full substitution
(g)	$q(t) = 3.9e^{-0.05t} \times 10^6$ $q'(t) = -0{\cdot}05(3{\cdot}9e^{-0{\cdot}05t} \times 10^6)$ $q'(8) = -0{\cdot}05\left(3{\cdot}9e^{-0{\cdot}05(8)} \times 10^6\right)$ $= -130712$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit* • $q'(t)$ *High Partial Credit* • $q'(t)$ fully substituted

Q3	Model Solution – 25 Marks	Marking Notes
(a)	$\dfrac{2\cos\frac{7A+A}{2}\cos\frac{7A-A}{2}}{2\cos\frac{7A+A}{2}\sin\frac{7A-A}{2}}$ $\dfrac{2\cos 4A\cos 3A}{2\cos 4A\sin 3A}$ $=\dfrac{\cos 3A}{\sin 3A}$ $=\cot 3A$	Scale 15C (0, 5, 10, 15) *Low Partial Credit* • sum to product formula with some substitution *High Partial Credit* • sum to product formula fully substituted
(b)	Method 1: $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$ $=\frac{1}{2}\left(1+\frac{1}{9}\right)=\frac{5}{9}$ $\cos\theta=\pm\frac{\sqrt{5}}{3}$ **or** Method 2: $\cos 2\theta=1-2\sin\theta=\frac{1}{9}$ $9-18\sin^2\theta=1$ $\sin^2\theta=\frac{4}{9}\Rightarrow\sin\theta=\pm\frac{2}{3}\Rightarrow\cos\theta=\pm\frac{\sqrt{5}}{3}$ **or** Method 3: $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\dfrac{1}{9}$ $9-9\tan^2\theta=1+\tan^2\theta$ $\tan^2\theta=\frac{4}{5}$ $\Rightarrow\tan\theta=\pm\frac{2}{\sqrt{5}}\Rightarrow\cos\theta=\pm\frac{\sqrt{5}}{3}$	Scale 10D (0, 3, 5, 8, 10) *Low Partial Credit* • Use of a relevant formula in $\cos 2\theta$ • $\cos^{-1}\left(\frac{1}{9}\right)=83{\cdot}62°$ • $\theta=41{\cdot}8°$ *Mid Partial Credit* • correct substitution (method 1) • expression in $\sin^2\theta$ (method 2) • expression in $\tan^2\theta$ (method 3) • expression in $\cos^2\theta$ (method 4) • $\theta=41{\cdot}8°$ and $\theta=132{\cdot}2°$ or $\theta=221{\cdot}8°$ *High Partial Credit* • one value only (e.g. $+\frac{\sqrt{5}}{3}$) • values found for $\cos 41{\cdot}8°$ and $\cos 138{\cdot}2°$ or $\cos 221{\cdot}8°$

	or Method 4: $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ $1-\cos^2\theta = \frac{1}{2}(1-\cos 2\theta)$ $2-2\cos^2\theta = 1-\cos 2\theta$ $\cos^2\theta = \frac{1+\cos 2\theta}{2} = \frac{1+\frac{1}{9}}{2}$ $\cos^2\theta = \frac{5}{9}$ $\cos\theta = \pm\frac{\sqrt{5}}{3}$	

Question 3 (2016)

(b) **(i)**	$p = \log_a 2$, $q = \log_a 3$ $\log_a \frac{8}{3} = \log_a 8 - \log_a 3$ $= \log_a (2)^3 - \log_a 3$ $= 3\log_a 2 - \log_a 3$ $= 3p - q$	Scale 5C (0, 2, 4, 5) *low Partial Credit* • $\log_a 8 - \log_a 3$ *High Partial Credit* • $\log_a 8 = 3\log_a 2$ (and/or $= 3p$)
(ii)	$\log_a \frac{9a^2}{16} = \log_a (3a)^2 - \log_a (2)^4$ $= 2\log_a 3 + 2\log_a a - 4\log_a 2$ $= 2q + 2(1) - 4p$ $= 2q + 2 - 4p$	Scale 5D (0, 2, 3, 4, 5) *Low Partial Credit* • $\log_a 9a^2 - \log_a 16$ *Mid Partial Credit* • $2\log_a 3$ • $2\log_a a$ • $4\log_a 2$ • $4p$ or $2q$ or 2 *High Partial Credit* • $2(\log_a 3 + \log_a a) - 4\log_a 2$ or equivalent

First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$. Now suppose that the statement is true for some $n \geq 1$. So

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Now, add $n+1$ to both sides and we get

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first $n+1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n.

(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

By part (i), we know that

$$1 + 2 + \cdots + 100 = \frac{100(101)}{2} = 5050.$$

We also know that

$$1 + 2 + \cdots + 50 = \frac{50(51)}{2} = 1275.$$

Subtracting the second equation from the first yields

$$51 + 52 + \cdots + 100 = 5050 - 1275 = 3775.$$

(b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c(cx)$ in terms of p.

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2}\log_c x = \frac{1}{2}p$$

using the power law for logarithms.
Also,

$$\log_c(cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.
But $\log_c c = 1$ since $c^1 = c$. Therefore

$$\log_c \sqrt{x} + \log_c(cx) = \frac{1}{2}p + 1 + p = \frac{3p}{2} + 1.$$

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Question 5 (2013)

$$Q = e^{-\frac{0{\cdot}693t}{5730}} = e^{-\frac{0{\cdot}693\times 2000}{5730}} = 0\cdot 7851$$

$$Q = e^{-\frac{0{\cdot}693t}{5730}} = 0\cdot 3402$$

$$\Rightarrow -\frac{0\cdot 693t}{5730} = \ln 0\cdot 3402$$

$$\Rightarrow t = -\frac{5730\times \ln 0\cdot 3402}{0\cdot 693} \approx 8915 \approx 8900 \text{ years}$$

Question 6 (2012)