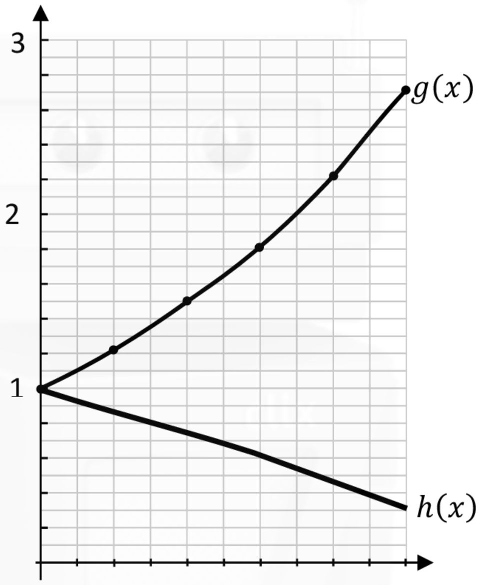
MarkingScheme

IntegrationH

Question 1 (2017)

(a)



$$g(x) = e^x \quad h(x) = e^{-x} = \frac{1}{e^x}$$

$g(x) = e^x$:

x	0	0·2	0·4	0·6	0·8	1·0
y	1	1·22	1·49	1·82	2·23	2·72

$h(x) = \frac{1}{e^x}$:

x	0	0·2	0·4	0·6	0·8	1·0
y	1	0·82	0·67	0·55	0·45	0·37

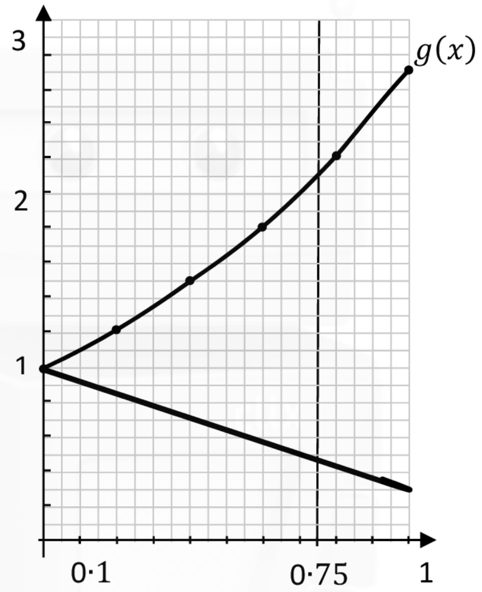
Scale 15C (0, 5, 10, 15)

Low Partial Credit:

- one point correct

High Partial Credit

- Graph not in required domain

(b)	 $$A = \int_0^{0\cdot75} e^x dx - \int_0^{0\cdot75} e^{-x} dx$$ $$= \int_0^{0\cdot75} (e^x - e^{-x}) dx$$ $$= e^x + e^{-x}$$ $$e^{0\cdot75} + e^{-0\cdot75} - [e^0 + e^0]$$ $$= 0\cdot5894$$	**Scale 10C (0, 5, 8, 10)** *Low Partial Credit:* • Formulates integration for area under one curve with limits *High Partial Credit* • integrates twice for correct area under both curves **Note**: Trapezoidal rule must have at least 5 divisions <u>AND</u> fully correct work gets Low Partial Credit

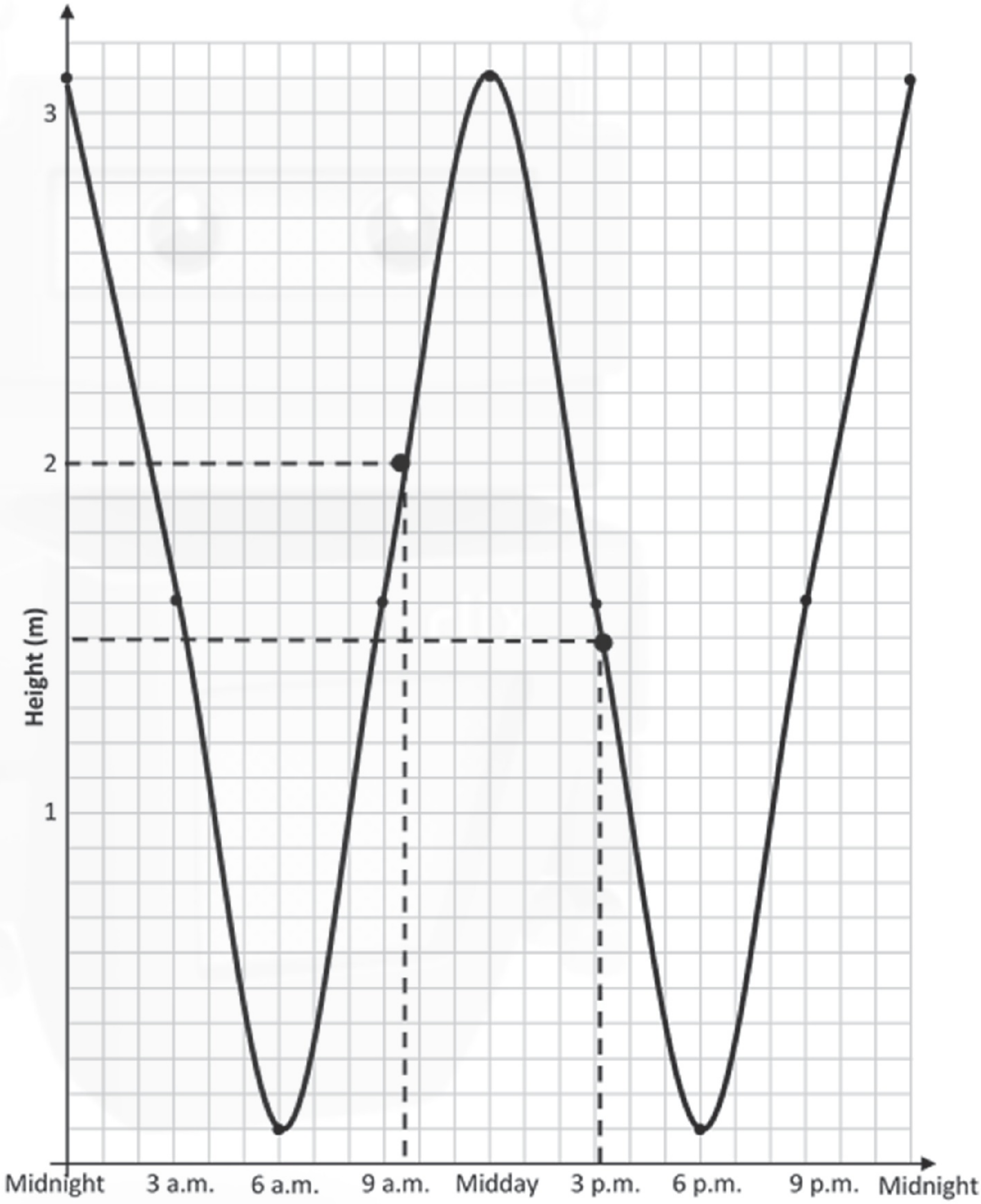
Q8	Model Solution – 45 Marks	Marking Notes
(a)	Period = $\frac{2\pi}{\frac{\pi}{6}}$ = 12 hours Range = $[1{\cdot}6 - 1{\cdot}5\,, 1{\cdot}6 + 1{\cdot}5\,] = [0{\cdot}1 \text{ m}, 3{\cdot}1 \text{ m}]$	Scale 5C (0, 2,4, 5) *Low Partial Credit* • some use of 2π or $\frac{\pi}{6}$ • range of cos function *High partial credit* • period or range correct **Note:** *Accept correct period and/or range without work*
(b)	Max = 1·6 + 1·5(1) =3·1 m. **or** 3·1 m from range	Scale 5B (0,2, 5) *Partial Credit* • max occurs when cos A = 1 or t = 0 • effort at $h'(t)$ **Note:** *Accept correct answer without work*
(c)	$h'(t) = 1{\cdot}5(-\sin\frac{\pi t}{6})\frac{\pi}{6}$ $h'(2) = 1{\cdot}5(-\sin\frac{2\pi}{6})\frac{\pi}{6}$ = – 0·68017 =–0·68 m/h Tide is going out at a rate of 0·68 m per hour at 2 am	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • effort at differentiation *High Partial Credit* • correct numerical answer but not in context

(d)(i)

$$h(t) = 1 \cdot 6 + 1 \cdot 5 \cos\left(\frac{\pi}{6}t\right)$$

Time	12 am	3 am	6 am	9 am	12 pm	3 pm	6 pm	9 pm	12 am
t	0	3	6	9	12	15	18	21	24
Height	3·1	1·6	·1	1·6	3·1	1·6	·1	1·6	3·1

(d) **(i)**		Scale 10C (0, 3, 7, 10) *Low Partial Credit* • one correct height *High Partial Credit* • five correct heights

(d) **(ii)**	

(d) **(ii)**	Graph	Scale 10C (0, 3,7,10) *Low Partial Credit* • one correct plot *High Partial Credit* • at least 7 correct plots • plots correct but graph not sketched or sketched incorrectly

(e)	Low tide = 0·1 m High tide = 3·1 m Difference = 3·1 – 0·1 =3 m	Scale 5B (0, 2, 5) *Partial Credit* • height of Low tide or High tide correctly identified **Notes:** (i) *candidates may show work for this section on graph* (ii) *accept values from candidate's graph* (iii) *accept correct answer from graph without work*
(f)	Enter port at 9:30 approx Leave port before 15:15 approx Time = 15:15 – 9:30 = 5 hr 45 min approx.	Scale 5B (0, 2, 5) *Partial Credit* • time of entry to port or leave port correctly identified • value(s) for *h* = 2 and/or *h* = 1·5 on sketch • time estimated using relevant values other than those required for the maximum time. **Notes:** (i) *candidates may show relevant work for this section on graph* (ii) *accept values from candidate's graph*

(a) **(i)** Complete Table 1 below.

Table 1							
x	3	4	5	6	7	8	9
$f(x)$	0	5	8	9	8	5	0

(ii) Use Table 1 and the trapezoidal rule to find the approximate area of the region bounded by the graph of f and the x-axis.

$$A = \frac{h}{2}\left[y_1 + y_n + 2\left(y_2 + y_3 + \cdots + y_{n-1}\right)\right]$$
$$= \frac{1}{2}\left[0 + 0 + 2\left(5 + 8 + 9 + 8 + 5\right)\right]$$
$$= 35 \text{ square units}$$

(b) **(i)** Find $\int_3^9 f(x)dx$.

$$\int_3^9 \left(-x^2 + 12x - 27\right) dx$$
$$= \left[\frac{-x^3}{3} + \frac{12x^2}{2} - 27x\right]_3^9$$
$$= (-243 + 486 - 243) - (-9 + 54 - 81)$$
$$= 36$$

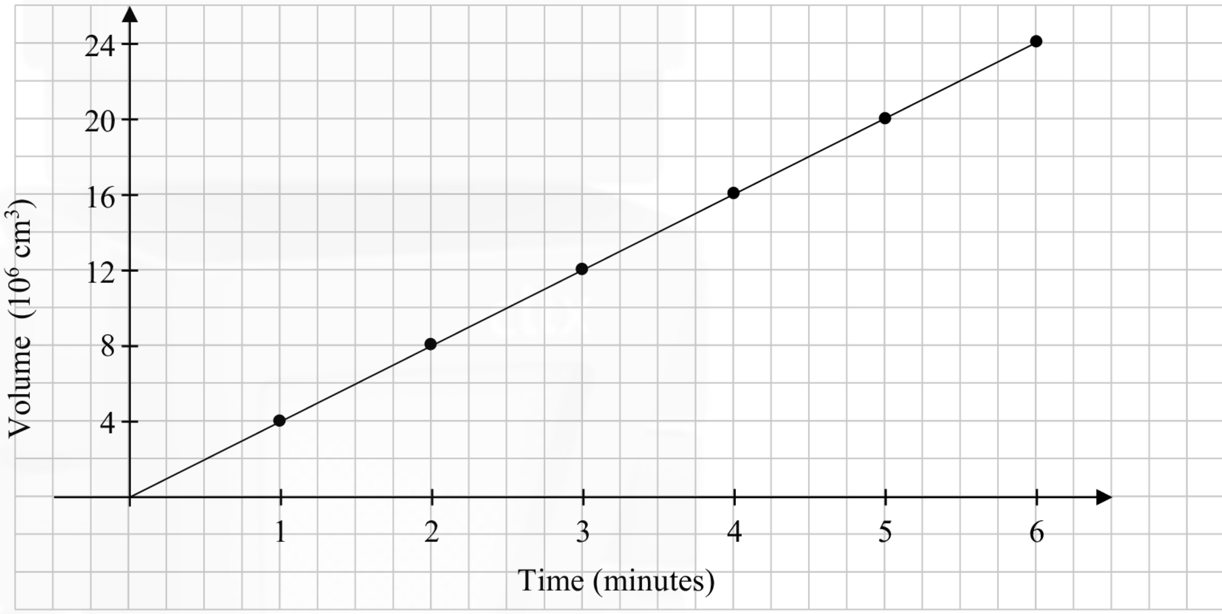
(ii) Use your answers above to find the percentage error in your approximation of the area, correct to one decimal place.

$$\frac{1}{36} \times 100 = 2 \cdot 8\%$$

(a) **(i)** Complete the table below to show the total volume of oil on the water after each of the first 6 minutes of the oil-spill.

Time (minutes)	1	2	3	4	5	6
Volume (10^6 cm^3)	4	8	12	16	20	24

(ii) Draw a graph to show the total volume of oil on the water over the first 6 minutes.



(iii) Write an equation for $V(t)$, the volume of oil on the water, in cm^3, after t minutes.

Line, slope 4×10^6, passing through (0, 0).

$V(t) = \left(4\times 10^6\right) t$

(b) The spilled oil forms a circular oil slick **1 millimetre** thick.

(i) Write an equation for the volume of oil in the slick, in cm^3, when the radius is r cm.

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi r^2 (0\cdot 1) \\ &= 0\cdot 1 \pi r^2 \text{ cm}^3 \end{aligned}$$

(ii) Find the rate, in cm per minute, at which the radius of the oil slick is increasing when the radius is 50 m.

$$\frac{dV}{dt} = 4\times10^6 \text{ cm}^3 \text{ per minute}$$

$$V = \pi r^2 h \text{ where } h = 0\cdot1\text{ cm}$$

$$\frac{dV}{dr} = 2\pi rh$$

$$\frac{dV}{dr} = 0\cdot2\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dV}\frac{dV}{dt} = \frac{1}{0\cdot2\pi r}\times4\times10^6$$

$$= \frac{4\times10^6}{0\cdot2\pi(5000)} = 1273\cdot3 \text{ cm per minute}$$

(c) Show that the area of water covered by the oil slick is increasing at a constant rate of 4×10^7 cm^2 per minute.

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r\frac{4\times10^6}{0\cdot2\pi r} = 4\times10^7 \text{ cm}^2 \text{ per minute}$$

or

$$(0\cdot1)\pi r^2 = (4\times10^6)t$$

$$\Rightarrow A = \pi r^2 = (4\times10^7)t$$

$$\frac{dA}{dt} = 4\times10^7$$

(d) The nearest land is 1 km from the point at which the oil-spill began. Find how long it will take for the oil slick to reach land. Give your answer correct to the nearest hour.

$$A = \pi r^2 = \pi(10^5)^2 = \pi10^{10} \text{ cm}^2$$

$$t = \frac{\pi10^{10}}{4\times10^7} = \frac{\pi10^3}{4} = 785\cdot398 \text{ minutes}$$

$$= 13\cdot09 = 13 \text{ hours}$$

(a) Find the length of the day in Galway on June 5th (76 days after March 21st). Give your answer in hours and minutes, correct to the nearest minute.

$$f(t) = 12 \cdot 25 + 4 \cdot 75 \sin\left(\frac{2\pi}{365}t\right)$$

$$f(76) = 12 \cdot 25 + 4 \cdot 75 \sin\left(\frac{2\pi}{365} \times 76\right)$$

$=12 \cdot 25 + 4 \cdot 587 = 16 \cdot 837 = 16$ hours 50 minutes

(b) Find a date on which the length of the day in Galway is approximately 15 hours.

$$f(t) = 12 \cdot 25 + 4 \cdot 75 \sin\left(\frac{2\pi}{365}t\right) = 15$$

$$\Rightarrow \sin\left(\frac{2\pi}{365}t\right) = 0 \cdot 578947$$

$$\Rightarrow \frac{2\pi}{365}t = 0 \cdot 6174371$$

$$\Rightarrow t = 35 \cdot 87$$

36 days after March 21 is April 26.

(c) Find $f'(t)$, the derivative of $f(t)$.

$$f(t) = 12 \cdot 25 + 4 \cdot 75 \sin\left(\frac{2\pi}{365}t\right)$$

$$f'(t) = 0 + 4 \cdot 75 \times \frac{2\pi}{365} \cos\left(\frac{2\pi}{365}t\right)$$

$$= \frac{9 \cdot 5\pi}{365} \cos\left(\frac{2\pi}{365}\right)t$$

(d) Hence, or otherwise, find the length of the longest day in Galway.

$f(t)$ is a maximum when $\sin\left(\frac{2\pi}{365}t\right)$ is a maximum of 1.

$t = 12 \cdot 25 + 4 \cdot 75 = 17$ hours

or

$$f'(t) = 0 \Rightarrow \frac{9 \cdot 5\pi}{365}\cos\left(\frac{2\pi}{365}t\right) = 0$$

$$\Rightarrow \cos\left(\frac{2\pi}{365}t\right) = 0$$

$$\Rightarrow \frac{2\pi}{365}t = \frac{\pi}{2}$$

$$\Rightarrow t = \frac{365}{4} = 91 \cdot 25$$

$$f(91 \cdot 25) = 12 \cdot 25 + 4 \cdot 75\sin\left(\frac{2\pi}{365} \times 91 \cdot 25\right)$$

$$= 12 \cdot 25 + 4 \cdot 75\sin\frac{\pi}{2}$$

=17 hours

(e) Use integration to find the average length of the day in Galway over the six months from March 21st to September 21st (184 days). Give your answer in hours and minutes, correct to the nearest minute.

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{184}\int_0^{184}\left(12 \cdot 25 + 4 \cdot 75\sin\left(\frac{2\pi}{365}t\right)\right)dt$$

$$= \frac{1}{184}\left[12 \cdot 25t - 4 \cdot 75 \times \frac{365}{2\pi}\cos\left(\frac{2\pi}{365}t\right)\right]_0^{184}$$

$$= \frac{1}{184}\left[(2254 + 275 \cdot 843) - (0 - 275 \cdot 934)\right]$$

$$= \frac{1}{184}\left[2805 \cdot 777\right]$$

$$= 15 \cdot 24879$$

$$= 15 \text{ hours } 15 \text{ minutes}$$

Question 6 (2014)

(a) Find $\int 5\cos 3x\, dx$.

$$\int 5\cos 3x\, dx = \tfrac{5}{3}\sin 3x + c$$

(b) The slope of the tangent to a curve $y = f(x)$ at each point (x, y) is $2x - 2$.
The curve cuts the x-axis at $(-2, 0)$.

(i) Find the equation of $f(x)$.

$$\int dy = \int (2x-2)dx$$

$$\Rightarrow y = x^2 - 2x + c$$

At $x = -2,\ y = 0 \quad \Rightarrow \quad 0 = 4 + 4 + c \quad \Rightarrow \quad c = -8$

Hence, $y = x^2 - 2x - 8$

(ii) Find the average value of f over the interval $0 \le x \le 3,\ x \in \mathbb{R}$.

Average value: $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$

$$\frac{1}{3-0}\int_0^3 \left(x^2 - 2x - 8\right)dx = \frac{1}{3}\left[\frac{x^3}{3} - x^2 - 8x\right]_0^3$$

$$= \frac{1}{3}\left[\frac{27}{3} - 9 - 24\right] = -8$$

Question 7 (2014)

(ii) Explain what is meant by the indefinite integral of a function f.

The indefinite integral of f is the general form of a function whose derivative is f.

Alternative answer: The indefinite integral of f is $F(x) + C$ where $F' = f$ and C is constant (the constant of integration).

(iii) Write down the indefinite integral of g, the function in part **(i)**.

Answer: $\displaystyle\int g(x)dx = \frac{1}{4}x^4 - x^3 + 3x + C.$

(b) **(i)** Let $h(x) = x\ln x$, for $x \in \mathbb{R}$, $x > 0$.
Find $h'(x)$.

Using the product rule we see that

$$h'(x) = (x)' \ln x + x(\ln x)'.$$

But $(x)' = 1$ and $(\ln x)' = \frac{1}{x}$. Therefore

$$\begin{aligned} h'(x) &= (1)\ln x + x\left(\frac{1}{x}\right) \\ &= \ln x + 1. \end{aligned}$$

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(ii) Hence, find $\int \ln x dx$.

We know that $h'(x) = \ln x + 1$. Also, we know that $(x)' = 1$. So if $F(x) = h(x) - x$, then

$$F'(x) = h'(x) - (x)' = \ln x + 1 - 1 = \ln x.$$

Therefore $\int \ln x \, dx = F(x) + c$. But $F(x) = h(x) - x = x \ln x - x$. Therefore

$$\int \ln x \, dx = x \ln x - x + C.$$

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(i) Find the value of $f(0.2)$

Substituting 0.2 for x gives

$$f(0.2) = -0.5(0.2)^2 + 5(0.2) - 0.98 = -0.5(0.04) + 1 - 0.98 = 0$$

(ii) Show that f has a local maximum point at $(5, 11.52)$.

First we calculate the derivative of f:

$$f'(x) = -0.5(2x) + 5(1) - 0 = -x + 5.$$

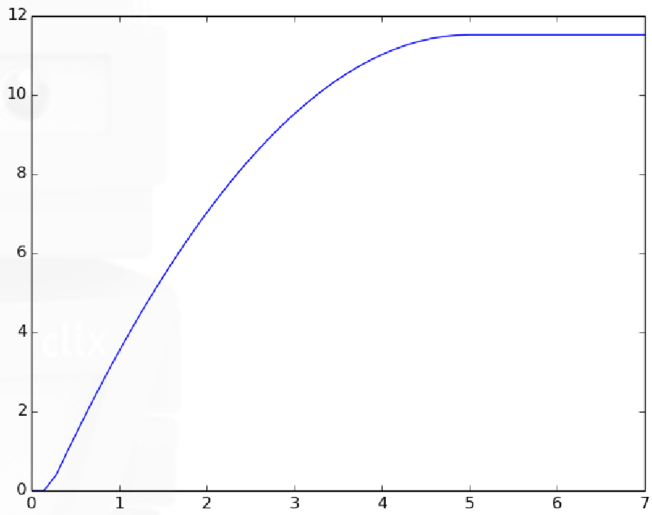
Now $f'(5) = -5 + 5 = 0$. Therefore $x = 5$ is a stationary point.
Now

$$f''(x) = -1.$$

So $f''(5) = -1 < 0$. That means that $x = -5$ is a local maximum. Finally,

$$f(5) = -0.5(5^2) + 5(5) - 0.98 = 11.52.$$

Therefore the graph of f has a local maximum point at $(5, 11.52)$.



Note that between $t = 0$ and $t = 0.2$ the graph is just a horizontal line along the t-axis. Likewise, for $t \geq 5$ the graph is a horizontal line at height $v = 11.52$. In between $t = 0.2$ and $t = 5$ the function is a quadratic so the graph must be a parabola. We can sketch this by evaluating the function at three or four points. For example $v(1) = 3.52$, $v(2) = 7.02$, $v(3) = 9.52$ and $v(4) = 11.02$. So we plot the points $(1, 3.52)$, $(2, 7.02)$, $(3, 9.52)$ and $(4, 11.02)$ and then join them by a smooth curve. Make sure that this parabolic arc starts at $(0.2, 0)$ and ends at $(5, 11.52)$.

(ii) Find the distance travelled by the sprinter in the first 5 seconds of the race.

The distance travelled in the first 5 seconds of the race is given by

$$\int_0^5 v(t) \quad dt.$$

Now

$$\begin{aligned}
\int_0^5 v(t) \quad dt &= \int_0^{0.2} v(t) \quad dt + \int_{0.2}^5 v(t) \quad dt \\
&= \int_0^{0.2} 0 \quad dt + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= 0 + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \Big|_{0.2}^5 \\
&= \frac{0.5(5^3)}{3} + \frac{5(5^2)}{2} - 0.98(5) - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2) \right) \\
&= 36.864
\end{aligned}$$

So the sprinter travels 36.864 metres in the first 5 seconds of the race.

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(iii) Find the sprinter's finishing time for the race. Give your answer correct to two decimal places.

We have just seen that the sprinter travels 36.864 metres in the first 5 seconds of the race. So he has 63.136metres left to travel to complete the race at that point. Also after 5 seconds, his velocity is a constant 11.52 metres per second. Therefore it will take him a further $\frac{63.136}{11.52}$ seconds to complete the race. Now $\frac{63.136}{11.52} = 5.48$ correct to two decimal places. So his total time is $5 + 5.48 = 10.48$ seconds, correct to two decimal places.

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After 7 seconds the sprinter's velocity doesn't change. So the first step is to calculate how far he has travelled in the first 7 seconds of the race. As in part (ii) this is equal to the value of the defnite integral

$$\int_{0.2}^{7} (-0.5t^2 + 5t - 0.98) \quad dt.$$

Now

$$\begin{aligned}
\int_{0}^{7} (-0.5t^2 + 5t - 0.98) \quad dt &= \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \Big|_{0.2}^{7} \\
&= \frac{0.5(7^3)}{3} + \frac{5(7^2)}{2} - 0.98(7) \\
&\quad - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2) \right) \\
&= 58.571
\end{aligned}$$

So he travels 58.571 metres in 7 seconds. Therefore, he has $100 - 58.571 - 41.429$ metres left to travel at that point. His velocity for rest of the race is 11.52 metres per second. Therefore it will take him another $\frac{41.429}{11.52} = 3.596$ seconds to complete the race. So his total time for the race is $7 + 3.596 = 10.596$. So it takes him 10.60 seconds to finish the race, correct to two decimal places.

(c) A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

(i) Prove that the radius of the snowball is decreasing at a constant rate.

Let t be time. Let r be the radius, A the surface area and V the volume of the snowball. From the Formula and Tables booklet we know that $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$. In particular,

$$\frac{dV}{dr} = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2 = A.$$

Now we are told that the rate of change of volume with respect to time is proportional to the surface area. In other words,

$$\frac{dV}{dt} = kA \tag{1}$$

for some constant k. Clearly $k < 0$ since the volume of the snowball is decreasing as it melts. On the other hand, using the chain rule, we see that

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt} \\ &= A\frac{dr}{dt} \end{aligned} \tag{2}$$

Therefore by combining (1) and (2) , we see that

$$A\frac{dr}{dt} = kA.$$

Now dividing across by A yields

$$\frac{dr}{dt} = k$$

where k is a constant, as required.

(ii) If the snowball loses half of its volume in an hour, how long more will it take for it to melt completely?

Give your answer to the nearest minute.

Let r_0 be the initial radius and let r_2 be the radius after 1 hour.
So the initial volume is $\frac{4}{3}\pi r_0^3$. Therefore after one hour, the volume is $\frac{2}{3}\pi r_0^3$. Therefore

$$\frac{4}{3}\pi r_1^3 = \frac{2}{3}\pi r_0^2.$$

Therefore

$$\left(\frac{r_1}{r_0}\right)^3 = \frac{1}{2}$$

or

$$r_1 = \frac{1}{\sqrt[3]{2}} r_0.$$

Now the radius is decreasing at a constant rate and we have found that it takes 1 hour for it to decrease from r_0 to $\frac{1}{\sqrt[3]{2}} r_0$. Therefore the rate of change of the radius is $r_0 - \frac{1}{\sqrt[3]{2}} r_0$ units per hour.
Now the snowball will have melted completely when the radius reaches 0. So we calculate the time required to to change from r_0 to 0. This will be

$$\frac{\text{total change}}{\text{rate of change}} = \frac{r_0 - 0}{r_0 - \frac{1}{\sqrt[3]{2}} r_0} = \frac{1}{1 - \frac{1}{\sqrt[3]{2}}} \text{ hours.}$$

This is equal to 4.8473 hours (correct to four decimal places). So it will take 3.8473 more hours (after the 1 already elapsed) for it to melt completely.
Now 3.8473 hours is equal $3.8473 \times 60 = 230.84$.
So, to the nearest minute, it will take a further 231 minutes for the snowball to melt completely.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	0	0·5	0·866	1	0·866	0·5	0

(b) Use the trapezoidal rule to find the approximate area of the region enclosed between the curve and the x-axis in the given domain.

$$A = \frac{h}{2}\left[y_1 + y_n + 2(y_2 + y_3 + y_4 + \cdots + y_{n-1})\right]$$
$$= \frac{\pi}{12}\left[0 + 0 + 2(0\cdot 5 + 0\cdot 866 + 1 + 0\cdot 866 + 0\cdot 5)\right]$$
$$= 1\cdot 95407$$

(c) Use integration to find the actual area of the region shown above.

$$\int_0^{\pi} \sin x dx = \left[-\cos x\right]_0^{\pi} = -\left[-1-1\right] = 2$$

(d) Find the percentage error in your answer to **(a)** above.

Percentage error $\frac{2 - 1\cdot 95407}{2} \times 100 = 2\cdot 2965 = 2\cdot 3\%$