MarkingScheme

InductionH

Question 1 (2016)

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<tr><th>Q4</th><th>Model Solution – 25 Marks</th><th>Marking Notes</th></tr>
<tr><td>(a)</td><td>
P_1: $8^1 - 1 = 7$ (divisible by 7)
P_k: Assume $8^k - 1$ is divisible by 7
$8^k - 1 = 7M$
$8^k = 7M + 1$
P_{k+1}: $8^{k+1} - 1 = 8(8^k) - 1$
$= 8(7M + 1) - 1$
$= 56M + 7$
$= 7(8M + 1)$
P_{k+1} is divisible by 7

P_1 is true
P_k true $\Longrightarrow$ P_{k+1} is true
So, P_{k+1} true whenever P_k true.
Since P_1 true, then, by induction, P_n is true for all natural numbers ≥ 1

Or

$P_{k+1} = 8^{k+1} - 1$
$= 8.8^k - 1$
$= (7 + 1).8^k - 1$
$= 7(8^k) + (8^k - 1)$
Obviously divisible by 7   From P_k
So, P_{k+1} true whenever P_k true.
Since P_1 true, then, by induction, P_n is true for all natural numbers ≥ 1
</td><td>
Scale 15D (0, 4, 7, 11, 15)
Low Partial Credit
• P_1 step

Mid Partial Credit
• P_k step
• P_{k+1} step

High Partial Credit
• use of P_k step to prove P_{k+1} step

Note: accept P_1 step, P_k step and P_{k+1} step in any order
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(a) Prove, by induction, that the sum of the first n natural numbers, $1+2+3+\cdots+n$, is $\frac{n(n+1)}{2}$.

To Prove: $P(n)=1+2+3+\cdots+n=\frac{n(n+1)}{2}$

$P(1)$: $1 = \frac{1(1+1)}{2}=1$, True

Assume P(n) is true for n = k, and prove P(n) is true for n = k + 1.

n = k: $1+2+3+\cdots+k=\frac{k(k+1)}{2}$

To prove P(k+1) $=\frac{(k+1)}{2}(k+2)$

L.H.S. $=1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$

$=\frac{(k+1)}{2}(k+2)=$ R.H.S

But P(1) is true, so P(2) is true etc.
Hence, P(n) is true for all n.

(b) Hence, or otherwise, prove that the sum of the first n even natural numbers, $2+4+6+\cdots+2n$, is n^2+n.

a = 2 and d = 2.
$S_n=\frac{n}{2}(2a+(n-1)d)=\frac{n}{2}(4+(n-1)2)=\frac{n}{2}(2n+2)=n^2+n$

OR

$$\begin{aligned}S_n &= 2+4+6+\ldots\ldots\ldots\ldots+2n\\ &= 2(1+2+3+\ldots\ldots\ldots+n)\\ &= 2\left[\frac{n(n+1)}{2}\right]\\ &= n(n+1)\\ &= n^2+n\end{aligned}$$

(c) Using the results from **(a)** and **(b)** above, find an expression for the sum of the first n odd natural numbers in its simplest form.

$$1+2+3+\cdots+2n=\frac{2n(2n+1)}{2}=2n^2+n$$

$$\Rightarrow (1+3+5+\cdots n\text{ terms})+(2+4+6+\cdots n\text{ terms})=2n^2+n$$

$$\Rightarrow (1+3+5+\cdots n\text{ terms})+(n^2+n)=2n^2+n$$

$$\Rightarrow 1+3+5+\cdots n\text{ terms}=2n^2+n-(n^2+n)=n^2$$

OR

$$S_A=1+2+3+\ldots\ldots\ldots\ldots\ldots+(2n-1)+(2n)=2n^2+n$$

$$S_B=2+4+6+8+\ldots\ldots\ldots\ldots+2n=n^2+n$$

$$S_A-S_B=1+3+5+\ldots\ldots\ldots\ldots+(2n-1)=n^2$$

First we check that the statement is true for $n=1$. The sum of the first 1 natural numbers is 1, and when $n=1$ we have $\frac{n(n+1)}{2}=\frac{1(1+1)}{2}=\frac{2}{2}=1$. So the statement is true for $n=1$. Now suppose that the statement is true for some $n\geq 1$. So

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Now, add $n+1$ to both sides and we get

$$\begin{aligned}1+2+\cdots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1)\\ &= \frac{n(n+1)}{2}+\frac{2(n+1)}{2}\\ &= \frac{n(n+1)+2(n+1)}{2}\\ &= \frac{(n+2)(n+1)}{2}\\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

So the sum of the first $n+1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n.

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(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

By part (i), we know that

$$1+2+\cdots+100=\frac{100(101)}{2}=5050.$$

We also know that

$$1+2+\cdots+50=\frac{50(51)}{2}=1275.$$

Subtracting the second equation from the first yields

$$51+52+\cdots+100=5050-1275=3775.$$

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(b) Given that $p=\log_c x$, express $\log_c\sqrt{x}+\log_c(cx)$ in terms of p.

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2}\log_c x = \frac{1}{2}p$$

using the power law for logarithms.
Also,

$$\log_c(cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.
But $\log_c c = 1$ since $c^1 = c$. Therefore

$$\log_c \sqrt{x} + \log_c(cx) = \frac{1}{2}p + 1 + p = \frac{3p}{2} + 1.$$

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Question 4 (2013)

Question 2 **(25 marks)**

(a) Prove by induction that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$.

Now suppose that the statement is true for some $n \geq 1$. Remember that $\sum_{r=1}^{n} r = 1 + 2 + \cdots + n$. So

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Now, add $n+1$ to both sides and we get

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first $n+1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n.

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(b) State the range of values for which the series $\sum_{r=2} (4x-1)^r$ is convergent, and write the infinite sum in terms of x.

This is a geometric series i.e. a series of the form $T_n = ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$ where $a = 1$ and $r = 4x - 1$.
However, given that it starts from r=2, this series is missing the first two terms a and ar (1 and $4x-1$). If this series is convergent we must have $|4x-1| < 1$ which means

$$\begin{aligned} -1 < 4x - 1 < 1 \\ 0 < 4x < 2 \\ 0 < x < \frac{1}{2} \end{aligned}$$

This is the required range.

The sum to infinity of a geometric series is given by $S_\infty = \dfrac{a}{1-r}$ where $|r| < 1$. Since this series is missing the first two terms we get

$$\begin{aligned} S_\infty &= \frac{1}{1-(4x-1)} - 1 - (4x-1) \\ &= \frac{1}{2-4x} - 4x \\ &= \frac{1}{2-4x} - \frac{(2-4x)4x}{2-4x} \\ &= \frac{1-8x+16x^2}{2-4x} \end{aligned}$$

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Question 5 (2012)

$P(n)$: $\quad a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$

Check $P(1)$: $\quad a = \dfrac{a(1-r)}{1-r}$, which is true.

Assume $P(k)$: $\quad a + ar + ar^2 + \cdots + ar^{k-1} = \dfrac{a(1-r^k)}{1-r}$

Then: $\quad \underbrace{a + ar + ar^2 + \cdots + ar^{k-1}} + ar^k$

$$\begin{aligned} &= \frac{a(1-r^k)}{1-r} + ar^k \\ &= \frac{a(1-r^k) + ar^k(1-r)}{1-r} \\ &= \frac{a(1-\cancel{r^k}+\cancel{r^k}-r^{k+1})}{1-r} \\ &= \frac{a(1-r^{k+1})}{1-r} \end{aligned}$$

which establishes $P(k+1)$.

Since we have $P(1) \wedge \{\forall k \in \mathbb{N}, (P(k) \Rightarrow P(k+1)\}$, it follows that $P(n)$ holds $\forall n \in \mathbb{N}$.

$$5\cdot\dot{2}\dot{1} = 5 + \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \cdots$$

$$= 5 + [\text{geometric series with } a = \tfrac{21}{100}, \quad r = \tfrac{1}{100}].$$

$$= 5 + \frac{\frac{21}{100}}{1 - \frac{1}{100}} = 5 + \frac{21}{100 - 1} = 5\tfrac{21}{99} = 5\tfrac{7}{33}.$$

Question 6 (2012)