

MarkingScheme

InductionH

Question 1 (2016)

Q4	Model Solution – 25 Marks	Marking Notes
(a)	<p> $P_1: 8^1 - 1 = 7$ (divisible by 7) P_k: Assume $8^k - 1$ is divisible by 7 $8^k - 1 = 7M$ $8^k = 7M + 1$ $P_{k+1}: 8^{k+1} - 1 = 8(8^k) - 1$ $= 8(7M + 1) - 1$ $= 56M + 7$ $= 7(8M + 1)$ P_{k+1} is divisible by 7 P_1 is true P_k true $\implies P_{k+1}$ is true So, P_{k+1} true whenever P_k true. Since P_1 true, then, by induction, P_n is true for all natural numbers ≥ 1 <p style="text-align: center;">Or</p> $P_{k+1} = 8^{k+1} - 1$ $= 8 \cdot 8^k - 1$ $= (7 + 1) \cdot 8^k - 1$ $= 7(8^k) + (8^k - 1)$ <div style="display: flex; justify-content: space-around; margin-top: 10px;"> <div style="text-align: center;"> \swarrow Obviously divisible by 7 </div> <div style="text-align: center;"> \searrow From P_k </div> </div> So, P_{k+1} true whenever P_k true. Since P_1 true, then, by induction, P_n is true for all natural numbers ≥ 1 </p>	<p>Scale 15D (0, 4, 7, 11, 15)</p> <p><i>Low Partial Credit</i></p> <ul style="list-style-type: none"> P_1 step <p><i>Mid Partial Credit</i></p> <ul style="list-style-type: none"> P_k step P_{k+1} step <p><i>High Partial Credit</i></p> <ul style="list-style-type: none"> use of P_k step to prove P_{k+1} step <p>Note: accept P_1 step, P_k step and P_{k+1} step in any order</p>

Question 2 (2014)

- (a) Prove, by induction, that the sum of the first n natural numbers, $1+2+3+\dots+n$, is $\frac{n(n+1)}{2}$.

$$\text{To Prove: } P(n) = 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$P(1): 1 = \frac{1(1+1)}{2} = 1, \text{ True}$$

Assume $P(n)$ is true for $n = k$, and prove $P(n)$ is true for $n = k + 1$.

$$n = k: \quad 1+2+3+\dots+k = \frac{k(k+1)}{2}$$

$$\text{To prove } P(k+1) = \frac{(k+1)}{2}(k+2)$$

$$\begin{aligned} \text{L.H.S.} &= 1+2+3+\dots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)}{2}(k+2) = \text{R.H.S} \end{aligned}$$

But $P(1)$ is true, so $P(2)$ is true etc.

Hence, $P(n)$ is true for all n .

- (b) Hence, or otherwise, prove that the sum of the first n even natural numbers, $2+4+6+\dots+2n$, is n^2+n .

$$a = 2 \text{ and } d = 2.$$

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(4 + (n-1)2) = \frac{n}{2}(2n+2) = n^2 + n$$

OR

$$\begin{aligned} S_n &= 2+4+6+\dots+2n \\ &= 2(1+2+3+\dots+n) \\ &= 2\left[\frac{n(n+1)}{2}\right] \\ &= n(n+1) \\ &= n^2 + n \end{aligned}$$

- (c) Using the results from (a) and (b) above, find an expression for the sum of the first n odd natural numbers in its simplest form.

$$1 + 2 + 3 + \dots + 2n = \frac{2n(2n+1)}{2} = 2n^2 + n$$

$$\Rightarrow (1 + 3 + 5 + \dots n \text{ terms}) + (2 + 4 + 6 + \dots n \text{ terms}) = 2n^2 + n$$

$$\Rightarrow (1 + 3 + 5 + \dots n \text{ terms}) + (n^2 + n) = 2n^2 + n$$

$$\Rightarrow 1 + 3 + 5 + \dots n \text{ terms} = 2n^2 + n - (n^2 + n) = n^2$$

OR

$$S_A = 1 + 2 + 3 + \dots + (2n-1) + (2n) = 2n^2 + n$$

$$S_B = 2 + 4 + 6 + 8 + \dots + 2n = n^2 + n$$

$$S_A - S_B = 1 + 3 + 5 + \dots + (2n-1) = n^2$$

Question 3 (2014)

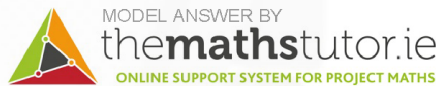
First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$. Now suppose that the statement is true for some $n \geq 1$. So

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Now, add $n + 1$ to both sides and we get

$$\begin{aligned} 1 + 2 + \dots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first $n + 1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n .



(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

By part (i), we know that

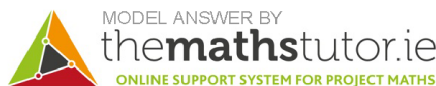
$$1 + 2 + \dots + 100 = \frac{100(101)}{2} = 5050.$$

We also know that

$$1 + 2 + \dots + 50 = \frac{50(51)}{2} = 1275.$$

Subtracting the second equation from the first yields

$$51 + 52 + \dots + 100 = 5050 - 1275 = 3775.$$



(b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c(cx)$ in terms of p .

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2} \log_c x = \frac{1}{2} p$$

using the power law for logarithms.

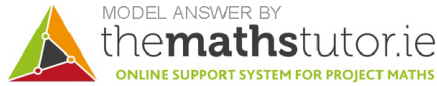
Also,

$$\log_c(cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.

But $\log_c c = 1$ since $c^1 = c$. Therefore

$$\log_c \sqrt{x} + \log_c(cx) = \frac{1}{2} p + 1 + p = \frac{3p}{2} + 1.$$



Question 4 (2013)

Question 2

(25 marks)

- (a) Prove by induction that $\sum_{r=1}^n r = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$.

Now suppose that the statement is true for some $n \geq 1$. Remember that $\sum_{r=1}^n r = 1 + 2 + \dots + n$.

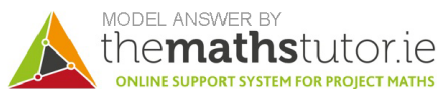
So

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Now, add $n + 1$ to both sides and we get

$$\begin{aligned} 1 + 2 + \dots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first $n + 1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n .



- (b) State the range of values for which the series $\sum_{r=2}^{\infty} (4x - 1)^r$ is convergent, and write the infinite sum in terms of x .

This is a geometric series i.e. a series of the form $T_n = ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$ where $a = 1$ and $r = 4x - 1$.

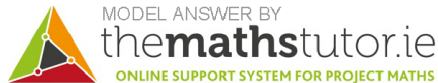
However, given that it starts from $r=2$, this series is missing the first two terms a and ar (1 and $4x - 1$). If this series is convergent we must have $|4x - 1| < 1$ which means

$$\begin{aligned} -1 < 4x - 1 < 1 \\ 0 < 4x < 2 \\ 0 < x < \frac{1}{2} \end{aligned}$$

This is the required range.

The sum to infinity of a geometric series is given by $S_\infty = \frac{a}{1-r}$ where $|r| < 1$. Since this series is missing the first two terms we get

$$\begin{aligned} S_\infty &= \frac{1}{1-(4x-1)} - 1 - (4x-1) \\ &= \frac{1}{2-4x} - 4x \\ &= \frac{1}{2-4x} - \frac{(2-4x)4x}{2-4x} \\ &= \frac{1-8x+16x^2}{2-4x} \end{aligned}$$



Question 5 (2012)

$$P(n): a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

Check $P(1)$: $a = \frac{a(1-r)}{1-r}$, which is true.

Assume $P(k)$: $a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$

Then:

$$\begin{aligned} &\underbrace{a + ar + ar^2 + \dots + ar^{k-1}} + ar^k \\ &= \frac{a(1-r^k)}{1-r} + ar^k \\ &= \frac{a(1-r^k) + ar^k(1-r)}{1-r} \\ &= \frac{a(1-r^k + r^k - r^{k+1})}{1-r} \\ &= \frac{a(1-r^{k+1})}{1-r} \end{aligned}$$

which establishes $P(k+1)$.

Since we have $P(1) \wedge \{\forall k \in \mathbb{N}, (P(k) \Rightarrow P(k+1))\}$, it follows that $P(n)$ holds $\forall n \in \mathbb{N}$.

$$\begin{aligned} 5 \cdot 2i &= 5 + \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \dots \\ &= 5 + [\text{geometric series with } a = \frac{21}{100}, r = \frac{1}{100}]. \\ &= 5 + \frac{\frac{21}{100}}{1 - \frac{1}{100}} = 5 + \frac{21}{100 - 1} = 5 \frac{21}{99} = 5 \frac{7}{33}. \end{aligned}$$

Question 6 (2012)
