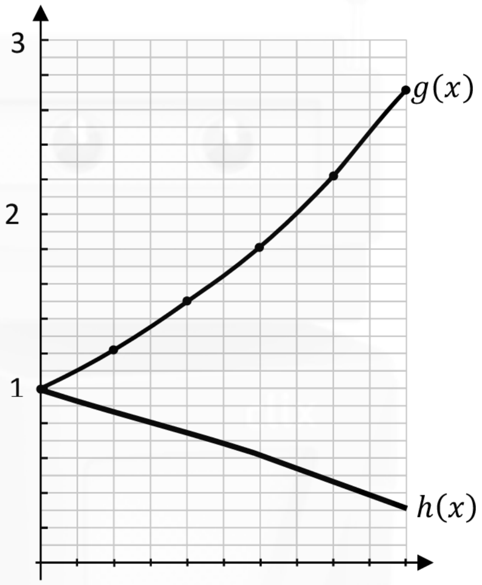
Question 1 (2017)

(a)



$$g(x) = e^x \quad h(x) = e^{-x} = \frac{1}{e^x}$$

$g(x) = e^x$:

x	0	0·2	0·4	0·6	0·8	1·0
y	1	1·22	1·49	1·82	2·23	2·72

$h(x) = \frac{1}{e^x}$:

x	0	0·2	0·4	0·6	0·8	1·0
y	1	0·82	0·67	0·55	0·45	0·37

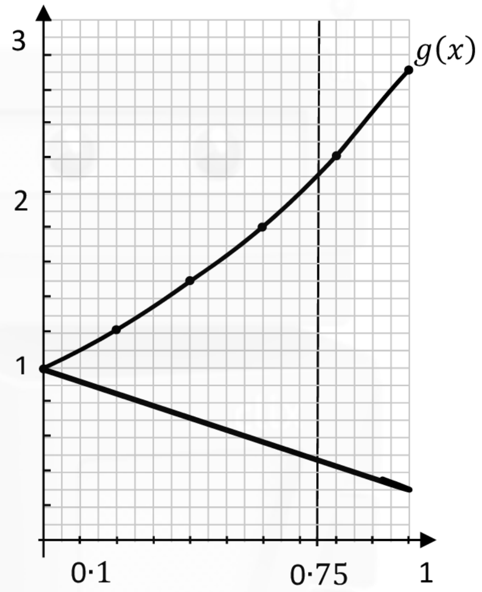
Scale 15C (0, 5, 10, 15)

Low Partial Credit:

- one point correct

High Partial Credit

- Graph not in required domain

(b)	 $A = \int_0^{0\cdot75} e^x dx - \int_0^{0\cdot75} e^{-x} dx$ $= \int_0^{0\cdot75} (e^x - e^{-x}) dx$ $= e^x + e^{-x}$ $e^{0\cdot75} + e^{-0\cdot75} - [e^0 + e^0]$ $= 0\cdot5894$	**Scale 10C (0, 5, 8, 10)** *Low Partial Credit:* • Formulates integration for area under one curve with limits *High Partial Credit* • integrates twice for correct area under both curves **Note**: Trapezoidal rule must have at least 5 divisions AND fully correct work gets Low Partial Credit

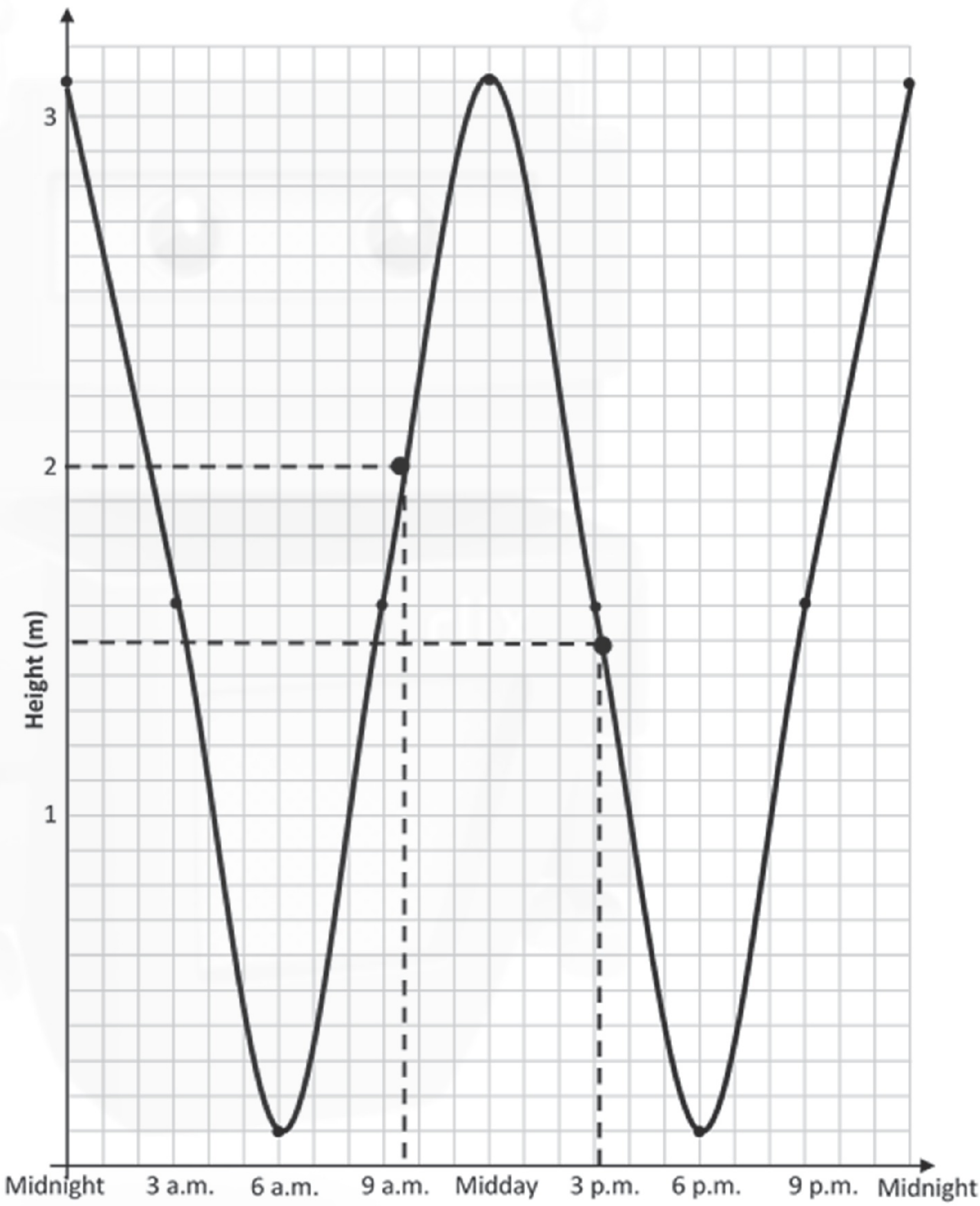
Q8	Model Solution – 45 Marks	Marking Notes
(a)	Period = $\dfrac{2\pi}{\frac{\pi}{6}}$ = 12 hours Range = $[1{\cdot}6 - 1{\cdot}5\,, 1{\cdot}6 + 1{\cdot}5\,] = [0{\cdot}1\text{ m}, 3{\cdot}1\text{ m}]$	Scale 5C (0, 2,4, 5) *Low Partial Credit* • some use of 2π or $\dfrac{\pi}{6}$ • range of cos function *High partial credit* • period or range correct **Note:** *Accept correct period and/or range without work*
(b)	Max = 1·6 + 1·5(1) =3·1 m. **or** 3·1 m from range	Scale 5B (0,2, 5) *Partial Credit* • max occurs when cos A = 1 or t = 0 • effort at $h'(t)$ **Note:** *Accept correct answer without work*
(c)	$h'(t) = 1{\cdot}5(-\sin\dfrac{\pi t}{6})\dfrac{\pi}{6}$ $h'(2) = 1{\cdot}5(-\sin\dfrac{2\pi}{6})\dfrac{\pi}{6}$ = – 0·68017 =–0·68 m/h Tide is going out at a rate of 0·68 m per hour at 2 am	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • effort at differentiation *High Partial Credit* • correct numerical answer but not in context

(d)(i)

$$h(t) = 1 \cdot 6 + 1 \cdot 5\cos\left(\frac{\pi}{6}t\right)$$

Time	12 am	3 am	6 am	9 am	12 pm	3 pm	6 pm	9 pm	12 am
t	0	3	6	9	12	15	18	21	24
Height	3·1	1·6	·1	1·6	3·1	1·6	·1	1·6	3·1

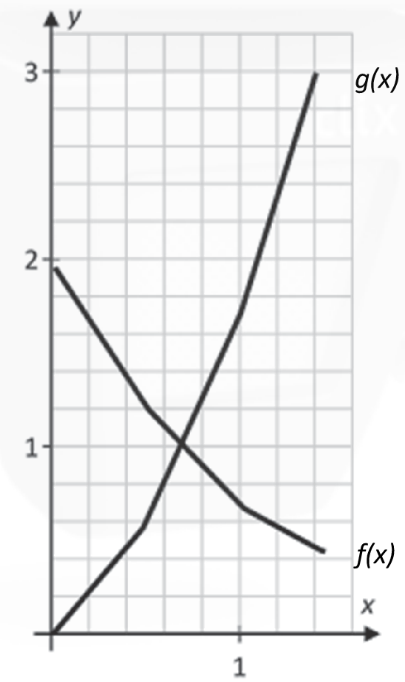
(d) **(i)**		Scale 10C (0, 3, 7, 10) *Low Partial Credit* • one correct height *High Partial Credit* • five correct heights

(d) **(ii)**	 	
(d) **(ii)**	Graph	Scale 10C (0, 3,7,10) *Low Partial Credit* • one correct plot *High Partial Credit* • at least 7 correct plots • plots correct but graph not sketched or sketched incorrectly

(e)	Low tide = 0·1 m High tide = 3·1 m Difference = 3·1 − 0·1 =3 m	Scale 5B (0, 2, 5) *Partial Credit* • height of Low tide or High tide correctly identified **Notes:** (i) *candidates may show work for this section on graph* (ii) *accept values from candidate's graph* (iii) *accept correct answer from graph without work*
(f)	Enter port at 9:30 approx Leave port before 15:15 approx Time = 15:15 – 9:30 = 5 hr 45 min approx.	Scale 5B (0, 2, 5) *Partial Credit* • time of entry to port or leave port correctly identified • value(s) for h = 2 and/or h = 1·5 on sketch • time estimated using relevant values other than those required for the maximum time. **Notes:** (i) *candidates may show relevant work for this section on graph* (ii) *accept values from candidate's graph*

Question 3 (2016)

<table>
<tr><th>Q3</th><th>Model Solution – 25 Marks</th><th>Marking Notes</th></tr>
<tr><td>(a)
(i)</td><td>
<table>
<tr><td>x</td><td>0</td><td>0·5</td><td>1</td><td>$\ln(4)$</td></tr>
<tr><td>$f(x)=\frac{2}{e^x}$</td><td>2</td><td>1·21</td><td>0·74</td><td>0·5</td></tr>
<tr><td>$g(x)=e^x-1$</td><td>0</td><td>0·65</td><td>1·72</td><td>3</td></tr>
</table>
</td><td>Scale 5C (0, 2, 4, 5)
Low Partial Credit
• one entry correct

High Partial Credit
• 5 entries correct</td></tr>
<tr><td>(ii)</td><td>


</td><td>Scale 5C (0, 2, 4, 5)
Low Partial Credit
• one plot correct

High Partial Credit
• 5 plots correct
• one correct graph
• no labelling

Notes:
- straight lines <u>NOT</u> acceptable
- one clear label merits full credit
- one ambiguous label merits High Partial Credit at most</td></tr>
<tr><td>(iii)</td><td>$f(x)=g(x)$ when $x\approx 0\cdot 7$</td><td>Scale 5B (0, 2, 5)
Partial Credit
• point of intersection clearly indicated on graph, but value of x not stated</td></tr>
</table>

Q3	Model Solution – Continued	Marking Notes
(b)	$\frac{e^x - 1}{1} = \frac{2}{e^x}$ $e^{2x} - e^x = 2$ $(e^x)^2 - e^x - 2 = 0$ $(e^x - 2)(e^x + 1) = 0$ $e^x = 2$ **or** $e^x = -1$ $x = \ln 2$ or $x = 0{\cdot}693$ **Or** $(e^x)^2 - e^x - 2 = 0$ Let $y = e^x \Rightarrow y^2 - y - 2 = 0$ $y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$ $= \frac{1 \pm \sqrt{1+8}}{2}$ $= \frac{1 \pm 3}{2}$ $\Rightarrow y = 2$ or $y = -1$ (not possible) $y = e^x \Rightarrow e^x = 2$ $x = \ln 2$ or $x = 0{\cdot}693$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • substitution correct *High Partial Credit* • correct factors of quadratic • root formula correctly substituted $e^x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$ **Note:** *oversimplification of equation (i.e. not treating as quadratic) merits Low Partial Credit at most* **Or** Scale 10C (0, 3, 7, 10) *Low Partial Credit* • substitution correct *High Partial Credit* • root formula correctly substituted $y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$ **Note:** *oversimplification of equation (i.e. not treating as quadratic) merits Low Partial Credit at most*

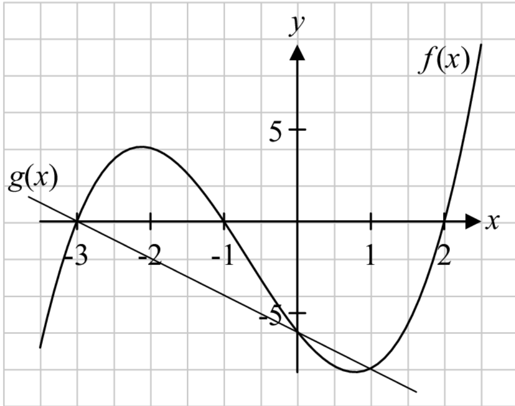
Q8	Model Solution – 55 Marks	Marking Notes
(a)		
(i)	$f(x) = -0{\cdot}274x^2 + 1{\cdot}193x + 3{\cdot}23$ $f'(x) = -0{\cdot}548x + 1{\cdot}193 = 0$ $x = 2{\cdot}177\text{ m}$ $f(2{\cdot}177) = -0{\cdot}274(2{\cdot}177)^2 + 1{\cdot}193(2{\cdot}177) + 3{\cdot}23$ $= -1{\cdot}2986 + 2{\cdot}5972 + 3{\cdot}23$ $= 4{\cdot}529\text{ m}$ *or* $-0{\cdot}274(x^2 - \frac{1193}{274}x - \frac{1615}{137})$ $-0{\cdot}274(x - \frac{1193}{548})^2 + 4.5285$ Max Height $= 4{\cdot}529$ m	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • any correct differentiation • effort made at completing square • trial and error with more than one value of x tested *High Partial Credit* • x value correct **Note:** *if correct answer by trial and error, must show points on each side of max point to be lower to earn full credit*
(ii)	$\tan\theta = -0{\cdot}548(4{\cdot}5)+1{\cdot}193$ $\tan\theta = -1{\cdot}273$ $\theta = 51{\cdot}8° = 52°$	Scale 5B (0, 2, 5) *Partial Credit* • tan **Note:** *right angled triangles may appear in diagram given in equation*
(iii)	Map $A \to C$ $(-0{\cdot}5, 2{\cdot}565) \to (0, 2)$ $2{\cdot}177 — (-0{\cdot}5) = 2{\cdot}677$ $4{\cdot}529 - 0{\cdot}565 = 3{\cdot}964$ $(2{\cdot}177, 4{\cdot}529) \to (2{\cdot}677, 3{\cdot}964)$	Scale 5B (0, 2, 5) *Partial Credit* • $(-0{\cdot}5, 2{\cdot}565) \to (0, 2)$

(iv)	$$g(x) = ax^2 + bx + c$$ $$C(0, 2) \in g(x) => c = 2$$ $B(4{\cdot}5, 3{\cdot}05)\ \epsilon\ g(x)$ $3{\cdot}05 = a(4{\cdot}5)^2 + b(4{\cdot}5) + 2$ $\Longrightarrow 20{\cdot}25a + 4{\cdot}5b = 1{\cdot}05$ ... **(i)** $g'(x) = 2ax + b = 0$ $\Longrightarrow 2a(2{\cdot}677) + b = 0$ $5{\cdot}354a + b = 0$ ... **(ii)** From **(i)** and **(ii)** $a = -0{\cdot}273$ $b = 1{\cdot}462$ $$g(x) = -0{\cdot}273x^2 + 1{\cdot}462x + 2$$ *[Note: a third equation that could be used is* $3{\cdot}964 = a(2{\cdot}677)^2 + b(2{\cdot}677) + 2$ ... **(iii)**]	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • *c* value found • relevant equation in *a*, *b* and/or *c* *Mid Partial Credit* • formulated correctly any two equations *High Partial Credit* • formulated correctly any three equations **Note**: $ax^2 + bx + c$ *not in an equation merits 0 marks*
	Or	**Or**
	Equation of parabola with vertex (h, k): $$g(x) = a(x - h)^2 + k$$ $C(0, 2)$ on curve: $(h, k) = (2{\cdot}677, 3.964)$ $$2 = a(-2{\cdot}677)^2 + 3{\cdot}964$$ $$-1{\cdot}964 = a(7{\cdot}166329)$$ $$a = -0{\cdot}27405 = -0{\cdot}274$$ Parabola: $$g(x) = -0{\cdot}274[(x - 2{\cdot}677)^2] + 3{\cdot}964$$ **or** $$g(x) = f(x - 0{\cdot}5) - 0{\cdot}565$$ $$g(x) = -0{\cdot}274(x - 0{\cdot}5)^2 + 1{\cdot}193(x - 0{\cdot}5) + 3{\cdot}23 - 0{\cdot}565$$ $$g(x) = -0{\cdot}274x^2 + 1{\cdot}467x + 2$$	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • equation of curve • use of C *Mid Partial Credit* • using peak value *High Partial Credit* • value of *a* found

(a) The graph of a cubic function $f(x)$ cuts the x-axis at $x = -3$, $x = -1$ and $x = 2$, and the y-axis at $(0, -6)$, as shown.

Verify that $f(x)$ can be written as

$$f(x) = x^3 + 2x^2 - 5x - 6.$$



$x = -3, \quad x = -1, \quad x = 2$

$f(x) = (x+3)(x+1)(x-2) = x^3 + 2x^2 - 5x - 6$

OR

$f(x) = x^3 + 2x^2 - 5x - 6$

$f(-3) = -27 + 18 + 15 - 6 = 0 \Rightarrow (x+3)$ is a factor

$f(-1) = -1 + 2 + 5 - 6 = 0 \Rightarrow (x+1)$ is a factor

$f(2) = 8 + 8 - 10 - 6 = 0 \Rightarrow (x-2)$ is a factor

$f(x) = (x+3)(x+1)(x-2) = x^3 + 2x^2 - 5x - 6$

(b) **(i)** The graph of the function $g(x) = -2x - 6$ intersects the graph of the function $f(x)$ above. Let $f(x) = g(x)$ and solve the resulting equation to find the co-ordinates of the points where the graphs of $f(x)$ and $g(x)$ intersect.

$f(x) = g(x)$

$x^3 + 2x^2 - 5x - 6 = -2x - 6$

$\Rightarrow x^3 + 2x^2 - 3x = 0$

$\Rightarrow x(x^2 + 2x - 3) = 0$

$\Rightarrow x(x-1)(x+3) = 0$

$\Rightarrow x = 0, \quad x = 1, \quad x = -3$

$\Rightarrow y = -6, \;\; y = -8, \;\; y = 0$

Points: $(-3, 0), (0, -6), (1, -8)$

(ii) Draw the graph of the function $g(x) = -2x - 6$ on the diagram above.

$$g(x) = -2x - 6$$
$$g(-3) = -2(-3) - 6 = 6 - 6 = 0 \Rightarrow (-3, 0)$$
$$g(0) = -2(0) - 6 = -6 \Rightarrow (0, -6)$$

Question 6 (2014)

(a) Using the co-ordinate plane, with $A(0, 0)$ and $B(48, 0)$, the equation of the parabola is $y = -0\cdot 013x^2 + 0\cdot 624x$. Find the co-ordinates of C, the highest point of the arch.

$$y = -0\cdot 013x^2 + 0\cdot 624x$$
$$\Rightarrow \frac{dy}{dx} = -0\cdot 026x + 0\cdot 624 = 0 \Rightarrow x = 24$$

$$y = -0\cdot 013x^2 + 0\cdot 624x = -0\cdot 013(24)^2 + 0\cdot 624(24) = 7\cdot 488.$$

$C(24, 7\cdot 488)$

OR

Max height at C when x = 24

$$y = -0\cdot 013x^2 + 0\cdot 624x$$
$$= -0\cdot 013(24)^2 + (0\cdot 624)(24)$$
$$= 7\cdot 488$$

$C(24, 7\cdot 488)$

(b) The perpendicular distance between the walking deck, $[DE]$, and $[AB]$ is 5 metres.
Find the co-ordinates of *D* and of *E*. Give your answers correct to the nearest whole number.

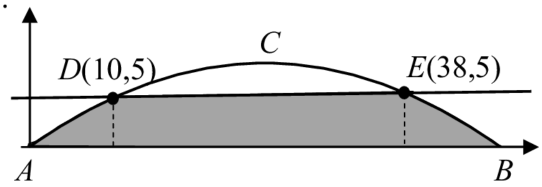
Equation *DE*: $y = 5$

Equation of the parabola: $y = -0\cdot 013x^2 + 0\cdot 624x$.

$$5 = -0\cdot 013x^2 + 0\cdot 624x$$
$$\Rightarrow 0\cdot 013x^2 - 0\cdot 624x + 5 = 0$$
$$x = \frac{0\cdot 624 \pm \sqrt{0\cdot 624^2 - 4(0\cdot 013)5}}{2(0\cdot 013)} = \frac{0\cdot 624 \pm 0\cdot 360}{0\cdot 026}$$
$$x = 37\cdot 8 \text{ or } x = 10\cdot 15$$

$D(10, 5)$, $E(38, 5)$

(c) Using integration, find the area of the shaded region, *ABED,* shown in the diagram below. Give your answer correct to the nearest whole number.



$$\text{Area } ABED = \int_0^{10} y\,dx + \text{Area of rectangle} + \int_{38}^{48} y\,dx$$

$$= 2\int_0^{10} y\,dx + (38-10)\times 5$$

$$= 2\int_0^{10} (-0{\cdot}013x^2 + 0{\cdot}624x)dx + 140$$

$$= 2\left[\frac{-0{\cdot}013x^3}{3} + \frac{0{\cdot}624x^2}{2}\right]_0^{10} + 140$$

$$= 2\left[-\frac{0{\cdot}013(10)^3}{3} + \frac{0{\cdot}624(10)^2}{2} - 0\right] + 140$$

$$= 2\left[-\frac{13}{3} + 31{\cdot}2\right] + 140$$

$$= 193{\cdot}7$$

$$\approx 194 \text{ m}^2$$

OR

Area under curve between A and B:

$$= \int_0^{48} (-0 \cdot 013x^2 + 0 \cdot 624x)dx$$

$$= \left[\frac{-0 \cdot 013x^3}{3} + \frac{0 \cdot 624x^2}{2} \right]_0^{48}$$

$$= \left[-\frac{0 \cdot 013(48)^3}{3} + \frac{0 \cdot 624(48)^2}{2} - 0 \right]$$

$$= 239 \cdot 616$$

Translate curve vertically downwards and find area under the curve between D and E:

$$= \int_{10}^{38} (-0 \cdot 013x^2 + 0 \cdot 624x - 5)dx$$

$$= \left[\frac{-0 \cdot 013x^3}{3} + \frac{0 \cdot 624x^2}{2} - 5x \right]_{10}^{38}$$

$$= \left[-\frac{0 \cdot 013(38)^3}{3} + \frac{0 \cdot 624(38)^2}{2} - 5(38) \right] - \left[\frac{-0 \cdot 013(10)^3}{3} + \frac{0 \cdot 624(10)^2}{2} - 5(10) \right]$$

$$= 22 \cdot 7493 + 23 \cdot 133$$

$$= 45 \cdot 8826$$

$$\text{Shaded area} = 239 \cdot 616 - 45 \cdot 8826$$
$$= 193 \cdot 73$$
$$\approx 194 \text{ m}^2$$

(d) Write the equation of the parabola in part **(a)** in the form $y - k = p(x-h)^2$, where k, p, and h are constants.

$$y = -0 \cdot 013x^2 + 0 \cdot 624x$$
$$= -0 \cdot 013(x^2 - 48x)$$
$$= -0 \cdot 013(x^2 - 48x + (-24)^2 - (-24)^2)$$
$$= -0 \cdot 013(x-24)^2 + 7 \cdot 488$$
$$\Rightarrow y - 7 \cdot 488 = -0 \cdot 013(x-24)^2$$

(e) Using what you learned in part **(d)** above, or otherwise, write down the equation of a parabola for which the coefficient of x^2 is -2 and the co-ordinates of the maximum point are $(3, -4)$.

Given function: coefficient of x^2, $-0\cdot013$; maximum point $(24, 7\cdot488)$
New function: coefficient of x^2, -2; maximum point $(3, -4)$
Function: $y+4=-2(x-3)^2$

OR

Given function: $y-7\cdot488=-0\cdot013(x-24)^2$

$$y-(\text{max height})=(\text{coefficient of } x^2)(x-x_{\text{max}})^2$$

New parabola: max height: -4
coefficient of x^2: -2
x_{max} 3

$$y-(-4)=-2(x-3)^2$$
$$y+4=-2(x-3)^2$$

Question 7 (2014)

(a) Write down the value of the temperature difference, y, when the water boils, and find the value of A.

$y=100-23=77$ at $t=0$

$y = Ae^{kt} \Rightarrow 77 = Ae^0 \Rightarrow A = 77$

(b) After five minutes, the temperature of the water is $88°C$.
Find the value of k, correct to three significant figures.

At $t=5$, $y=88-23=65$

$$y=77e^{kt} \Rightarrow 65=77e^{5k} \Rightarrow 5k=\ln\tfrac{65}{77}=-0\cdot169418$$
$$\Rightarrow k=-0\cdot03388\approx-0\cdot0339$$

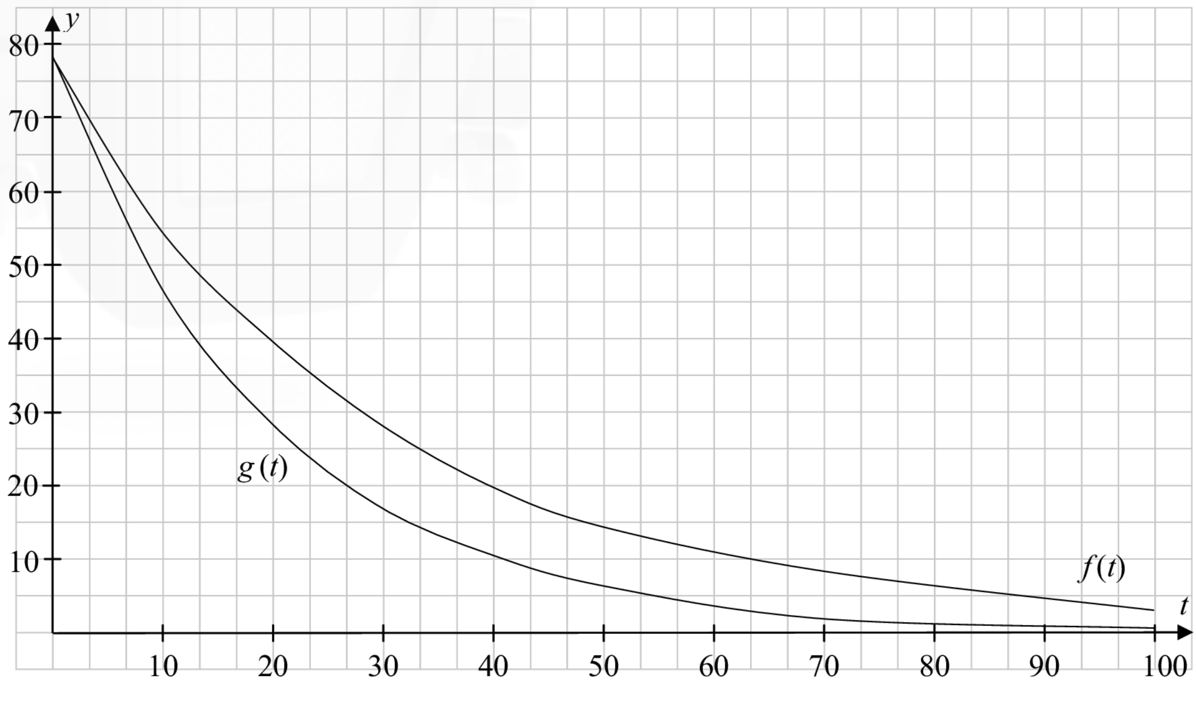
(c) Ciarán prepares the food for his baby when the water has cooled to 50°C. How long does it take, correct to the nearest minute, for the water to cool to this temperature?

$$y = 50 - 23 = 27$$

$$27 = 77e^{-0.0339t} \Rightarrow 0.0339t = \ln\tfrac{77}{27} = 1.047969$$
$$\Rightarrow t = 30.9 \approx 31 \text{ minutes}$$

(d) Using your values for A and k, sketch the curve $f(t) = Ae^{kt}$ for $0 \le t \le 100$, $t \in \mathbb{R}$.

$t = 0 \Rightarrow y = 77e^{-0.0339(0)} = 77$; (0, 77)
$t = 10 \Rightarrow y = 77e^{-0.0339(10)} = 54.9$; (10, 55)
$t = 20 \Rightarrow y = 77e^{-0.0339(20)} = 39.1$; (20, 39)
$t = 30 \Rightarrow y = 77e^{-0.0339(30)} = 27.9$; (30, 28)
$t = 40 \Rightarrow y = 77e^{-0.0339(40)} = 19.8$; (40, 20)
$t = 50 \Rightarrow y = 77e^{-0.0339(50)} = 14.1$; (50, 14)
$t = 60 \Rightarrow y = 77e^{-0.0339(60)} = 10.1$; (60, 10)
$t = 70 \Rightarrow y = 77e^{-0.0339(70)} = 7.2$; (70, 7)
$t = 80 \Rightarrow y = 77e^{-0.0339(80)} = 5.1$; (80, 5)
$t = 90 \Rightarrow y = 77e^{-0.0339(90)} = 3.6$; (90, 4)
$t = 100 \Rightarrow y = 77e^{-0.0339(100)} = 2.6$; (100, 3)



(e) **(i)** On the same diagram, sketch a curve $g(t) = Ae^{mt}$, showing the water cooling at a *faster* rate, where A is the value from part **(a)**, and m is a constant. Label each graph clearly.

(ii) Suggest one possible value for m for the sketch you have drawn and give a reason for your choice.

Test $m = -0\cdot 02$, $m = k = -0.0339$ and $m = -0\cdot 05$

$m = -0\cdot 02$, $t = 10 \Rightarrow y = 77e^{-0\cdot 02(10)} = 63\cdot 0$
$m = k = -0\cdot 0339 \Rightarrow y = 54\cdot 9$ (from table)
$m = -0\cdot 05$, $t = 10 \Rightarrow y = 77e^{-0\cdot 05(10)} = 46\cdot 7$

Any value of $m < k$ for faster decay.

(f) **(i)** Find the rates of change of the function $f(t)$ after 1 minute and after 10 minutes. Give your answers correct to two decimal places.

$$y = 77e^{-0\cdot 0339t} \Rightarrow \frac{dy}{dt} = -2\cdot 6103e^{-0\cdot 0339t}$$

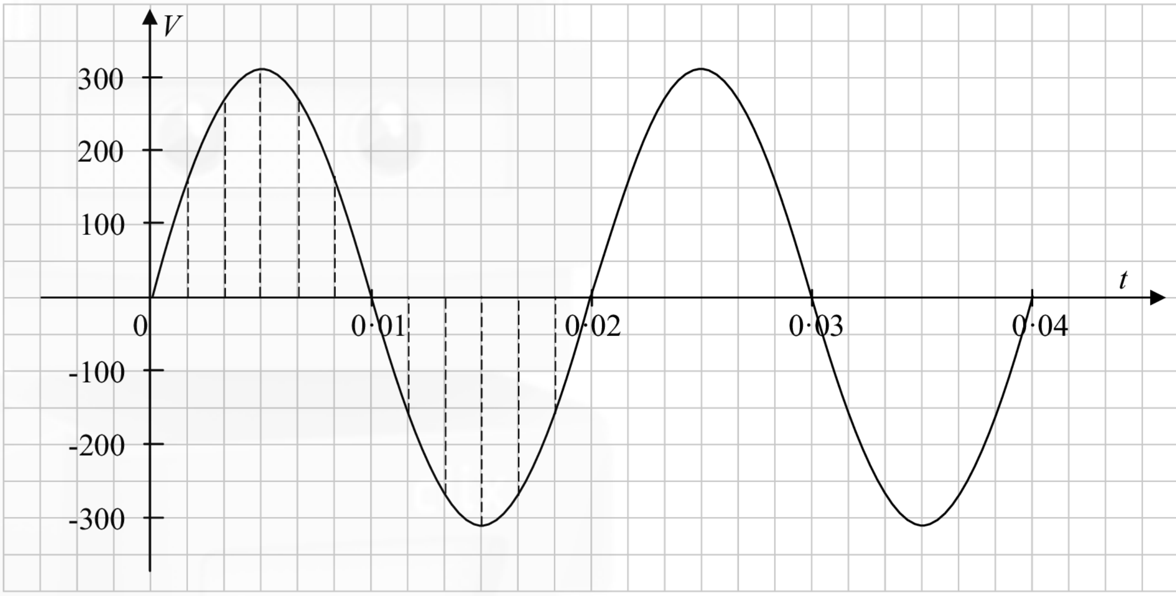
$$t = 1, \ \frac{dy}{dt} = -2\cdot 6103e^{-0\cdot 0339} = -2\cdot 52$$

$$t = 10, \ \frac{dy}{dt} = -2\cdot 6103e^{-0\cdot 339} = -1\cdot 86$$

(ii) Show that the rate of change of $f(t)$ will always increase over time.

$$\frac{d^2y}{dt^2} = 0\cdot 088e^{-0\cdot 0339t} > 0 \Rightarrow \frac{dy}{dt} \text{ is increasing}$$

The graph below shows the voltage, V, in an electric circuit as a function of time, t. The voltage is given by the formula $V = 311\sin(100\pi t)$, where V is in volts and t is in seconds.



(a) **(i)** Write down the range of the function.

Range: $[-311, 311]$

(ii) How many complete periods are there in one second?

$$\frac{100\pi}{2\pi} = 50 \text{ periods per second}$$

Or

Time for 1 period = 0·02 seconds

Number of periods in 1 second $= \dfrac{1}{0\cdot 02} = 50$

(b) **(i)** The table below gives the voltage, correct to the nearest whole number, at equally spaced intervals from t_0 to t_{12} over one complete period (as shown by the dashed lines on the diagram). Use the entries given in the table and the properties of the function to complete the table.

T	t_1	t_2	t_3	t_4	t_5	$t_6 = 0{\cdot}01$	t_7	t_8	t_9	t_{10}	t_{11}	$t_{12} = 0{\cdot}02$
V	156	269	311	269	156	0	−156	−269	−311	−269	−156	0

(ii) Using a calculator, or otherwise, calculate the standard deviation, σ, of the twelve values of V in the table, correct to the nearest whole number.

$\sigma = 219{\cdot}89 = 220$

(c) **(i)** The standard deviation, σ, of closely spaced values of any function of the form $V = a\sin(bt)$, over 1 full period, is given by $k\sigma = V_{\max}$, where k is a constant that does not depend on a or b, and $V_{\max}$ is the maximum value of the function. Use the function $V = 311\sin(100\pi t)$ to find an approximate value for k correct to three decimal places.

$$k = \frac{V_{\max}}{\sigma} = \frac{311}{220} \approx 1{\cdot}414$$

(ii) Using your answer in part **(c) (i)**, or otherwise, find the value of b required so that the function $V = a\sin(bt)$ has 60 complete periods in one second and the approximate value of a so that it has a standard deviation of 110 volts.

$$\frac{b}{2\pi} = 60 \Rightarrow b = 120\pi = 377$$

$$k\sigma = V_{\max} \quad \Rightarrow \quad V_{\max} = 1{\cdot}414 \times 110 = 155{\cdot}54 \Rightarrow a = 156$$

Let l be the length of the box and let w be the width of the box, both in centimetres. Then by adding up dimensions as we move left to right across the diagram above, we see that $1+l+h+l+h=31$. Therefore, by isolating l in this equation we obtain

$$l = 15 - h.$$

Going top to bottom, we see that $1+h+w+h+1=22$ and by isolating w, we see that

$$w = 20 - 2h.$$

Therefore

height =	h cm
length =	$15-h$ cm
width =	$20-2h$ cm

(b) Write an expression for the capacity of the box in cubic centimetres, in terms of h.

Capacity = length $\times$ width $\times$ height $=(15-h)(20-2h)h = 2h^3-50h^2+300h\text{cm}^3$.

(c) Show that the value of h that gives a box with a square bottom will give the correct capacity.

The bottom of the box is square if and only if length = width. In other words, if and only if $15-h=20-2h$. This is equivalent to $h=5$. From the solution to part (b), we calculate that, when $h=5$, the capacity of the box will be $(15-5)(20-2(5))5 = 10(10)(5) = 500\text{cm}^3$, as required.

(d) Find, correct to one decimal place, the other value of h that gives a box of the correct capacity.

We must solve $2h^3-50h^2+300h=500$, or

$$2h^3-50h^2+300h-500=0.$$

From part (c), we know that $h=5$ is one solution. Therefore, by the Factor Theorem, $(h-5)$ is a factor of $2h^3-50h^2+300h-500$. Factorising yields

$$2h^3-50h^2+300h-500=(h-5)(2h^2-40h+100).$$

Now, we solve $2h^2-40h+100=0$ using the quadratic formula. So

$$h=\frac{40\pm\sqrt{40^2-4(2)(100)}}{2(2)}=\frac{40\pm\sqrt{800}}{4}=10\pm\sqrt{50}$$

So, correct to one decimal place, $h=17.1$ or $h=2.9$.
Now, however, we observe that since the length of the box is $15-h$, we must have $15-h>0$ or $h<15$. Therefore $h\neq 17.1$. So the other value of h that gives the correct capacity is 2.9cm.

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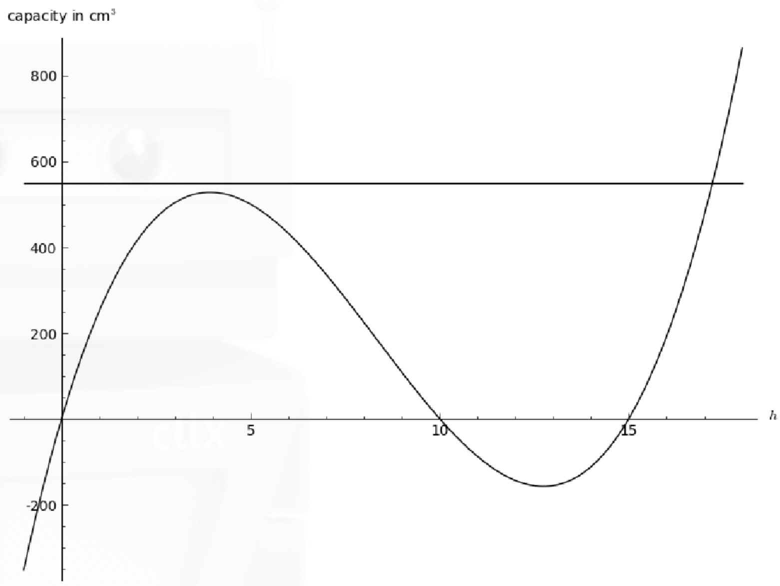

The capacity of the new box will be $1.1\times 500=550\text{cm}^3$. On the diagram above we have drawn a horizontal line representing the equation

$$\text{Capacity}=550.$$

We can see from the diagram that this horizontal line only meets the cubic curve at one point and that the h-co-ordinate of that point is greater than 15.
However, as we observed above, for any box constructed as described in the question, we must have $h<15$. Therefore it is not possible to make the bigger box from the same piece of cardboard as before.

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(i) Find the value of $f(0.2)$

Substituting 0.2 for x gives

$$f(0.2) = -0.5(0.2)^2 + 5(0.2) - 0.98 = -0.5(0.04) + 1 - 0.98 = 0$$

(ii) Show that f has a local maximum point at $(5, 11.52)$.

First we calculate the derivative of f:

$$f'(x) = -0.5(2x) + 5(1) - 0 = -x + 5.$$

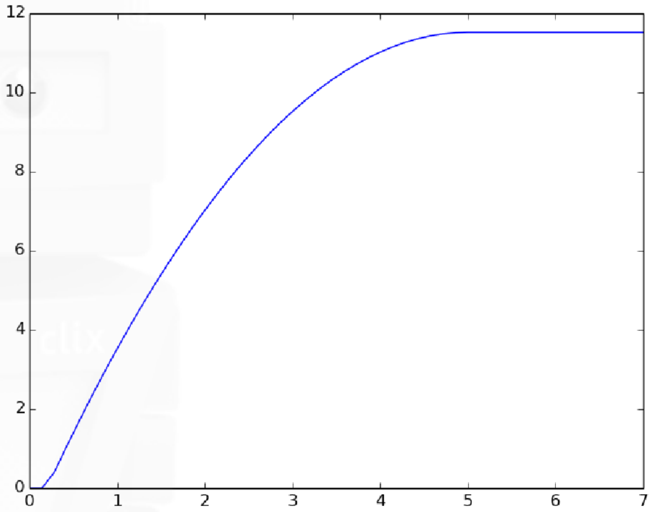
Now $f'(5) = -5 + 5 = 0$. Therefore $x = 5$ is a stationary point.
Now

$$f''(x) = -1.$$

So $f''(5) = -1 < 0$. That means that $x = -5$ is a local maximum. Finally,

$$f(5) = -0.5(5^2) + 5(5) - 0.98 = 11.52.$$

Therefore the graph of f has a local maximum point at $(5, 11.52)$.



Note that between $t = 0$ and $t = 0.2$ the graph is just a horizontal line along the t-axis. Likewise, for $t \geq 5$ the graph is a horizontal line at height $v = 11.52$. In between $t = 0.2$ and $t = 5$ the function is a quadratic so the graph must be a parabola. We can sketch this by evaluating the function at three or four points. For example $v(1) = 3.52$, $v(2) = 7.02$, $v(3) = 9.52$ and $v(4) = 11.02$. So we plot the points $(1, 3.52)$, $(2, 7.02)$, $(3, 9.52)$ and $(4, 11.02)$ and then join them by a smooth curve. Make sure that this parabolic arc starts at $(0.2, 0)$ and ends at $(5, 11.52)$.

(ii) Find the distance travelled by the sprinter in the first 5 seconds of the race.

The distance travelled in the first 5 seconds of the race is given by

$$\int_0^5 v(t) \quad dt.$$

Now

$$\begin{aligned}
\int_0^5 v(t) \quad dt &= \int_0^{0.2} v(t) \quad dt + \int_{0.2}^5 v(t) \quad dt \\
&= \int_0^{0.2} 0 \quad dt + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= 0 + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \Big|_{0.2}^5 \\
&= \frac{0.5(5^3)}{3} + \frac{5(5^2)}{2} - 0.98(5) - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2)\right) \\
&= 36.864
\end{aligned}$$

So the sprinter travels 36.864 metres in the first 5 seconds of the race.

(iii) Find the sprinter's finishing time for the race. Give your answer correct to two decimal places.

We have just seen that the sprinter travels 36.864 metres in the first 5 seconds of the race. So he has 63.136metres left to travel to complete the race at that point. Also after 5 seconds, his velocity is a constant 11.52 metres per second. Therefore it will take him a further $\frac{63.136}{11.52}$ seconds to complete the race. Now $\frac{63.136}{11.52} = 5.48$ correct to two decimal places. So his total time is $5 + 5.48 = 10.48$ seconds, correct to two decimal places.

After 7 seconds the sprinter's velocity doesn't change. So the first step is to calculate how far he has travelled in the first 7 seconds of the race. As in part (ii) this is equal to the value of the defnite integral

$$\int_{0.2}^{7} (-0.5t^2 + 5t - 0.98) \quad dt.$$

Now

$$\begin{aligned}\int_{0}^{7} (-0.5t^2 + 5t - 0.98) \quad dt &= \left.\frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t\right|_{0.2}^{7} \\ &= \frac{0.5(7^3)}{3} + \frac{5(7^2)}{2} - 0.98(7) \\ &\quad - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2)\right) \\ &= 58.571\end{aligned}$$

So he travels 58.571 metres in 7 seconds. Therefore, he has $100 - 58.571 - 41.429$ metres left to travel at that point. His velocity for rest of the race is 11.52 metres per second. Therefore it will take him another $\frac{41.429}{11.52} = 3.596$ seconds to complete the race. So his total time for the race is $7 + 3.596 = 10.596$. So it takes him 10.60 seconds to finish the race, correct to two decimal places.

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(c) A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

(i) Prove that the radius of the snowball is decreasing at a constant rate.

Let t be time. Let r be the radius, A the surface area and V the volume of the snowball. From the Formula and Tables booklet we know that $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$. In particular,

$$\frac{dV}{dr} = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2 = A.$$

Now we are told that the rate of change of volume with respect to time is proportional to the surface area. In other words,

$$\frac{dV}{dt} = kA \tag{1}$$

for some constant k. Clearly $k < 0$ since the volume of the snowball is decreasing as it melts. On the other hand, using the chain rule, we see that

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt} \\ &= A\frac{dr}{dt} \end{aligned} \tag{2}$$

Therefore by combining (1) and (2) , we see that

$$A\frac{dr}{dt} = kA.$$

Now dividing across by A yields

$$\frac{dr}{dt} = k$$

where k is a constant, as required.

(ii) If the snowball loses half of its volume in an hour, how long more will it take for it to melt completely?

Give your answer to the nearest minute.

Let r_0 be the initial radius and let r_2 be the radius after 1 hour.
So the initial volume is $\frac{4}{3}\pi r_0^3$. Therefore after one hour, the volume is $\frac{2}{3}\pi r_0^3$. Therefore

$$\frac{4}{3}\pi r_1^3 = \frac{2}{3}\pi r_0^2.$$

Therefore

$$\left(\frac{r_1}{r_0}\right)^3 = \frac{1}{2}$$

or

$$r_1 = \frac{1}{\sqrt[3]{2}} r_0.$$

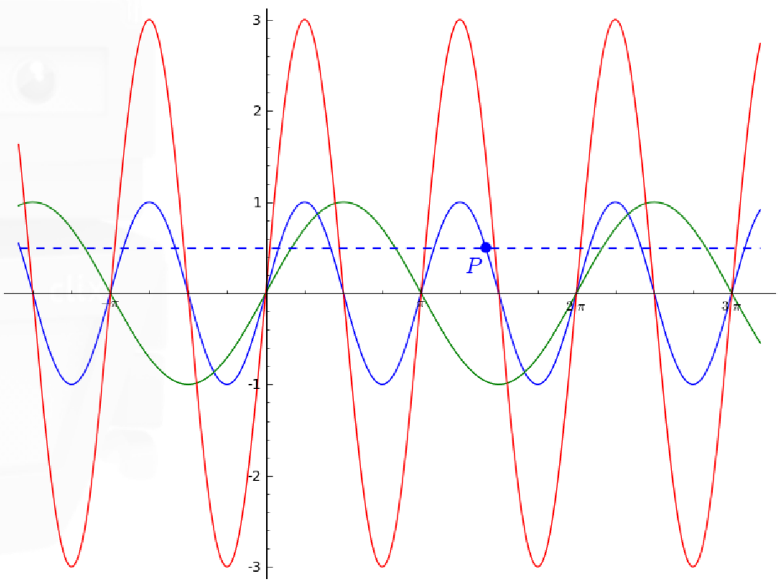
Now the radius is decreasing at a constant rate and we have found that it takes 1 hour for it to decrease from r_0 to $\frac{1}{\sqrt[3]{2}} r_0$. Therefore the rate of change of the radius is $r_0 - \frac{1}{\sqrt[3]{2}} r_0$ units per hour.
Now the snowball will have melted completely when the radius reaches 0. So we calculate the time required to to change from r_0 to 0. This will be

$$\frac{\text{total change}}{\text{rate of change}} = \frac{r_0 - 0}{r_0 - \frac{1}{\sqrt[3]{2}} r_0} = \frac{1}{1 - \frac{1}{\sqrt[3]{2}}} \text{ hours.}$$

This is equal to 4.8473 hours (correct to four decimal places). So it will take 3.8473 more hours (after the 1 already elapsed) for it to melt completely.
Now 3.8473 hours is equal $3.8473 \times 60 = 230.84$.
So, to the nearest minute, it will take a further 231 minutes for the snowball to melt completely.

Question 11 (2014)

The diagram below shows the graph of the function $f : x \mapsto \sin 2x$. The line $2y = 1$ is also shown.



The graph of f is in blue. The graph of g is in green and the graph of h is in red.

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P is a point of intersection of the line $y = \frac{1}{2}$ and the curve $y = \sin 2x$. So the x-co-ordinate of P is a solution of the equation $\sin 2x = \frac{1}{2}$.
Therefore either

$$2x = \frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}$$

or

$$2x = \frac{5\pi}{6} + 2n\pi, n \in \mathbb{Z}.$$

In the first case, we get

$$x = \frac{\pi}{12} + n\pi, n \in \mathbb{Z}$$

and in the second case we get

$$x = \frac{5\pi}{12} + n\pi, n \in \mathbb{Z}.$$

So x is one of the following numbers

$$\ldots, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{25\pi}{12}, \frac{29\pi}{12}, \ldots$$

From the diagram we see that

$$\frac{5\pi}{4} \leq x \leq \frac{3\pi}{2}$$

so the only possibility is that

$$x = \frac{17\pi}{12}.$$

Clearly the y-co-ordinate of P is $\frac{1}{2}$ since it lies on the line $2y = 1$. So the co-ordinates of P are $\left(\frac{17\pi}{12}, \frac{1}{2}\right)$.

Question 12 (2012)

$$f(x) = g(x)$$
$$2x^2 - 3x + 2 = x^2 + x + 7$$
$$x^2 - 4x - 5 = 0$$
$$(x+1)(x-5) = 0$$
$$x = -1, \quad x = 5.$$

$$f(-1) = 7 \Rightarrow (-1, 7)$$
$$f(5) = 37 \Rightarrow (5, 37)$$

(b) Find the area of the region enclosed between the two curves.

$$A = \int_{-1}^{5} \left(g(x) - f(x)\right)dx$$
$$= \int_{-1}^{5} \left(-x^2 + 4x + 5\right)dx$$
$$= \left[\frac{-x^3}{3} + 2x^2 + 5x\right]_{-1}^{5}$$
$$= \left(\tfrac{-125}{3} + 50 + 25\right) - \left(\tfrac{1}{3} + 2 - 5\right)$$
$$= 36.$$

Question 13 (2012)

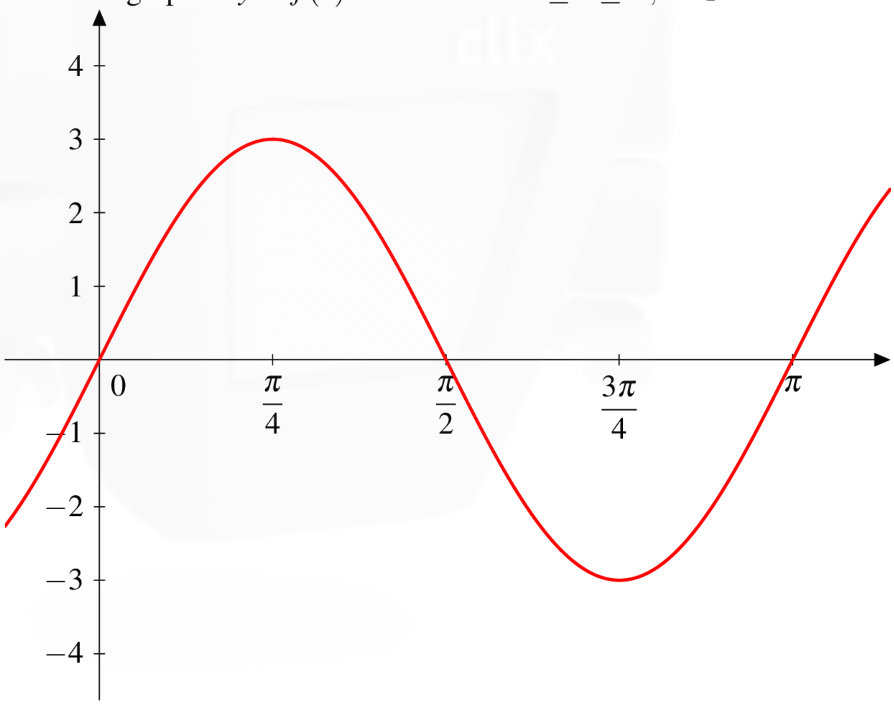
Question 5 **(25 marks)**

The function $f : x \mapsto 3\sin(2x)$ is defined for $x \in \mathbb{R}$.

(a) Complete the table below

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin(2x)$	0	1	0	-1	0
$3\sin(2x)$	0	3	0	-3	0

(b) Draw the graph of $y = f(x)$in the domain $0 \leq x \leq \pi,\ \ x \in \mathbb{R}$.



(c) Write down the range and period of f.

Range $= [-3, 3]$ Period $= \pi$

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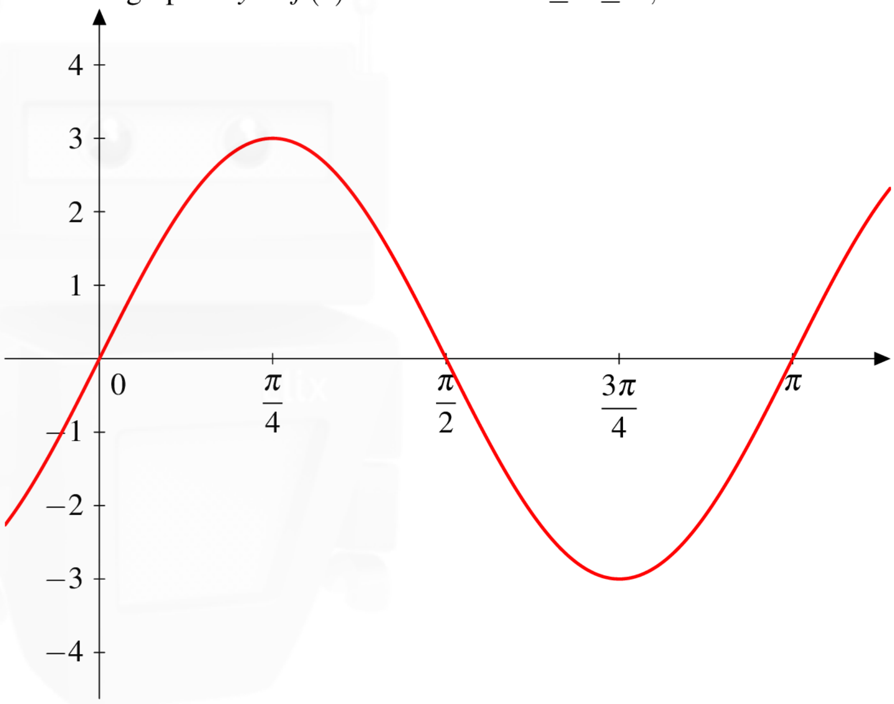

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