MarkingScheme

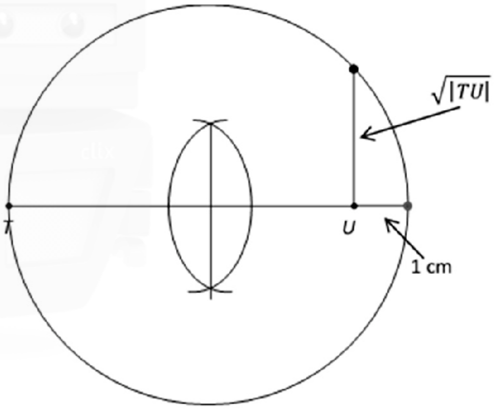
GeometryProofConstructons

Question 1 (2017)

(a)	Proof: $\lvert\angle AEF\rvert = \lvert\angle AED\rvert$ ... $right\ angles$ $\lvert\angle FAE\rvert + \lvert\angle EAD\rvert = 90^o$ $\lvert\angle EAD\rvert + \lvert\angle ADE\rvert = 90^o$ $remaining\ angles\ in\ \Delta AED$ $\therefore \lvert\angle FAE\rvert = \lvert\angle ADE\rvert$ ***or*** $\therefore \lvert\angle AFE\rvert = \lvert\angle DAE\rvert$ $\therefore \Delta AFE\ and\ \Delta DAE\ equiangular$ $\therefore similar$	**Scale 10C (0, 4, 5, 10)** *Low Partial Credit:* • Identifies one angle of same size in each triangle *High Partial Credit:* • Identifies second angle of same size in each triangle • Implies triangles are similar without justifying $\lvert\angle FAE\rvert = \lvert\angle ADE\rvert$
(b)	$\frac{\lvert AD\rvert}{13} = \frac{12}{5}$ $\lvert AD\rvert = 31{\cdot}2$ cm	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* • $\lvert AF\rvert = 13$ • One set of corresponding sides identified, e.g . $\frac{\lvert AD\rvert}{13}$ **or** $\frac{12}{5}$ *High Partial Credit:* • $\frac{\lvert AD\rvert}{13} = \frac{12}{5}$ or equivalent
(c)	$\frac{39}{13} = \frac{\lvert AB\rvert}{12}$ $\lvert AB\rvert = 3 \times 12 = 36$ cm	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* • $\lvert AG\rvert = 39$ • One set of corresponding sides identified *High Partial Credit:* • $\frac{\mathbf{39}}{\mathbf{13}} = \frac{\lvert \boldsymbol{AB}\rvert}{\mathbf{12}}$ or equivalent

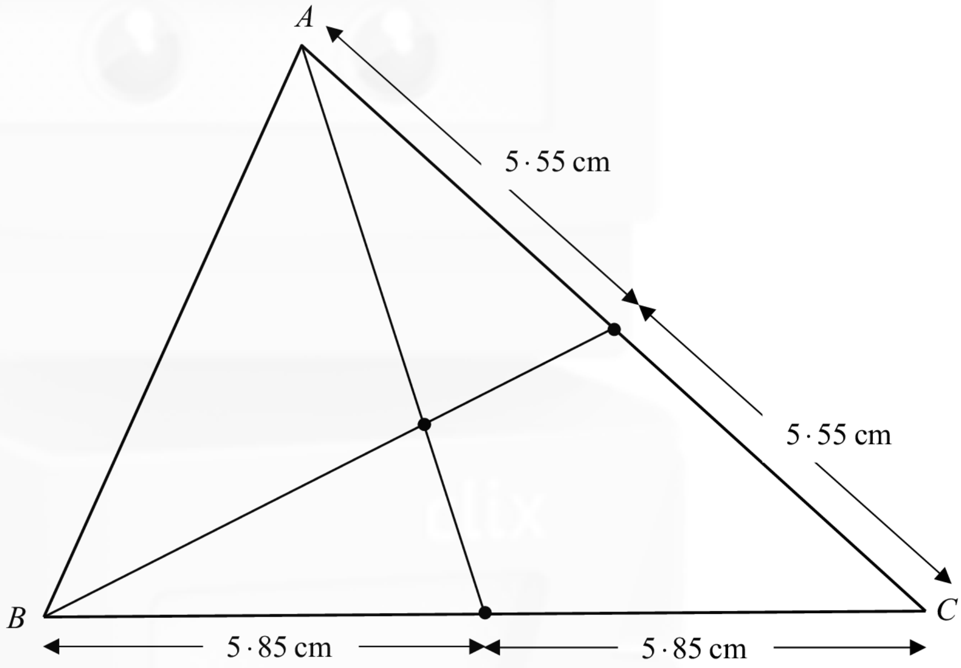
(d)	Area = Area$ABCD$ – AreaΔAFD $-\Delta$AreaABG + AreaΔAFE $= (31{\cdot}2)(36) - \frac{1}{2}(31{\cdot}2)(13)$ $-\frac{1}{2}(36)(15) + \frac{1}{2}(5)(12)$ $= 680{\cdot}4$ cm^2 **or** (method 2) Area = Area$ABCD$ – AreaΔABG – AreaΔAED $= (31{\cdot}2)(36) - \frac{1}{2}(36)(15)$ $-\frac{1}{2}(12)\sqrt{31{\cdot}2^2 - 12^2}$ $= 1123{\cdot}2 - 270 - 172{\cdot}8$ $= 680{\cdot}4$ cm^2 **or** (method 3) Area = AreaΔDCG + AreaΔGED $= \frac{1}{2}(36)(16{\cdot}2) + \frac{1}{2}(27)\sqrt{31{\cdot}2^2 - 12^2}$ $= 291{\cdot}6 + 388{\cdot}8$ $= 680{\cdot}4$ cm^2	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* • One relevant area formulated • Relevant equation for area *GCDE* *High Partial Credit:* • Relevant individual areas found but fails to finish • Area calculated but with one relevant area omitted (except method 3)

Q4	Model Solution – 25 Marks	Marking Notes
(a) **(i)**	$\|\angle ABD\| = \|\angle CBD\| = 90°$.......(i) $\|\angle BDC\| + \|\angle BCD\| = 90°$ ...*angles in triangle sum to* $180°$ $\|\angle ADB\| + \|\angle BDC\| = 90°$ .... *angle in semicircle* $\|\angle ADB\| + \|\angle BDC\| = \|\angle BDC\| + \|\angle BCD\|$ $\|\angle ADB\| = \|\angle BCD\|$........(ii) $\therefore$ *Triangles are equiangular (or similar)* **or** $\|\angle ABD\| = \|\angle CBD\| = 90°$.......(i) $\|\angle DAB\| = \|\angle DAC\|$ *same angle* $\Rightarrow \|\angle ADB\| = \|\angle DCA\|$ *(reasons as above) which is also* $\angle DCB$.........(ii)	Scale 15C (0, 5, 10, 15) *Low Partial Credit* • identifies one angle of same size in each triangle *High Partial Credit* • identifies second angle of same size in each triangle • implies triangles are similar without justifying (ii) in model solution or equivalent
(a) **(ii)**	$\frac{y}{1} = \frac{x}{y}$ $\Rightarrow y^2 = x$ $y = \sqrt{x}$ **or** $\|AD\|^2 + \|DC\|^2 = \|AC\|^2$ $\|AD\| = \sqrt{x^2 + y^2}$ $\|DC\| = \sqrt{y^2 + 1}$ $x^2 + y^2 + y^2 + 1 = (x+1)^2$ $2y^2 = 2x$ $y = \sqrt{x}$ **Or** $\frac{\sqrt{x^2+y^2}}{\sqrt{y^2+1}} = \frac{y}{1} \Rightarrow x^2 + y^2 = y^2(y^2+1)$ $y^4 = x^2 \Rightarrow y^2 = x \Rightarrow y = \sqrt{x}$	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • one set of corresponding sides identified • indicates relevant use of Pythagoras *High Partial Credit* • corresponding sides fully substituted • expression in y^2 or y^4, i.e. fails to finish

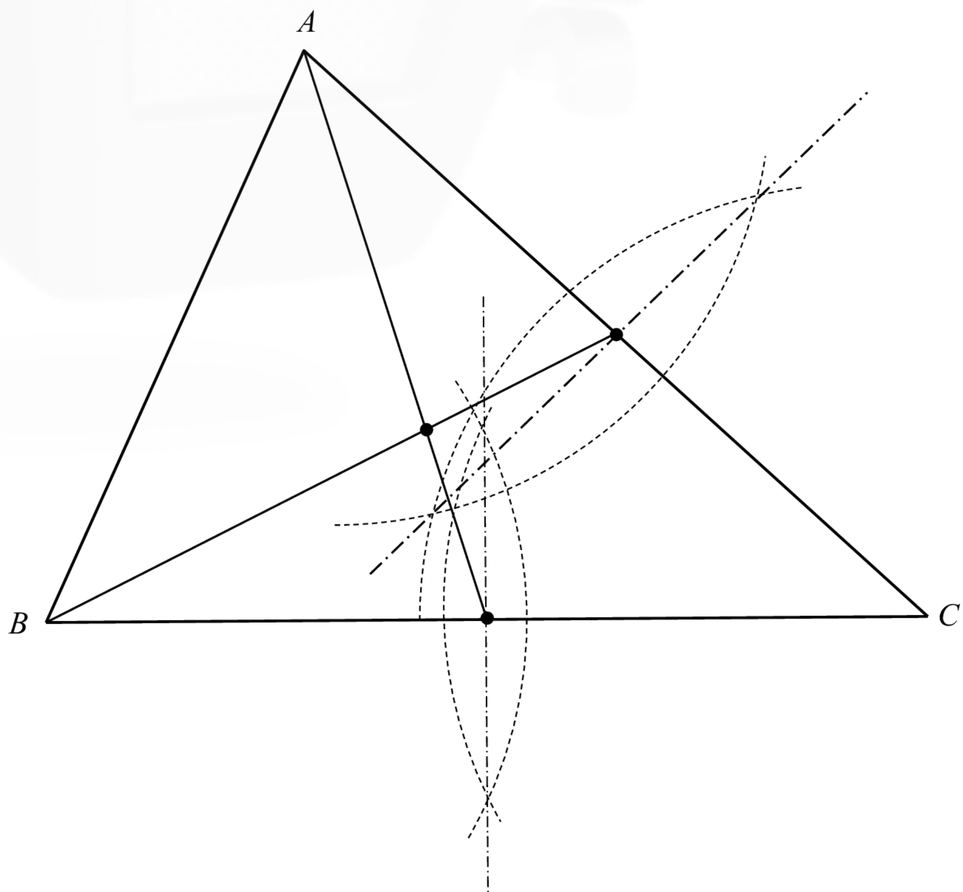
(b)		
	Construction 	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • perpendicular line drawn at *U* or *T* • relevant use of 1 cm length • mid point of incorrect extended segment constructed *High Partial Credit* • correct mid-point constructed

(a) Construct the centroid of the triangle ABC below. Show all construction lines.
(Where measurement is used, show all relevant measurements and calculations clearly.)

$|AC| = 11 \cdot 1$ cm; $|BC| = 11 \cdot 7$ cm

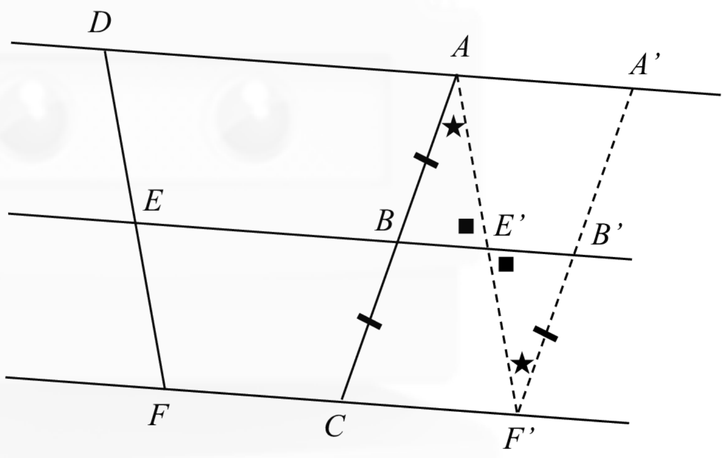


or



(b) Prove that, if three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal line.

Diagram:



Given:	$AD \parallel BE \parallel CF$, as in the diagram, with $\lvert AB \rvert = \lvert BC \rvert$
To Prove:	$\lvert DE \rvert = \lvert EF \rvert$
Construction:	Draw $AE' \parallel DE$, cutting EB at E' and CF at F' Draw $F'B' \parallel AB$, cutting EB at B', as in diagram.

Proof:

$\lvert B'F' \rvert = \lvert BC \rvert$ (opposite sides in a parallelogram)

$= \lvert AB \rvert$ (by assumption)

$\lvert \angle BAE' \rvert = \lvert \angle E'F'B' \rvert$ (alternate angles)

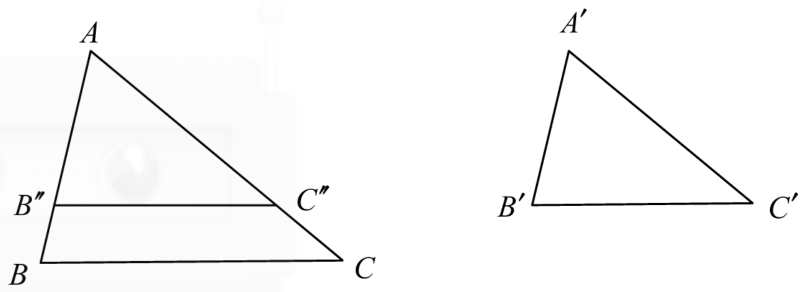
$\lvert \angle AE'B \rvert = \lvert \angle F'E'B' \rvert$ (vertically opposite angles)

$\therefore \Delta ABE'$ is congruent to $\Delta F'B'E'$ (ASA)

$\therefore \lvert AE' \rvert = \lvert F'E' \rvert$

But $\lvert AE' \rvert = \lvert DE \rvert$ and $\lvert F'E' \rvert = \lvert FE \rvert$ (opposite sides in a parallelogram)

$\therefore \lvert DE \rvert = \lvert EF \rvert$



Given: The similar triangles ΔABC and $\Delta A'B'C'$.

To Prove: $\dfrac{|AB|}{|A'B'|} = \dfrac{|BC|}{|B'C'|} = \dfrac{|CA|}{|C'A'|}$.

Construction: Mark B'' on $|AB|$ such that $|AB''| = |A'B'|$.
Mark C'' on $|AC|$ such that $|AC''| = |A'C'|$.
Join $B''\, C''$

Proof: $\Delta AB''C''$ is congruent to $\Delta A'B'C'$. . SAS

$\Rightarrow |\angle AB''C''| = |\angle ABC|$

$\Rightarrow B''C'' \parallel BC$ ... corresponding angles

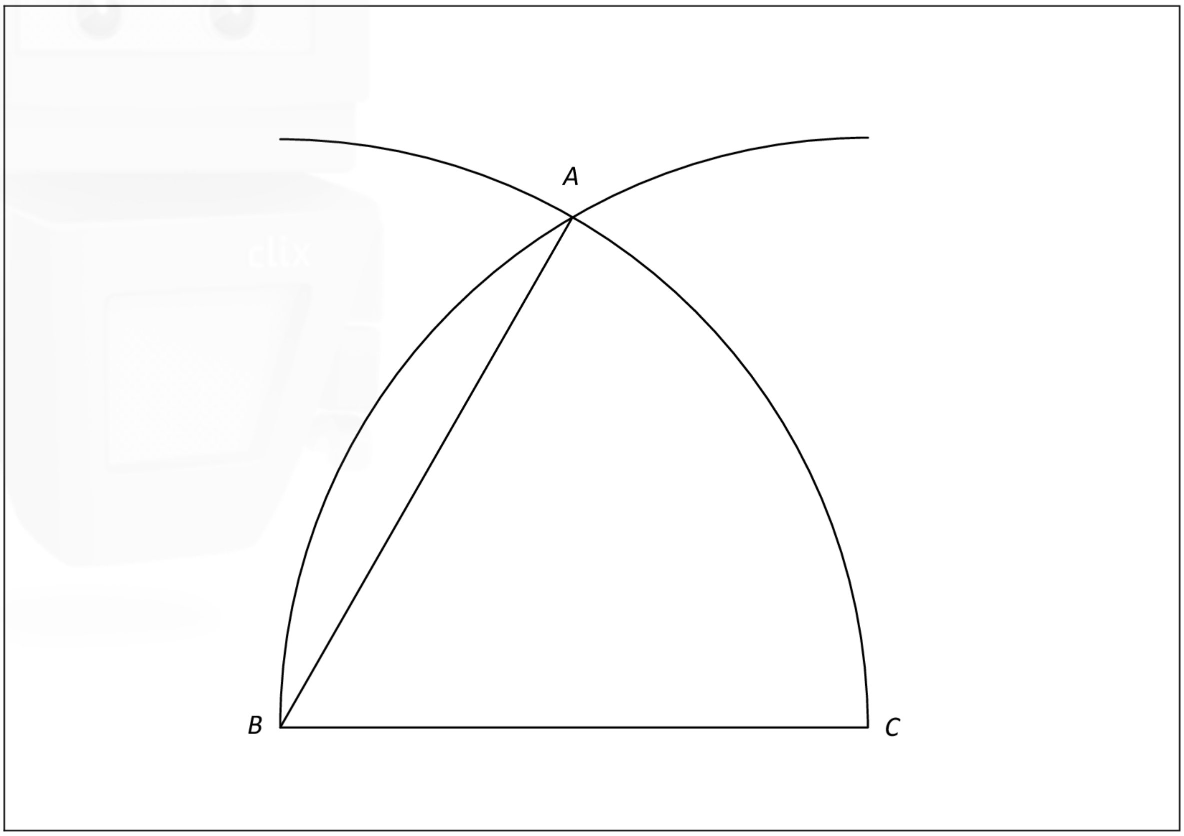
Hence, $\dfrac{|AB|}{|AB''|} = \dfrac{|AC|}{|AC''|}$ ...Theorem

$\Rightarrow \dfrac{|AB|}{|A'B'|} = \dfrac{|AC|}{|A'C'|}$.

Similarly, $\dfrac{|BC|}{|B'C'|} = \dfrac{|AB|}{|A'B'|}$

Hence, $\dfrac{|AB|}{|A'B'|} = \dfrac{|BC|}{|B'C'|} = \dfrac{|CA|}{|C'A'|}$.

(b) Given the line segment $[BC]$, construct, without using a protractor or set square, a point A such that $|\angle ABC| = 60°$. Show your construction lines.



Question 5 (2014)

(a) Name two similar triangles in the diagram above and give reasons for your answer.

ΔADE and ΔBCE are similar
$|\angle EAD| = |\angle BCE|$, on arc BD
$|\angle DEA| = |\angle CEB|$, same angle
$|\angle ADE| = |\angle EBC|$, third angle

Also (i) ΔAXB and ΔDXC are similar, where $AD \cap CB = \{X\}$
and (ii) ΔAXC and ΔBXD are similar, where $AD \cap CB = \{X\}$

(b) Prove that $|EA|.|EB| = |EC|.|ED|$.

ΔADE and ΔBCE are similar.

Hence, $\dfrac{|EA|}{|EC|} = \dfrac{|ED|}{|EB|}$

$\Rightarrow |EA|.|EB| = |EC|.|ED|$

(c) Given that $|EB| = 6\cdot 25$, $|ED| = 5\cdot 94$ and $|CB| = 10$, find $|AD|$.

$$\frac{|ED|}{|EB|} = \frac{|AD|}{|CB|} \Rightarrow \frac{5\cdot 94}{6\cdot 25} = \frac{|AD|}{10}$$

$$\Rightarrow |AD| = \frac{5\cdot 94 \times 10}{6\cdot 25} = 9\cdot 504$$

Question 6 (2014)

Explanation:

To prove a statement by contradiction, we assume that the statement is false and then prove that this assumption contradicts another statement that is known to be true.

Example:

Consider the statement

> The lines $y = x$ and $y = x + 1$ do not intersect.

We can prove this by contradiction as follows.
Assume that that the statement is false. So there is some point (a, b) that lies on both lines. Therefore $b = a$ since the point is on the line $y = x$ and $b = a + 1$ since the point is on the line $y = x + 1$.
Combining these equation, we get

$$a = a + 1$$

which implies that

$$0 = 1.$$

However this contradicts that fact that $0 \neq 1$. Therefore our original assumption is false. So we have proved that the lines $y = x$ and $y = x + 1$ do not intersect.

(i) The circumcentre of a triangle is the point of intersection of

the perpendicular bisectors of the sides of the triangle

(ii) The incentre of a triangle is the point of intersection of

the bisectors of the angles of the triangle

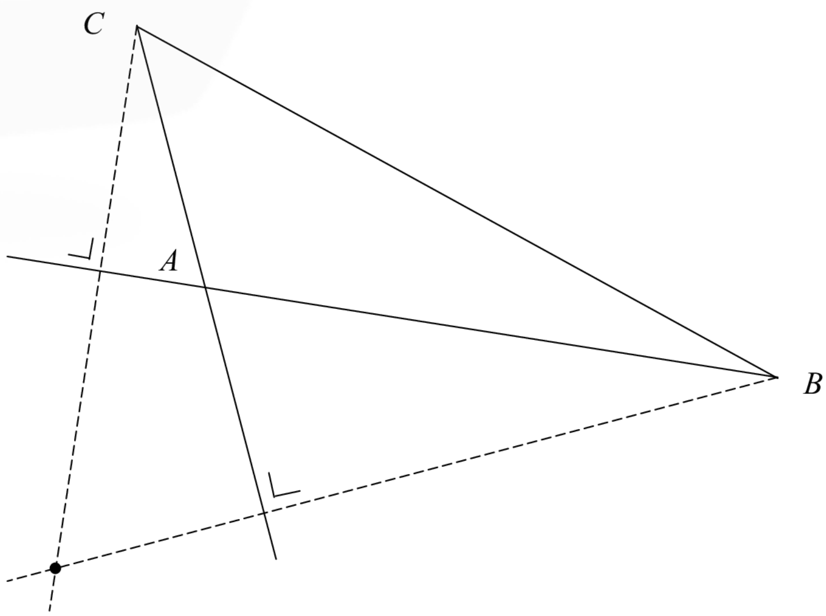
(iii) The centroid is the point of intersection of

the medians of the triangle

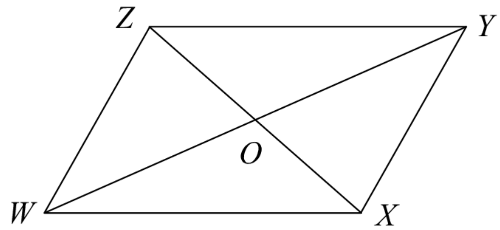
(b) In an equilateral triangle, the circumcentre, the incentre and the centroid are all in the same place. Explain why this is the case.

In an equilateral triangle the medians are perpendicular to the opposite sides and bisect the angles. Therefore, the perpendicular bisectors of the sides, the bisectors of the angles and the median are the same line and intersect at one point.

(c) Construct the orthocentre of the triangle *ABC* below. Show all construction lines clearly.



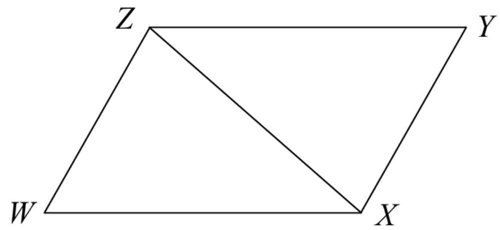
In the quadrilateral $WXYZ$, $WX \parallel ZY$ and $|WX| = |ZY|$
To Prove: $WXYZ$ is a parallelogram.



Join Z to X and Y to W
Proof:
In ΔZOY and ΔOWX ,
$|ZY| = |WX|$
$|\angle ZYO| = |\angle OWX| \ldots ZY \parallel WX$
$|\angle YZO| = |\angle OXW| \ldots ZY \parallel WX$
Hence, ΔZOY and ΔOWX are congruent since AAS
Hence, $|ZO| = |OX|$ and $|YO| = |OW|$
Hence, the diagonals of $WXYZ$ bisect each other $\Rightarrow$ $WXYZ$ is a parallelogram.

OR

In the quadrilateral $WXYZ$, $WX \parallel ZY$ and $|WX| = |ZY|$
To Prove: $WXYZ$ is a parallelogram.



Join Z to X
Proof:
In ΔWXZ and ΔYZX ,
|WX| = |ZY|
$|\angle YZX| = |\angle WXZ| \ldots ZY \parallel WX$
$|ZX| = |ZX| \ldots$ common to both
Hence, ΔWXZ and ΔZXY are congruent since SAS
$\Rightarrow WZ$ and XY parallel
$\Rightarrow$ $WXYZ$ is a parallelogram.

In the parallelogram $ABCD$,
$DE \perp AC$ and $AC \perp BF$ $\Rightarrow$ $DE \parallel BF$.

In the parallelogram $ABCD$,
area of ΔDAC = area of $\Delta ABC \Rightarrow$ $|DE| = |BF|$.

$DE \parallel BF$ and $|DE| = |BF|$ $\Rightarrow$ $EBFD$ is a parallelogram.

Question 6A **(25 marks)**

Explain with the aid of an example what is meant by *proof by contradiction*.

Note: you do not need to provide the full proof in your example. Give a sufficient outline to demonstrate how contradiction is used.

Explanation:

To prove a statement by contradiction, we assume that the statement is false and then prove that this assumption contradicts another statement that is known to be true.

Example:

Consider the statement

The lines $y = x$ and $y = x + 1$ do not intersect.

We can prove this by contradiction as follows.
Assume that that the statement is false. So there is some point (a, b) that lies on both lines.
Therefore $b = a$ since the point is on the line $y = x$ and $b = a + 1$ since the point is on the line $y = x + 1$.
Combining these equation, we get

$$a = a + 1$$

which implies that

$$0 = 1.$$

However this <u>contradicts</u> that fact that $0 \neq 1$. Therefore our original assumption is false.
So we have proved that the lines $y = x$ and $y = x + 1$ do not intersect.

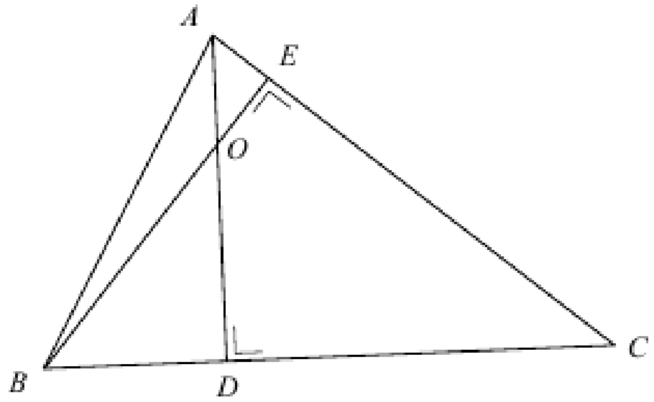
Question 6B **(25 marks)**

ABC is a triangle.

D is a point on BC such that $AD \perp BC$.
E is a point on AC such that $BE \perp AC$.

AD and BE intersect at 0.

Prove that $|\angle DOC| = |\angle DEC|$.



Consider the quadrilateral $DOEC$. We have

$$|\angle CDO| + |\angle OEC| = 90^\circ + 90^\circ = 180^\circ.$$

Therefore $DOEC$ is a cyclic quadrilateral (by the converse of Corollary 5).

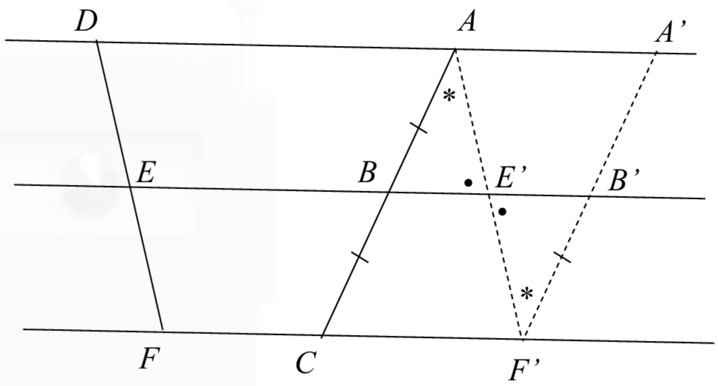
Therefore $\angle DOC$ and $\angle DEC$ are angles standing on the same arc of a circle (the circumcircle of $DOEC$).

Therefore, by Theorem 19

$$|\angle DOC| = |\angle DEC|$$

as required.

Diagram:



Given: $AD \parallel BE \parallel CF$, as in the diagram, with $|AB| = |BC|$

To prove: $|DE| = |EF|$

Construction:

Draw $AE' \parallel DE$, cutting EB at E' and CF at F'
Draw $F'B' \parallel AB$, cutting EB at B', as in the diagram.

Proof:

$|B'F'| = |BC|$ (opposite sides in a parallelogram)

$= |AB|$ (by assumption)

$|\angle BAE'| = |\angle E'F'B'|$ (alternate angles)

$|\angle AE'B| = |\angle F'E'B'|$ (vertically opposite angles)

$\therefore\ \Delta ABE'$ is congruent to $\Delta F'B'E'$ (ASA)

Therefore $|AE'| = |F'E'|$.

But $|AE'| = |DE|$ and $|F'E'| = |FE|$ (opposite sides in a parallelogram)

$\therefore\ |DE| = |EF|$.

$$\frac{|OP_3|}{|OP_2|} = \frac{|OQ_2|}{|OQ_1|} \qquad (P_3Q_2 \parallel P_2Q_1) \quad ^*$$

$$\frac{|OP_2|}{|OP_1|} = \frac{|OQ_2|}{|OQ_1|} \qquad (P_2Q_2 \parallel P_1Q_1)$$

$$\therefore \frac{|OP_3|}{|OP_2|} = \frac{|OP_2|}{|OP_1|}$$

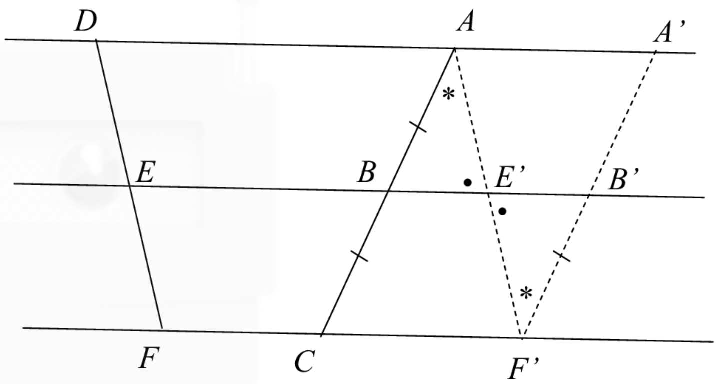
$$\frac{|OP_2|}{|OP_1|} = \frac{|P_2Q_2|}{|P_1Q_1|} \qquad (\textit{Similar triangles})$$

$$\frac{|OP_3|}{|OP_2|} = \frac{|P_3Q_3|}{|P_2Q_2|} \qquad (\textit{Similar triangles})$$

$$\therefore \frac{|P_2Q_2|}{|P_1Q_1|} = \frac{|P_3Q_3|}{|P_2Q_2|}$$

$$\therefore |P_1Q_1|\,|P_3Q_3| = |P_2Q_2|^2$$

Diagram:



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$|\angle AE'B| = |\angle F'E'B'|$ (vertically opposite angles)

$\therefore \Delta ABE'$ is congruent to $\Delta F'B'E'$ (ASA)

Therefore $|AE'| = |F'E'|$.

But $|AE'| = |DE|$ and $|F'E'| = |FE|$ (opposite sides in a parallelogram)

$\therefore |DE| = |EF|$.