MarkingScheme

FunctionsH

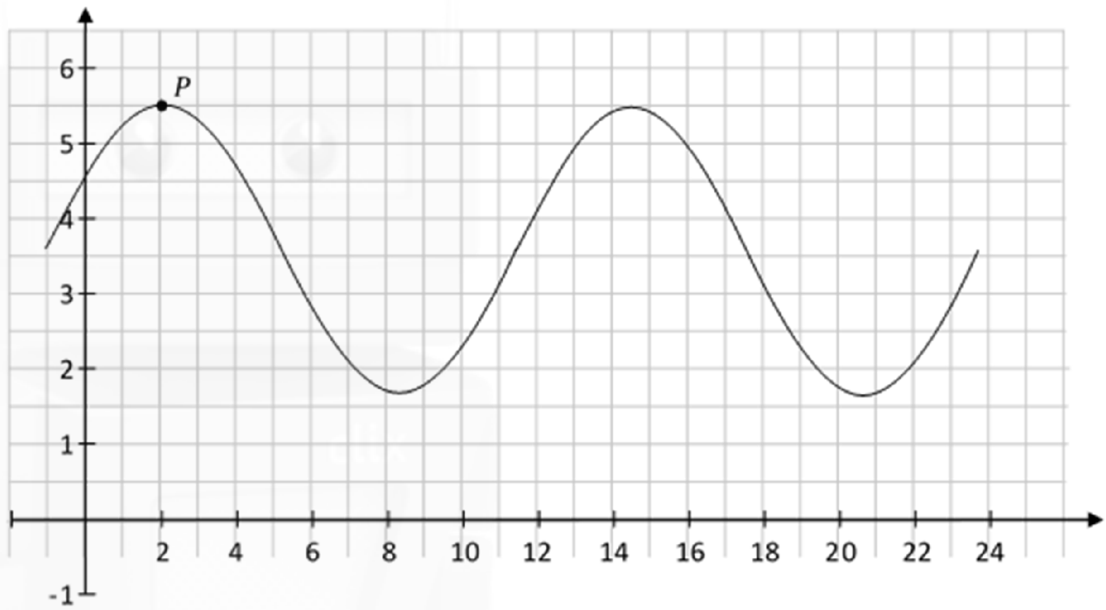
Question 1 (2017)

(a)	$r = \frac{42{\cdot}75}{95} = \frac{9}{20}$ $\quad T_n = ar^{n-1} < 0{\cdot}01$ $95\left(\frac{9}{20}\right)^{n-1} < 0{\cdot}01$ $\left(\frac{9}{20}\right)^{n-1} < \frac{0{\cdot}01}{95}$ $(n-1)\log\left(\frac{9}{20}\right) < \log\left(\frac{0.01}{95}\right)$ $(n-1) > \frac{\log\left(\frac{0.01}{95}\right)}{\log\left(\frac{9}{20}\right)}$ (since $\log\left(\frac{9}{20}\right)$ is negative) $n - 1 > 11{\cdot}47$ $n > 12{\cdot}47$ 12^{th} day	**Scale 15D (0, 5, 8, 12, 15)** *Low Partial Credit:* • r found • T_n of a GP with some substitution *Mid Partial Credit:* • Inequality in n written *High Partial Credit:* • Inequality in n simplified (log handled) *Full Credit:* • Accept $n = 12{\cdot}47$

Question 2 (2017)

(a)	$Se^{\cdot 1(0)} \times 10^6 = 1100000$ $S = 1{\cdot}1$	**Scale 10B (0, 4, 10)** *Partial Credit* • equation in S with substitution
(b)	$p(5) = 1{\cdot}1e^{0{\cdot}1(5)} \times 10^6$ $= 1{\cdot}813593 \times 10^6$ $= 1813593$	**Scale 10B (0, 4, 10)** *Partial Credit* • substitution into formula for $p(5)$
(c)	$p(6) = 1{\cdot}1e^{0{\cdot}6} \times 10^6$ $p(5) = 1{\cdot}1e^{0{\cdot}5} \times 10^6$ $p(6) - p(5) = (1{\cdot}1e^{0{\cdot}6} - 1{\cdot}1e^{0{\cdot}5}) \times 10^6$ $= 0{\cdot}1907372 \times 10^6$ $= 190737$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • substitution into formula for $p(6)$ • use of $p(5)$ from previous part • $p(6) - p(5)$ written or implied *High partial Credit* • Formulates $p(6) - p(5)$ with some substitution

(d)	$$q(t) = 3{\cdot}9e^{kt} \times 10^6$$ $$3709795 = 3{\cdot}9e^{k} \times 10^6$$ $$\frac{3{\cdot}709795}{3{\cdot}9} = e^k$$ $$\log_e \frac{3{\cdot}709795}{3{\cdot}9} = k$$ $$k = -0{\cdot}0499 = -0{\cdot}05$$	**Scale 15C (0, 5, 10, 15)** *Low Partial Credit* • Either substitution into formula for k • Verifies k value only. *High Partial Credit* • relevant equation in k
(e)	$$p(t) = q(t)$$ $$1{\cdot}1e^{0{\cdot}1t} \times 10^6 = 3{\cdot}9e^{-0{\cdot}05t} \times 10^6$$ $$1{\cdot}1e^{0{\cdot}1t} = 3{\cdot}9e^{-0{\cdot}05t}$$ $$\frac{e^{0{\cdot}1t}}{e^{-0{\cdot}05t}} = \frac{3{\cdot}9}{1{\cdot}1}$$ $$e^{0{\cdot}15t} = \frac{39}{11}$$ $$\ln\frac{39}{11} = 0{\cdot}15t$$ $t = 8{\cdot}44$ years In 2018 both populations equal	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit* • $p(t) = q(t)$ written or implied *High Partial Credit* • relevant equation in t
(f)	$$\frac{1}{15}\int_0^{15} 3{\cdot}9e^{-0{\cdot}05t} \times 10^6 dt$$ $$\frac{1}{15}\left[\frac{3{\cdot}9}{-0{\cdot}05}e^{-0{\cdot}05(15)} - \frac{3{\cdot}9}{-0{\cdot}05}e^{-0{\cdot}05(0)}\right]$$ $$\times 10^6$$ $$2{\cdot}743694 \times 10^6$$ $$2743694$$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • integral formulated (with limits) *High Partial Credit:* • integration with full substitution
(g)	$$q(t) = 3.9e^{-0.05t} \times 10^6$$ $$q'(t) = -0{\cdot}05(3{\cdot}9e^{-0{\cdot}05t} \times 10^6)$$ $$q'(8) = -0{\cdot}05\left(3{\cdot}9e^{-0{\cdot}05(8)} \times 10^6\right)$$ $$= -130712$$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit* • $q'(t)$ *High Partial Credit* • $q'(t)$ fully substituted

(a)		
(a)		**Scale 20C (0, 10, 18, 20)** *Low Partial Credit:* • Vertical axis drawn • Horizontal axis drawn. *High Partial Credit:* • Horizontal axis fully scaled and positioned **OR** • Vertical axis fully scaled Use relevant portions of axes **Note:** P can be on vertical axis

(b) **(i)**	$f(t) = a + b\cos ct$ $Range: \; [(a+b),(a-b)]$ $a + b = 5{\cdot}5 \quad a - b = 1{\cdot}7$ $a = 3{\cdot}6 \quad b = 1{\cdot}9$	**Scale 10C (0, 5, 8, 10)** *Low Partial Credit:* • one equation in a and b • Range in terms of a and b *High Partial Credit:* • a or b found **Note:** Accept correct answer without work
(b) **(ii)**	Time between two successive high tides is: $12\frac{34}{60}$ hours $\text{period} = 12\frac{34}{60}$ $\text{period} = \frac{2\pi}{c}$ $c = \frac{2\pi}{12\frac{34}{60}} = 0{\cdot}4999 = 0{\cdot}5$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • Period identified **or** $\frac{2\pi}{c}$ **or** $12{\cdot}34$ *High Partial Credit:* • equation in c with some substitution
(c)	$5{\cdot}2 = a + b\cos ct$ $5{\cdot}2 = 3{\cdot}6 + 1{\cdot}9\cos 0{\cdot}5t$ $0{\cdot}5t = \cos^{-1}\frac{1{\cdot}6}{1{\cdot}9} = 0{\cdot}569621319$ $0{\cdot}5t = 0{\cdot}5696$ $t = 1{\cdot}139$ hours (before and after high tide at 14:34) Time = 1 hour 8 minutes Times: (14:34) ± 1 hour 8 min ⇒ 13:26 and 15:42	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • equation with some substitution *High Partial Credit:* • solution for t **Note***:* Low partial at most if formula not used

Question 4 (2016)

(b) **(ii)**	$f(x) = 3x - 2$ $f^{-1}(x) = \frac{x+2}{3}$	Scale 5B (0, 2, 5) *Partial Credit* • any relevant transpose

Q7	Model Solution – 40 Marks	Marking Notes
(a) **(i)**	$v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dr} = 4\pi r^2$ $\frac{dv}{dt} = 250 \text{ cm}^3/\text{s}$ $\frac{dr}{dt} = \frac{dr}{dv}.\frac{dv}{dt} = \frac{1}{4\pi r^2}.250$ $\frac{dr}{dt} = \frac{250}{4\pi 400} = \frac{5}{32\pi} \text{ cm/s}$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • work towards $\frac{dv}{dr}$ or $\frac{dv}{dt}$ or $\frac{dr}{dt}$ *High Partial Credit* • correct expression for $\frac{dr}{dt}$
(ii)	$a = 4\pi r^2 \Rightarrow \frac{da}{dr} = 8\pi r$ $\frac{da}{dt} = \frac{da}{dr}.\frac{dr}{dt} = 8\pi r.\frac{5}{32\pi}$ $= \frac{5(20)}{4}$ $= 25 \text{ cm}^2/\text{s}$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • work towards $\frac{da}{dr}$ or $\frac{da}{dt}$ *High Partial Credit* • correct expression for $\frac{da}{dt}$
(b) **(i)**	$-x^2 + 10x = 0$ $x(-x + 10) = 0$ $x = 0$ or $x = 10$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • quadratic equation formed • gets $x = 0$ only *High Partial Credit* • quadratic factorised **Note:** $f'(x) = 0 \Rightarrow 2x - 10 = 0 \Rightarrow x = 5$ *merits 0 marks*
(ii)	$\frac{1}{10-0}\int_0^{10}(-x^2 + 10x)dx$ $= \frac{1}{10}\left[\frac{-x^3}{3} + 5x^2\right]_0^{10}$ $= \frac{1}{10}\left[\left(\frac{-1000}{3} + 500\right) - 0\right]$ $= \frac{-100}{3} + 50 = \frac{50}{3} \text{ m}$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • integration set up *High Partial Credit* • correct integration with some substitution

Q8	Model Solution – 55 Marks	Marking Notes
(a) **(i)**	$f(x) = -0{\cdot}274x^2 + 1{\cdot}193x + 3{\cdot}23$ $f'(x) = -0{\cdot}548x + 1{\cdot}193 = 0$ $x = 2{\cdot}177 \text{ m}$ $f(2{\cdot}177) = -0{\cdot}274(2{\cdot}177)^2 + 1{\cdot}193(2{\cdot}177) + 3{\cdot}23$ $= -1{\cdot}2986 + 2{\cdot}5972 + 3{\cdot}23$ $= 4{\cdot}529 \text{ m}$ *or* $-0{\cdot}274(x^2 - \frac{1193}{274}x - \frac{1615}{137})$ $-0{\cdot}274(x - \frac{1193}{548})^2 + 4.5285$ Max Height $= 4{\cdot}529$ m	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • any correct differentiation • effort made at completing square • trial and error with more than one value of x tested *High Partial Credit* • x value correct **Note:** *if correct answer by trial and error, must show points on each side of max point to be lower to earn full credit*
(ii)	$\tan\theta = -0{\cdot}548(4{\cdot}5) + 1{\cdot}193$ $\tan\theta = -1{\cdot}273$ $\theta = 51{\cdot}8° = 52°$	Scale 5B (0, 2, 5) *Partial Credit* • tan **Note:** *right angled triangles may appear in diagram given in equation*
(iii)	Map $A \rightarrow C$ $(-0{\cdot}5, 2{\cdot}565) \rightarrow (0, 2)$ $2{\cdot}177 - (-0{\cdot}5) = 2{\cdot}677$ $4{\cdot}529 - 0{\cdot}565 = 3{\cdot}964$ $(2{\cdot}177, 4{\cdot}529) \rightarrow (2{\cdot}677, 3{\cdot}964)$	Scale 5B (0, 2, 5) *Partial Credit* • $(-0{\cdot}5, 2{\cdot}565) \rightarrow (0, 2)$

(iv)	$$g(x) = ax^2 + bx + c$$ $$C(0,2) \in g(x) => c = 2$$ $B(4\cdot5, 3\cdot05) \epsilon\, g(x)$ $3\cdot05 = \mathrm{a}(4\cdot5)^2 + \mathrm{b}(4\cdot5) + 2$ $\Longrightarrow 20\cdot25a + 4\cdot5b = 1\cdot05$ ... **(i)** $g'(x) = 2ax + b = 0$ $\Longrightarrow 2a(2\cdot677) + b = 0$ $5\cdot354a + b = 0$ ... **(ii)** From **(i)** and **(ii)** $a = -0\cdot273$ $b = 1\cdot462$ $$g(x) = -0\cdot273x^2 + 1\cdot462x + 2$$ *[Note: a third equation that could be used is* $3\cdot964 = a(2\cdot677)^2 + b(2\cdot677) + 2$ ... **(iii)**]	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • c value found • relevant equation in *a*, *b* and/or *c* *Mid Partial Credit* • formulated correctly any two equations *High Partial Credit* • formulated correctly any three equations **Note**: $ax^2 + bx + c$ *not in an equation merits 0 marks*
	Or	**Or**
	Equation of parabola with vertex (h, k): $$g(x) = a(x-h)^2 + k$$ $C(0,2)$ on curve: $(h,k) = (2\cdot677, 3.964)$ $$2 = a(-2\cdot677)^2 + 3\cdot964$$ $$-1\cdot964 = a(7\cdot166329)$$ $$a = -0\cdot27405 = -0\cdot274$$ Parabola: $$g(x) = -0\cdot274[(x - 2\cdot677)^2] + 3\cdot964$$ **or** $$g(x) = f(x - 0\cdot5) - 0\cdot565$$ $$g(x) = -0\cdot274(x-0\cdot5)^2 + 1\cdot193(x - 0\cdot5) + 3\cdot23 - 0\cdot565$$ $$g(x) = -0\cdot274x^2 + 1\cdot467x + 2$$	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • equation of curve • use of C *Mid Partial Credit* • using peak value *High Partial Credit* • value of *a* found

(b) **(i)**	G_5 =Female,Male,Female,Female,Male	Scale 5B (0, 2, 5) *Partial Credit* • one correct entry
(b) **(ii)**	$G_6 = G_5 + G_4 = 5 + 3 = 8$ $G_7 = G_6 + G_5 = 8 + 5 = 13$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • $G_6 = G_5 + G_4$ • $G_7 = G_6 + G_5$ • G_7 or G_6 correct • 8 and/or 13 without work *High Partial Credit* • correct substitution in both
(b) **(iii)**	$G_3 = \frac{(1+\sqrt{5})^3 - (1-\sqrt{5})^3}{2^3\sqrt{5}} = 2$ $(1+\sqrt{5})^3 = \left(1 + 3\sqrt{5} + 3\sqrt{5}^2 + \sqrt{5}^3\right)$ $= 16 + 8\sqrt{5}$ $(1-\sqrt{5})^3 = \left(1 - 3\sqrt{5} + 3\sqrt{5}^2 - \sqrt{5}^3\right)$ $= 16 - 8\sqrt{5}$ $G_3 = \frac{6\sqrt{5} + 2\sqrt{5}^3}{8\sqrt{5}}$ $= \frac{6 + 2\sqrt{5}^2}{8} = \frac{16}{8} = 2 \quad \text{Q.E.D.}$	Scale 5B (0, 2, 5) *Partial Credit* • some correct substitution • using approximate value for $\sqrt{5}$ • $G_3 = 2$ • some effort at cubing **Note:** *use of* $\sqrt{5}$ *as approximation, even if rounded off to 2 at end of work merits at most Partial Credit*

(a) **(i)** Show that $d = 0$.

$$f(x) = 0\cdot0024x^3 + 0\cdot018x^2 + cx + d$$
$$f(0) = 0 + 0 + 0 + d = 0 \quad \Rightarrow \quad d = 0$$

(ii) Using the fact that P is the point $(-5,\ 0\cdot15)$, or otherwise, show that $c = 0$.

$$f(x) = 0\cdot0024x^3 + 0\cdot018x^2 + cx$$
$$f(-5) = 0\cdot0024(-5)^3 + 0\cdot018(-5)^2 + c(-5) = 0\cdot15$$
$$\Rightarrow 0\cdot15 - 5c = 0\cdot15 \quad \Rightarrow \quad c = 0$$

or

The plane lands horizontally at O $\Rightarrow$ $f'(x) = 0$ when $x = 0$

$$f'(x) = 0\cdot0072x^2 + 0\cdot036x + c$$
$$f'(0) = 0 + 0 + c = 0$$
$$\Rightarrow c = 0$$

(b) **(i)** Find the value of $f'(x)$, the derivative of $f(x)$, when $x = -4$.

$$f(x) = 0\cdot0024x^3 + 0\cdot018x^2 + cx + d$$
$$f'(x) = 0\cdot0072x^2 + 0\cdot036x$$
$$f'(-4) = 0\cdot0072(-4)^2 + 0\cdot036(-4)$$
$$= -0\cdot0288$$

(ii) Use your answer to part **(b) (i)** above to find the angle at which the plane is descending when it is 4 km from touchdown. Give your answer correct to the nearest degree.

$\tan\theta = f'(x) = -0\cdot 0288 \Rightarrow \theta = 178\cdot 3503°$
Angle of descent $\alpha = 1\cdot 6497° = 2°$

(c) Show that $(-2\cdot 5,\ 0\cdot 075)$ is the point of inflection of the curve $y = f(x)$.

$f'(x) = 0\cdot 0072x^2 + 0\cdot 036x$
$f''(x) = 0\cdot 0144\ x + 0\cdot 036 = 0$
$\Rightarrow x = -2\cdot 5$
$f(x) = 0\cdot 0024x^3 + 0\cdot 018x^2$
$f(-2\cdot 5) = 0\cdot 0024(-2\cdot 5)^3 + 0\cdot 018(-2\cdot 5)^2$
$= -0\cdot 0375 + 0\cdot 1125 = 0\cdot 075$
$(-2\cdot 5, 0\cdot 075)$

(d) **(i)** If (x, y) is a point on the curve $y = f(x)$, verify that $(-x-5, -y+0\cdot 15)$ is also a point on $y = f(x)$.

$f(x) = 0\cdot 0024x^3 + 0\cdot 018x^2$
$f(-x-5) = 0\cdot 0024(-x-5)^3 + 0\cdot 018(-x-5)^2$
$= 0\cdot 0024\left(-x^3 - 15x^2 - 75x - 125\right) + 0\cdot 018\left(x^2 + 10x + 25\right)$
$= -0\cdot 0024x^3 - 0\cdot 018x^2 + 0x + 0\cdot 15$
$= -y + 0\cdot 15$

(ii) Find the image of $(-x-5, -y+0\cdot 15)$ under symmetry in the point of inflection.

Point: $(-x-5,\ -y+0\cdot 15)$
Point of inflection: $(-2\cdot 5,\ 0\cdot 075)$

Change in x value: $(-2\cdot 5) - (-x-5) = x + 2\cdot 5$
Change in y value: $0\cdot 075 - (-y + 0\cdot 15) = y - 0\cdot 075$

Image of point of inflection:
x value: $-2\cdot 5 + (x + 2\cdot 5) = x$
y value: $0\cdot 075 + (y - 0\cdot 075) = y$
$\Rightarrow (x, y)$ is image

or

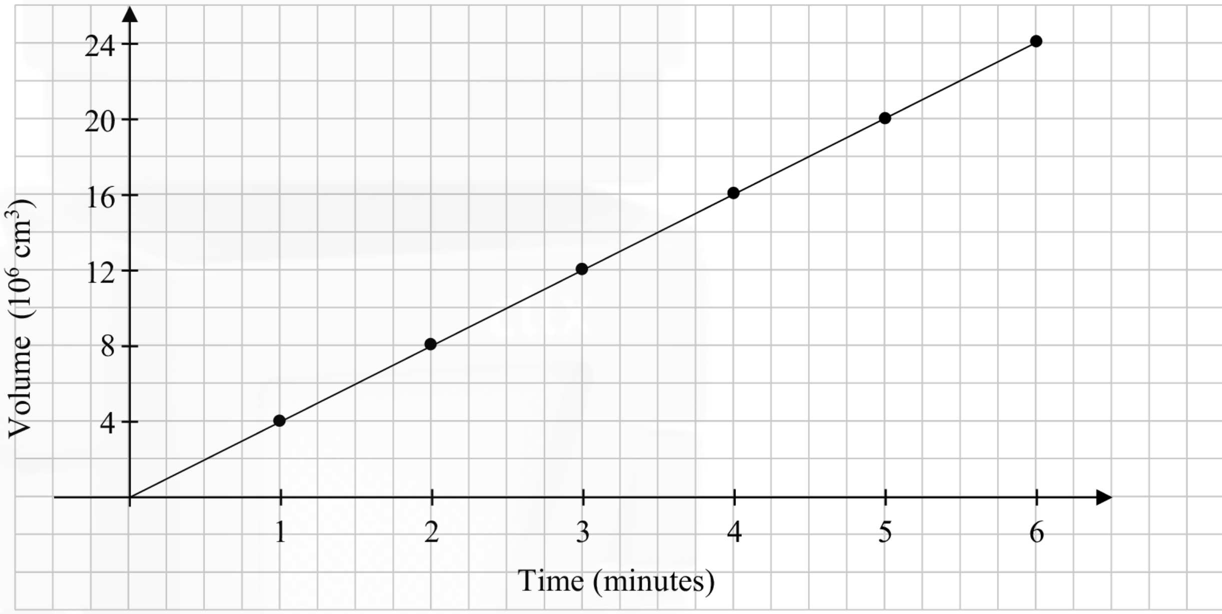
Let (x, y) be the image.

$$\left(\frac{-x-5+x}{2}, \frac{-y+0\cdot 15+y}{2}\right)=(-2\cdot 5, 0\cdot 075)$$, the point of inflection

(a) **(i)** Complete the table below to show the total volume of oil on the water after each of the first 6 minutes of the oil-spill.

Time (minutes)	1	2	3	4	5	6
Volume (10^6 cm ^{3})	4	8	12	16	20	24

(ii) Draw a graph to show the total volume of oil on the water over the first 6 minutes.



(iii) Write an equation for $V(t)$, the volume of oil on the water, in cm^3, after t minutes.

Line, slope 4×10^6, passing through (0, 0).
$V(t)=\left(4\times10^6\right)t$

(b) The spilled oil forms a circular oil slick **1 millimetre** thick.
(i) Write an equation for the volume of oil in the slick, in cm^3, when the radius is r cm.

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi r^2 (0\cdot1) \\ &= 0\cdot1\pi r^2 \text{ cm}^3 \end{aligned}$$

(ii) Find the rate, in cm per minute, at which the radius of the oil slick is increasing when the radius is 50 m.

$$\frac{dV}{dt} = 4\times10^6 \text{ cm}^3 \text{ per minute}$$

$$V = \pi r^2 h \text{ where } h = 0\cdot1 \text{ cm}$$

$$\frac{dV}{dr} = 2\pi rh$$

$$\frac{dV}{dr} = 0\cdot 2\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dV}\frac{dV}{dt} = \frac{1}{0\cdot 2\pi r}\times 4\times10^6$$

$$= \frac{4\times10^6}{0\cdot 2\pi(5000)} = 1273\cdot3 \text{ cm per minute}$$

(c) Show that the area of water covered by the oil slick is increasing at a constant rate of 4×10^7 cm^2 per minute.

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r\frac{4\times10^6}{0\cdot 2\pi r} = 4\times10^7 \text{ cm}^2 \text{ per minute}$$

or

$$(0\cdot1)\pi r^2 = (4\times10^6)t$$

$$\Rightarrow A = \pi r^2 = (4\times10^7)t$$

$$\frac{dA}{dt} = 4\times10^7$$

(d) The nearest land is 1 km from the point at which the oil-spill began. Find how long it will take for the oil slick to reach land. Give your answer correct to the nearest hour.

$$A = \pi r^2 = \pi(10^5)^2 = \pi 10^{10} \text{ cm}^2$$

$$t = \frac{\pi 10^{10}}{4\times10^7} = \frac{\pi 10^3}{4} = 785\cdot398 \text{ minutes}$$

$$= 13\cdot09 = 13 \text{ hours}$$

(a) Find the length of the day in Galway on June 5th (76 days after March 21st). Give your answer in hours and minutes, correct to the nearest minute.

$$f(t) = 12\cdot 25 + 4\cdot 75\sin\left(\frac{2\pi}{365}t\right)$$
$$f(76) = 12\cdot 25 + 4\cdot 75\sin\left(\frac{2\pi}{365}\times 76\right)$$
$=12\cdot 25+4\cdot 587 = 16\cdot 837 = 16$ hours 50 minutes

(b) Find a date on which the length of the day in Galway is approximately 15 hours.

$$f(t) = 12\cdot 25 + 4\cdot 75\sin\left(\frac{2\pi}{365}t\right) = 15$$
$$\Rightarrow \sin\left(\frac{2\pi}{365}t\right) = 0\cdot 578947$$
$$\Rightarrow \frac{2\pi}{365}t = 0\cdot 6174371$$
$$\Rightarrow t = 35\cdot 87$$
36 days after March 21 is April 26.

(c) Find $f'(t)$, the derivative of $f(t)$.

$$f(t) = 12\cdot 25 + 4\cdot 75\sin\left(\frac{2\pi}{365}t\right)$$
$$f'(t) = 0 + 4\cdot 75\times\frac{2\pi}{365}\cos\left(\frac{2\pi}{365}t\right)$$
$$= \frac{9\cdot 5\pi}{365}\cos\left(\frac{2\pi}{365}\right)t$$

(d) Hence, or otherwise, find the length of the longest day in Galway.

$f(t)$ is a maximum when $\sin\left(\frac{2\pi}{365}t\right)$ is a maximum of 1.

$t = 12\cdot 25 + 4\cdot 75 = 17$ hours

or

$$f'(t) = 0 \Rightarrow \frac{9\cdot 5\pi}{365}\cos\left(\frac{2\pi}{365}t\right) = 0$$
$$\Rightarrow \cos\left(\frac{2\pi}{365}t\right) = 0$$
$$\Rightarrow \frac{2\pi}{365}t = \frac{\pi}{2}$$
$$\Rightarrow t = \frac{365}{4} = 91\cdot 25$$
$$f(91\cdot 25) = 12\cdot 25 + 4\cdot 75\sin\left(\frac{2\pi}{365}\times 91\cdot 25\right)$$
$$= 12\cdot 25 + 4\cdot 75\sin\frac{\pi}{2}$$
$$= 17 \text{ hours}$$

(e) Use integration to find the average length of the day in Galway over the six months from March 21st to September 21st (184 days). Give your answer in hours and minutes, correct to the nearest minute.

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{184}\int_0^{184}\left(12\cdot 25 + 4\cdot 75\sin\left(\frac{2\pi}{365}t\right)\right)dt$$
$$= \frac{1}{184}\left[12\cdot 25t - 4\cdot 75 \times \frac{365}{2\pi}\cos\left(\frac{2\pi}{365}t\right)\right]_0^{184}$$
$$= \frac{1}{184}\left[(2254 + 275\cdot 843) - (0 - 275\cdot 934)\right]$$
$$= \frac{1}{184}[2805\cdot 777]$$
$$= 15\cdot 24879$$
$$= 15 \text{ hours } 15 \text{ minutes}$$

(a) Show that $-k$ is a root of f.

Substituting $-k$ for x we obtain

$$\begin{aligned} f(-k) &= (-k)^3 + (1-k^2)(-k) + k \\ &= -k^3 - k + k^3 + k \\ &= 0 \end{aligned}$$

Therefore $-k$ is a root of f.

(b) Find, in terms of k, the other two roots of f.

Since $-k$ is a root of f we know, by the Factor Theorem, that $(x+k)$ is a factor of $f(x)$. Now we carry out long division to find the other factor.

$$
\begin{array}{r|cccccccc}
 & x^2 & - & kx & + & 1 & & \\
\hline
x+k & x^3 & + & 0x^2 & + & (1-k^2)x & + & k \\
 & x^3 & + & kx^2 & & & & \\
\cline{2-4}
 & & - & kx^2 & + & (1-k^2)x & & \\
 & & - & kx^2 & - & k^2x & & \\
\cline{3-6}
 & & & & & x & + & k \\
 & & & & & x & + & k \\
\cline{6-8}
 & & & & & & & 0
\end{array}
$$

So

$$x^3+(1-k^2)x+k=(x+k)(x^2-kx+1).$$

Therefore the other two roots of f are solutions of the equation

$$x^2-kx+1=0.$$

Using the quadratic formula we get

$$x=\frac{k\pm\sqrt{k^2-4}}{2}.$$

So the other two roots of f are

$$\frac{k+\sqrt{k^2-4}}{2} \text{ and } \frac{k-\sqrt{k^2-4}}{2}.$$

(c) Find the set of values of k for which f has exactly one real root.

From the solution to part (b), we see that f has exactly one real root if and only if $k^2-4<0$. This is equivalent to $k^2<4$ or

$$-2<k<2$$

First we calculate the derivative of f. So

$$f'(x) = 3x^2 - 5(2x) + 3(1) + 0 = 3x^2 - 10x + 3$$

Now we solve $f'(x) = 0$ to find the stationary points of f. Now

$$3x^2 - 10x + 3 = 0 \Leftrightarrow x = \frac{10 \pm \sqrt{10^2 - 36}}{2(3)} \Leftrightarrow x = \frac{10 \pm 8}{6}.$$

So the stationary points of $f(x)$ are given by

$$x = 3 \text{ or } x = \frac{1}{3}.$$

Now both of these lie inside A. So the maximum/minimum value of $f(x)$ occurs at one $x = 0, \frac{1}{3}, 3$ or 5. Now

$$\begin{aligned} f(0) &= 5 \\ f(\frac{1}{3}) &= \frac{1}{27} - \frac{5}{9} + 1 + 5 = \frac{148}{27} \\ f(3) &= 27 - 5(9) + 3(3) + 5 = -4 \\ f(5) &= 125 - 5(25) + 3(5) + 5 = 20 \end{aligned}$$

Therefore

The maximum value of f is 20

and

The minimum value of f is -4

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(b) State whether f is injective. Give a reason for your answer.

f is NOT injective.
Reason: We see from our solution to part (a) that $f(\frac{1}{3}) > 0 > f(3)$ - in particular the sign of $f(x)$ changes between $x = \frac{1}{3}$ and $x = 3$. This means that there is some $a < 3$ such that $f(a) = 0$
Similarly, since $f(3) < 0 < f(5)$, there is some $b > 3$ such that $f(b) = 0$.
So $a \neq b$ since $a < 0$ and $b > 0$, but $f(a) = f(b) = 0$. Therefore f is not injective.

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