Question 1

Question 4 **(25 marks)**

(a) The amount of a substance remaining in a solution reduces exponentially over time. An experiment measures the percentage of the substance remaining in the solution. The percentage is measured at the same time each day. The data collected over the first 4 days are given in the table below. Based on the data in the table, estimate which is the first day on which the percentage of the substance in the solution will be less than 0·01%.

Day	**1**	**2**	**3**	**4**
Percentage of substance (%)	95	42·75	19·2375	8·6569

Question 7 **(55 marks)**

Sometimes it is possible to predict the future population in a city using a function.
The population in Sapphire City, over time, can be predicted using the following function:

$$p(t) = Se^{0\cdot 1t} \times 10^6.$$

The population in Avalon, over time, can be predicted using the following function:

$$q(t) = 3\cdot 9e^{kt} \times 10^6.$$

In the functions above, t is time, in years; $t = 0$ is the beginning of 2010; and both S and k are constants.

(a) The population in Sapphire City at the beginning of 2010 is 1 100 000 people.
Find the value of S.

(b) Find the predicted population in Sapphire City at the beginning of 2015.

(c) Find the predicted change in the population in Sapphire City during 2015.

(d) The predicted population in Avalon at the beginning of 2011 is 3 709 795 people.
Write down and solve an equation in k to show that $k = -0{\cdot}05$, correct to 2 decimal places.

(e) Find the year during which the populations in both cities will be equal.

(f) Find the predicted average population in Avalon from the beginning of 2010 to the beginning of 2025.

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(g) Use the function $q(t) = 3{\cdot}9e^{-0{\cdot}05t} \times 10^6$ to find the predicted rate of change of the population in Avalon at the beginning of 2018.

Question 9 **(40 marks)**

The depth of water, in metres, at a certain point in a harbour varies with the tide and can be modelled by a function of the form

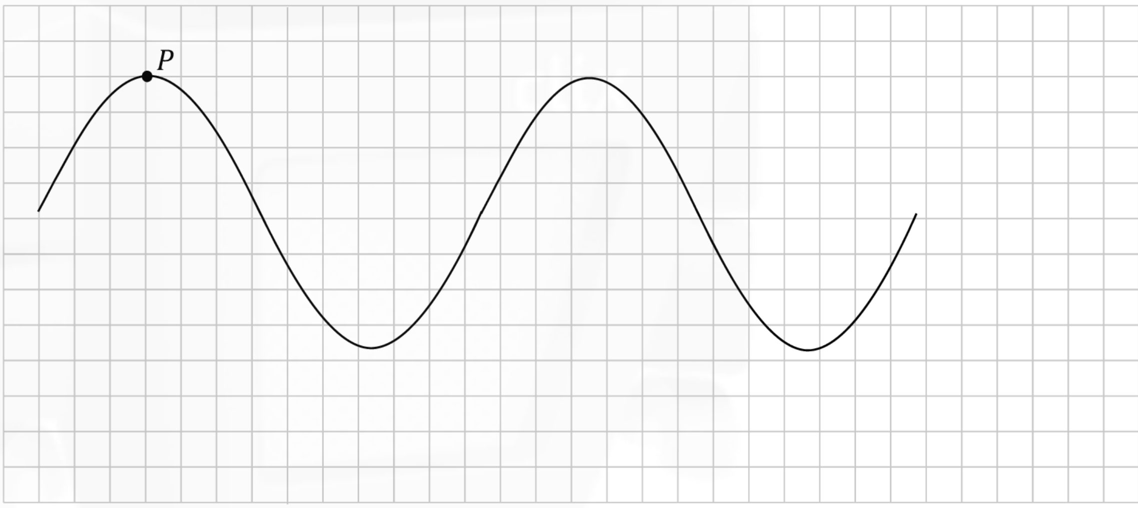
$$f(t) = a + b\cos ct$$

where t is the time in hours from the first high tide on a particular Saturday and a, b, and c are constants. (**Note**: ct is expressed in radians.)

On that Saturday, the following were noted:

- The depth of the water in the harbour at high tide was 5·5 m
- The depth of the water in the harbour at low tide was 1·7 m
- High tide occurred at 02:00 and again at 14:34.

(a) Use the information you are given to add, as accurately as you can, labelled and scaled axes to the diagram below to show the graph of f over a portion of that Saturday.
The point P should represent the depth of the water in the harbour at high tide on that Saturday morning.



(b) **(i)** Find the value of a and the value of b.

(ii) Show that $c = 0{\cdot}5$, correct to 1 decimal place.

(c) Use the equation $f(t) = a + b \cos ct$ to find the times on that Saturday **afternoon** when the depth of the water in the harbour was exactly 5·2 m.
Give each answer correct to the nearest minute.

(b) **(i)** Show that $f(x) = 3x - 2$, where $x \in \mathbb{R}$, is an injective function.

(ii) Given that $f(x) = 3x - 2$, where $x \in \mathbb{R}$, find a formula for f^{-1}, the inverse function of f. Show your work.

Question 5

(a) **(i)** Air is pumped into a spherical exercise ball at the rate of 250 cm^3 per second.
Find the rate at which the radius is increasing when the radius of the ball is 20 cm.
Give your answer in terms of π.

(ii) Find the rate at which the surface area of the ball is increasing when the radius of the ball is 20 cm.

(b) The inflated ball is kicked into the air from a point O on the ground. Taking O as the origin, $(x,\ f(x))$ approximately describes the path followed by the ball in the air, where

$$f(x) = -x^2 + 10x$$

and both x and $f(x)$ are measured in metres.

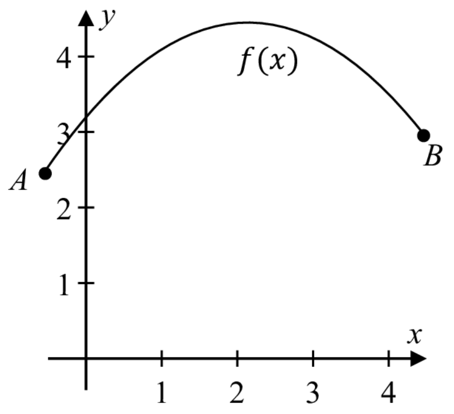
(i) Find the values of x when the ball is on the ground.

(ii) Find the average height of the ball above the ground, during the interval from when it is kicked until it hits the ground again.

(a) The diagram shows Sarah's first throw at the basket in a basketball game. The ball left her hands at A and entered the basket at B. Using the co-ordinate plane with $A(-0{\cdot}5, 2{\cdot}565)$ and $B(4{\cdot}5, 3{\cdot}05)$, the equation of the path of the centre of the ball is

$$f(x) = -0{\cdot}274x^2 + 1{\cdot}193x + 3{\cdot}23,$$

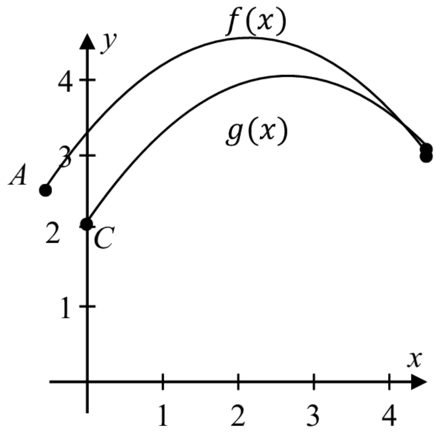
where both x and $f(x)$ are measured in metres.



(i) Find the maximum height reached by the centre of the ball, correct to three decimal places.

(ii) Find the acute angle to the horizontal at which the ball entered the basket. Give your answer correct to the nearest degree.

(iii) Sarah took a second throw. This throw followed the path of the parabola $g(x)$ as shown.
The ball left Sarah's hands at the point $C(0, 2)$.
The graph $y = g(x)$ is the image of the graph $y = f(x)$ under the translation which maps A onto C.
Using your result from part **a(i)**, show that the centre of this ball reached its maximum height at the point $(2{\cdot}677, 3{\cdot}964)$, correct to three decimal places.



(iv) Hence, or otherwise, find the equation of the parabola $g(x)$.

(b) A male bee comes from an unfertilised egg, i.e. he has a female parent but he does not have a male parent. A female bee comes from a fertilised egg, i.e. she has a female parent and a male parent.

(i) The following diagram shows the ancestors of a certain male bee. We identify his generation as G_1 and our diagram goes back to G_4. Continue the diagram to G_5.

G_1	G_2	G_3	G_4	G_5
		↗	Female	
	↗	Female ↘		
Male →	Female		Male	
	↘	Male →	Female	

(ii) The number of ancestors of this bee in each generation can be calculated by the formula

$$G_{n+2} = G_{n+1} + G_n,$$

where $G_1 = 1$ and $G_2 = 1$, as in the diagram.
Use this formula to calculate the number of ancestors in G_6 and in G_7.

G_6 G_7

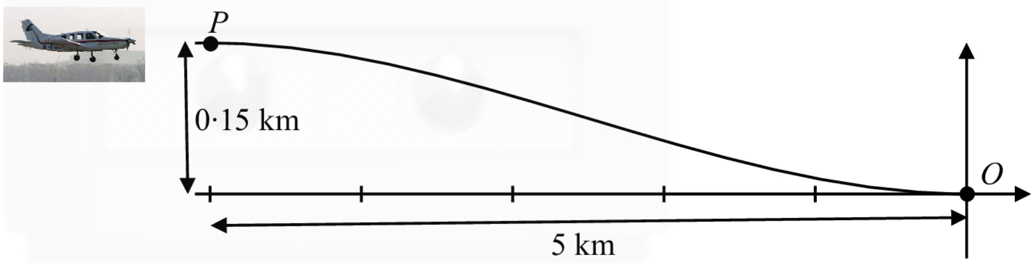
(iii) The number of ancestors in each generation can also be calculated by using the formula

$$G_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Use this formula to verify the number of ancestors in G_3.

Question 7 **(50 marks)**

A plane is flying horizontally at *P* at a height of 150 m above level ground when it begins its descent. *P* is 5 km, horizontally, from the point of touchdown *O*. The plane lands horizontally at *O*.



Taking *O* as the origin, $(x,\ f(x))$ approximately describes the path of the plane's descent where $f(x) = 0{\cdot}0024x^3 + 0{\cdot}018x^2 + cx + d$, $-5 \le x \le 0$, and both x and $f(x)$ are measured in km.

(a) **(i)** Show that $d = 0$.

(ii) Using the fact that *P* is the point $(-5,\ 0{\cdot}15)$, or otherwise, show that $c = 0$.

(b) **(i)** Find the value of $f'(x)$, the derivative of $f(x)$, when $x = -4$.

(ii) Use your answer to part **(b) (i)** above to find the angle at which the plane is descending when it is 4 km from touchdown. Give your answer correct to the nearest degree.

(c) Show that $(-2\cdot5,\ 0\cdot075)$ is the point of inflection of the curve $y = f(x)$.

(d) **(i)** If (x, y) is a point on the curve $y = f(x)$, verify that $(-x-5, -y+0\cdot15)$ is also a point on $y = f(x)$.

(ii) Find the image of $(-x-5, -y+0\cdot15)$ under symmetry in the point of inflection.

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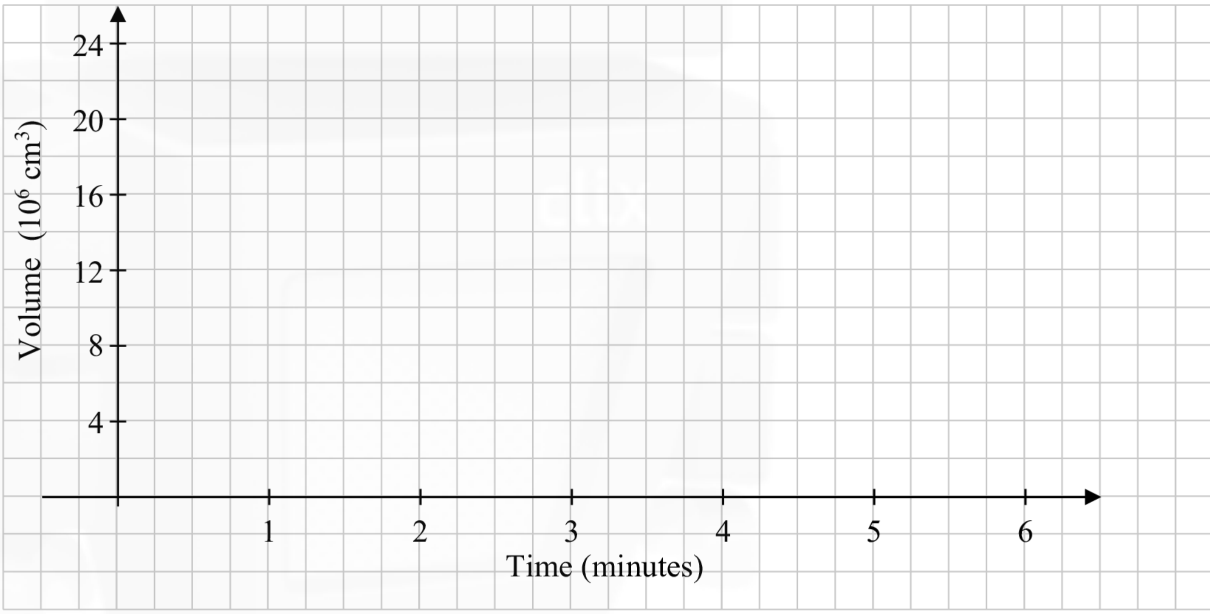
Question 8 **(50 marks)**

An oil-spill occurs off-shore in an area of calm water with no currents. The oil is spilling at a rate of 4×10^{6} cm^{3} per minute. The oil floats on top of the water.

(a) **(i)** Complete the table below to show the total volume of oil on the water after each of the first 6 minutes of the oil-spill.

Time (minutes)	1	2	3	4	5	6
Volume (10^6 cm^3)		8				

(ii) Draw a graph to show the total volume of oil on the water over the first 6 minutes.



(iii) Write an equation for $V(t)$, the volume of oil on the water, in cm^3, after t minutes.

(b) The spilled oil forms a circular oil slick **1 millimetre** thick.

(i) Write an equation for the volume of oil in the slick, in cm^3, when the radius is r cm.

(ii) Find the rate, in cm per minute, at which the radius of the oil slick is increasing when the radius is 50 m.

(c) Show that the area of water covered by the oil slick is increasing at a constant rate of 4×10^7 cm^2 per minute.

(d) The nearest land is 1 km from the point at which the oil-spill began. Find how long it will take for the oil slick to reach land. Give your answer correct to the nearest hour.

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Question 9 **(50 marks)**

The approximate length of the day in Galway, measured in hours from sunrise to sunset, may be calculated using the function

$$f(t) = 12{\cdot}25 + 4{\cdot}75\sin\left(\frac{2\pi}{365}t\right),$$

where t is the number of days after March 21st and $\left(\frac{2\pi}{365}t\right)$ is expressed in radians.

(a) Find the length of the day in Galway on June 5th (76 days after March 21st). Give your answer in hours and minutes, correct to the nearest minute.

(b) Find a date on which the length of the day in Galway is approximately 15 hours.

(c) Find $f'(t)$, the derivative of $f(t)$.

(d) Hence, or otherwise, find the length of the longest day in Galway.

(e) Use integration to find the average length of the day in Galway over the six months from March 21st to September 21st (184 days). Give your answer in hours and minutes, correct to the nearest minute.

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Question 3 **(25 marks)**

A cubic function f is defined for $x \in \mathbb{R}$ as

$$f : x \mapsto x^3 + (1 - k^2)x + k, \quad \text{where } k \text{ is a constant.}$$

(a) Show that $-k$ is a root of f.

(b) Find, in terms of k, the other two roots of f.

(c) Find the set of values of k for which f has exactly one real root.

Question 5 **(25 marks)**

A is the closed interval $[0,5]$. That is, $A=\{x \mid 0 \le x \le 5, x \in \mathbb{R}\}$.

The function f is defined on A by:

$$f : A \to \mathbb{R} : x \mapsto x^3 - 5x^2 + 3x + 5.$$

(a) Find the maximum and minimum values of f.

(b) State whether f is injective. Give a reason for your answer.