Question 1 (2017)

(a)	$f(x+h)=\frac{1}{3}(x+h)^2-(x+h)+3$ $f(x)=\frac{1}{3}x^2-x+3$ $f(x+h)-f(x)=\frac{2xh}{3}+\frac{h^2}{3}-h$ $\frac{f(x+h)-f(x)}{h}=\frac{2x}{3}+\frac{h}{3}-1$ $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{2x}{3}-1$	**Scale 20D (0, 5, 14, 17, 20)** *Low Partial Credit* • any $f(x+h)$ *Mid Partial Credit* • $f(x+h)-f(x)$ *with some correct work* *High Partial Credit* • $\frac{\frac{1}{3}(x+h)^2-(x+h)+3-\left(\frac{x^2}{3}-x+3\right)}{h}$ simplified **Notes**: • omission of limit sign penalised once only • answer not from 1st Principles merits 0 marks
(b)	$\frac{d(fg(x))}{dx}=$ $\frac{1}{(3(x+5)^2+2)}(6(x+5))$ $\frac{d(fg\left(\frac{1}{4}\right))}{dx}=\frac{6(\frac{21}{4})}{3(\frac{21}{4})^2+2}=\frac{504}{1355}$ $=0{\cdot}372$ **OR** $f(x)=\ln(3x^2+2)$ $g(x)=(x+5)$ $f[g(x)]=\ln[3(x+5)^2+2]$ $=\ln(3x^2+30x+77)$ $f'(x)=\frac{6x+30}{3x^2+30x+77}$ $x=\frac{1}{4}$: $f'(x)=\frac{31{\cdot}5}{84{\cdot}6875}=0{\cdot}3719$ $=0{\cdot}372$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • Any correct differentiation • $fg(x)$ formulated *High Partial Credit:* • $\frac{d(fg(x))}{dx}$ found **Note**: Work with $f(x)\times g(x)$ merits low partial credit at most

(a)	$r = \frac{42{\cdot}75}{95} = \frac{9}{20}$ $\quad T_n = ar^{n-1} < 0{\cdot}01$ $95\left(\frac{9}{20}\right)^{n-1} < 0{\cdot}01$ $\left(\frac{9}{20}\right)^{n-1} < \frac{0{\cdot}01}{95}$ $(n-1)\log\left(\frac{9}{20}\right) < \log\left(\frac{0{\cdot}01}{95}\right)$ $(n-1) > \frac{\log\left(\frac{0{\cdot}01}{95}\right)}{\log\left(\frac{9}{20}\right)}$ (since $\log\left(\frac{9}{20}\right)$ is negative) $n-1 > 11{\cdot}47$ $n > 12{\cdot}47$ 12th day	**Scale 15D (0, 5, 8, 12, 15)** *Low Partial Credit:* • r found • T_n of a GP with some substitution *Mid Partial Credit:* • Inequality in n written *High Partial Credit:* • Inequality in n simplified (log handled) *Full Credit:* • Accept $n = 12{\cdot}47$
(b)	$4(2) + 4\sqrt{2} + 4 + \cdots\cdots$ $a = 8 \quad r = \frac{1}{\sqrt{2}}$ $S_\infty = \frac{a}{1-r}$ $S_\infty = \frac{8}{1-\frac{1}{\sqrt{2}}}$ $S_\infty = \frac{8}{1-\frac{1}{\sqrt{2}}} \cdot \frac{1+\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$ $S_\infty = \frac{8\left(1+\frac{1}{\sqrt{2}}\right)}{\frac{1}{2}}$ $S_\infty = 16 + 8\sqrt{2}$	**Scale 10C (0, 5, 8, 10)** *Low Partial Credit:* • length of one side of new square *High Partial Credit:* • S_∞ fully substituted • Correct work with one side only

(a)	$f(x) = 2x^3 + 5x^2 - 4x - 3$ $f(-3) = 2(-3)^3 + 5(-3)^2 - 4(-3) - 3$ $= -54 + 45 + 12 - 3$ $f(-3) = 0$ $\Rightarrow (x+3)$ is a factor $\begin{array}{r} 2x^2 - x - 1 \\ x+3 \overline{) 2x^3 + 5x^2 - 4x - 3} \\ \underline{2x^3 + 6x^2} \qquad\qquad \\ -x^2 - 4x \quad\; \\ \underline{-x^2 - 3x} \quad\; \\ -x - 3 \\ \underline{-x - 3} \end{array}$ $f(x) = (x+3)(2x^2 - x - 1)$ $f(x) = (x+3)(2x+1)(x-1)$ $x = -3 \quad x = -\frac{1}{2} \quad x = 1$	**Scale 15C (0, 5, 10, 15)** *Low Partial Credit:* • Shows $f(-3) = 0$ *High Partial Credit:* • quadratic factor of $f(x)$ found **Note**: No remainder in division may be stated as reason for $x = -3$ as root
(b)	$y = 2x^3 + 5x^2 - 4x - 3$ $\frac{dy}{dx} = 6x^2 + 10x - 4 = 0$ $3x^2 + 5x - 2 = 0$ $(x+2)(3x-1) = 0$ $3x - 1 = 0 \quad x + 2 = 0$ $x = \frac{1}{3} \quad x = -2$ $f\left(\frac{1}{3}\right) = \frac{-100}{27} \quad f(-2) = 9$ $Max = (-2, 9) \quad Min = \left(\frac{1}{3}, \frac{-100}{27}\right)$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • $\frac{dy}{dx}$ found (Some correct differentiation) *High Partial Credit* • roots and one y value found **Note**: One of Max/Min must be identified for full credit
(c)	$a > \frac{100}{27}$ **or** $a < -9$	**Scale 5B (0, 3, 5)** *Partial Credit:* • one value identified • no range identified (from 2 values)

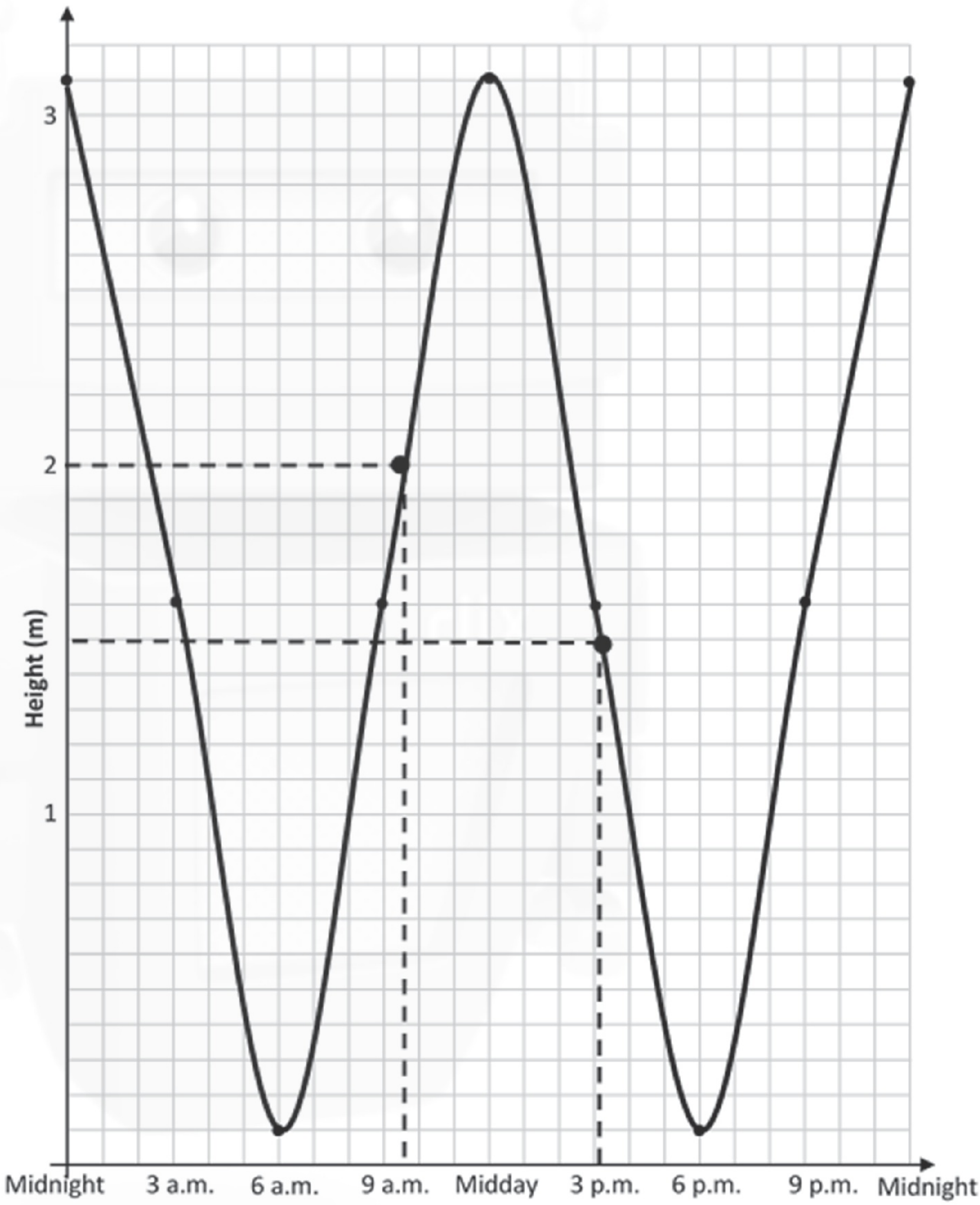
Q8	Model Solution – 45 Marks	Marking Notes
(a)	Period = $\dfrac{2\pi}{\frac{\pi}{6}}$ = 12 hours Range = $[1{\cdot}6 - 1{\cdot}5\,, 1{\cdot}6 + 1{\cdot}5\,] = [0{\cdot}1 \text{ m}, 3{\cdot}1 \text{ m}]$	Scale 5C (0, 2,4, 5) *Low Partial Credit* • some use of 2π or $\dfrac{\pi}{6}$ • range of cos function *High partial credit* • period or range correct **Note:** *Accept correct period and/or range without work*
(b)	Max = 1·6 + 1·5(1) =3·1 m. **or** 3·1 m from range	Scale 5B (0,2, 5) *Partial Credit* • max occurs when cos A = 1 or t = 0 • effort at $h'(t)$ **Note:** *Accept correct answer without work*
(c)	$h'(t) = 1{\cdot}5(-\sin\dfrac{\pi t}{6})\dfrac{\pi}{6}$ $h'(2) = 1{\cdot}5(-\sin\dfrac{2\pi}{6})\dfrac{\pi}{6}$ = – 0·68017 =–0·68 m/h Tide is going out at a rate of 0·68 m per hour at 2 am	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • effort at differentiation *High Partial Credit* • correct numerical answer but not in context

(d)(i)

$$h(t) = 1 \cdot 6 + 1 \cdot 5 \cos\left(\frac{\pi}{6}t\right)$$

Time	12 am	3 am	6 am	9 am	12 pm	3 pm	6 pm	9 pm	12 am
t	0	3	6	9	12	15	18	21	24
Height	3·1	1·6	·1	1·6	3·1	1·6	·1	1·6	3·1

(d) (i)		Scale 10C (0, 3, 7, 10) *Low Partial Credit* • one correct height *High Partial Credit* • five correct heights

<table>
<tr><td>(d)
(ii)</td><td colspan="2">
</td></tr>
<tr><td>(d)
(ii)</td><td>Graph</td><td>Scale 10C (0, 3,7,10)
Low Partial Credit
• one correct plot

High Partial Credit
• at least 7 correct plots
• plots correct but graph not sketched or sketched incorrectly</td></tr>
</table>

(e)	Low tide = $0{\cdot}1$ m High tide = $3{\cdot}1$ m Difference = $3{\cdot}1 - 0{\cdot}1 = 3$ m	Scale 5B (0, 2, 5) *Partial Credit* • height of Low tide or High tide correctly identified **Notes:** (i) *candidates may show work for this section on graph* (ii) *accept values from candidate's graph* (iii) *accept correct answer from graph without work*
(f)	Enter port at 9:30 approx Leave port before 15:15 approx Time = 15:15 – 9:30 = 5 hr 45 min approx.	Scale 5B (0, 2, 5) *Partial Credit* • time of entry to port or leave port correctly identified • value(s) for h = 2 and/or $h = 1{\cdot}5$ on sketch • time estimated using relevant values other than those required for the maximum time. **Notes:** (i) *candidates may show relevant work for this section on graph* (ii) *accept values from candidate's graph*

Q6	Model Solution – 25 Marks	Marking Notes
(a)	$f(x+h)-f(x)=(2x+2h+4)^2-(2x+4)^2$ $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=$ $\lim\limits_{h\to 0}\dfrac{(2x+2h+4)^2-(2x+4)^2}{h}$ $=\lim\limits_{h\to 0}\left(\dfrac{\begin{bmatrix}(4x^2+8hx+4h^2+16x+16h+16)\\ -(4x^2+16x+16)\end{bmatrix}}{h}\right)$ $=\lim\limits_{h\to 0}\dfrac{8hx+4h^2+16h}{h}$ $=8x+16$ **or** $f(x)=(2x+4)^2=4x^2+16x+16$ $f(x+h)=4(x+h)^2+16(x+h)+16$ $=4x^2+8hx+4h^2+16x+16h+16$ $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ $\lim\limits_{h\to 0}\dfrac{8hx+4h^2+16h}{h}$ $=8x+16$	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • any $f(x+h)$ *Mid Partial Credit* • limit of $\dfrac{f(x+h)-f(x)}{h}$ *High Partial Credit* • limit of $\dfrac{(2x+2h+4)^2-(2x+4)^2}{h}$ **Notes:** - *omission of limit sign penalised once only* - *answer not from 1st Principles merits 0 marks*
(b) **(i)+** **(ii)**	$y=x.\sin\dfrac{1}{x}$ $\dfrac{dy}{dx}=\sin\dfrac{1}{x}+x\left(\cos\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right)$ $\dfrac{dy}{dx}=\sin\dfrac{1}{x}-\dfrac{1}{x}\cos\dfrac{1}{x}$ $\dfrac{dy}{dx}=\sin\dfrac{\pi}{4}-\dfrac{\pi}{4}\cos\dfrac{\pi}{4}$ $=0{\cdot}15$	Scale 15D (0, 4, 7, 11, 15) *Low Partial Credit* • any correct differentiation *Mid Partial Credit* • product rule applied *High Partial Credit* • correct differentiation **Note:** *one penalty for calculator in wrong mode*

<table>
<tr><th>Q7</th><th>Model Solution – 40 Marks</th><th>Marking Notes</th></tr>
<tr><td>(a)
(i)</td><td>$$v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dr} = 4\pi r^2$$
$$\frac{dv}{dt} = 250\ \text{cm}^3/\text{s}$$
$$\frac{dr}{dt} = \frac{dr}{dv}.\frac{dv}{dt} = \frac{1}{4\pi r^2}.250$$
$$\frac{dr}{dt} = \frac{250}{4\pi 400} = \frac{5}{32\pi}\ \ \text{cm/s}$$</td><td>Scale 10C (0, 3, 7, 10)
Low Partial Credit
• work towards $\frac{dv}{dr}$ or $\frac{dv}{dt}$ or $\frac{dr}{dt}$

High Partial Credit
• correct expression for $\frac{dr}{dt}$</td></tr>
<tr><td>(ii)</td><td>$$a = 4\pi r^2 \Rightarrow \frac{da}{dr} = 8\pi r$$
$$\frac{da}{dt} = \frac{da}{dr}.\frac{dr}{dt} = 8\pi r.\frac{5}{32\pi}$$
$$= \frac{5(20)}{4}$$
$$= 25\ \text{cm}^2/\text{s}$$</td><td>Scale 10C (0, 3, 7, 10)
Low Partial Credit
• work towards $\frac{da}{dr}$ or $\frac{da}{dt}$

High Partial Credit
• correct expression for $\frac{da}{dt}$</td></tr>
<tr><td>(b)
(i)</td><td>$$-x^2 + 10x = 0$$
$$x(-x + 10) = 0$$
$$x = 0 \quad \text{or} \quad x = 10$$</td><td>Scale 10C (0, 3, 7, 10)
Low Partial Credit
• quadratic equation formed
• gets $x = 0$ only

High Partial Credit
• quadratic factorised

Note: $f'(x) = 0 \Rightarrow 2x - 10 = 0 \Rightarrow x = 5$
merits 0 marks</td></tr>
<tr><td>(ii)</td><td>$$\frac{1}{10-0}\int_0^{10} (-x^2 + 10x)dx$$
$$= \frac{1}{10}\left[\frac{-x^3}{3} + 5x^2\right]_0^{10}$$
$$= \frac{1}{10}\left[\left(\frac{-1000}{3} + 500\right) - 0\right]$$
$$= \frac{-100}{3} + 50 = \frac{50}{3}\text{m}$$</td><td>Scale 10C (0, 3, 7, 10)
Low Partial Credit
• integration set up

High Partial Credit
• correct integration with some substitution</td></tr>
</table>

Q8	Model Solution – 55 Marks	Marking Notes
(a) **(i)**	$f(x) = -0{\cdot}274x^2 + 1{\cdot}193x + 3{\cdot}23$ $f'(x) = -0{\cdot}548x + 1{\cdot}193 = 0$ $x = 2{\cdot}177 \text{ m}$ $f(2{\cdot}177) = -0{\cdot}274(2{\cdot}177)^2 + 1{\cdot}193(2{\cdot}177) + 3{\cdot}23$ $= -1{\cdot}2986 + 2{\cdot}5972 + 3{\cdot}23$ $= 4{\cdot}529 \text{ m}$ or $-0{\cdot}274(x^2 - \frac{1193}{274}x - \frac{1615}{137})$ $-0{\cdot}274(x - \frac{1193}{548})^2 + 4.5285$ Max Height $= 4{\cdot}529$ m	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • any correct differentiation • effort made at completing square • trial and error with more than one value of x tested *High Partial Credit* • x value correct **Note:** *if correct answer by trial and error, must show points on each side of max point to be lower to earn full credit*
(ii)	$\tan\theta = -0{\cdot}548(4{\cdot}5) + 1{\cdot}193$ $\tan\theta = -1{\cdot}273$ $\theta = 51{\cdot}8° = 52°$	Scale 5B (0, 2, 5) *Partial Credit* • tan **Note:** *right angled triangles may appear in diagram given in equation*
(iii)	Map $A \to C$ $(-0{\cdot}5, 2{\cdot}565) \to (0, 2)$ $2{\cdot}177 - (-0{\cdot}5) = 2{\cdot}677$ $4{\cdot}529 - 0{\cdot}565 = 3{\cdot}964$ $(2{\cdot}177, 4{\cdot}529) \to (2{\cdot}677, 3{\cdot}964)$	Scale 5B (0, 2, 5) *Partial Credit* • $(-0{\cdot}5, 2{\cdot}565) \to (0, 2)$

(iv)	$$g(x) = ax^2 + bx + c$$ $$C(0,2) \in g(x) => c = 2$$ $B(4{\cdot}5, 3{\cdot}05) \epsilon\, g(x)$ $3{\cdot}05 = a(4{\cdot}5)^2 + b(4{\cdot}5) + 2$ $\Longrightarrow 20{\cdot}25a + 4{\cdot}5b = 1{\cdot}05$ ... **(i)** $g'(x) = 2ax + b = 0$ $\Longrightarrow 2a(2{\cdot}677) + b = 0$ $5{\cdot}354a + b = 0$ ... **(ii)** From **(i)** and **(ii)** $a = -0{\cdot}273$ $b = 1{\cdot}462$ $$g(x) = -0{\cdot}273x^2 + 1{\cdot}462x + 2$$ *[Note: a third equation that could be used is* $3{\cdot}964 = a(2{\cdot}677)^2 + b(2{\cdot}677) + 2$ ... **(iii)**]	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • c value found • relevant equation in *a*, *b* and/or *c* *Mid Partial Credit* • formulated correctly any two equations *High Partial Credit* • formulated correctly any three equations **Note**: $ax^2 + bx + c$ *not in an equation merits 0 marks*
	Or	**Or**
	Equation of parabola with vertex (h, k): $$g(x) = a(x-h)^2 + k$$ $C(0,2)$ on curve: $(h,k) = (2{\cdot}677, 3.964)$ $$2 = a(-2{\cdot}677)^2 + 3{\cdot}964$$ $$-1{\cdot}964 = a(7{\cdot}166329)$$ $$a = -0{\cdot}27405 = -0{\cdot}274$$ Parabola: $$g(x) = -0{\cdot}274[(x-2{\cdot}677)^2] + 3{\cdot}964$$ **or** $$g(x) = f(x-0{\cdot}5) - 0{\cdot}565$$ $$g(x) = -0{\cdot}274(x-0{\cdot}5)^2 + 1{\cdot}193(x-0{\cdot}5) + 3{\cdot}23 - 0{\cdot}565$$ $$g(x) = -0{\cdot}274x^2 + 1{\cdot}467x + 2$$	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • equation of curve • use of C *Mid Partial Credit* • using peak value *High Partial Credit* • value of a found

(b) Differentiate $x-\sqrt{x+6}$ with respect to x.

$$f(x)=x-\sqrt{x+6}=x-(x+6)^{\frac{1}{2}}$$
$$f'(x)=1-\tfrac{1}{2}(x+6)^{-\frac{1}{2}}=1-\frac{1}{2\sqrt{x+6}}$$

(c) Find the co-ordinates of the turning point of the function $y=x-\sqrt{x+6},\ x\geq -6$.

$$f'(x)=0\Rightarrow 1-\frac{1}{2\sqrt{x+6}}=0$$
$$\Rightarrow 2\sqrt{x+6}=1$$
$$\Rightarrow x+6=\tfrac{1}{4}$$
$$\Rightarrow x=-5\tfrac{3}{4}$$
$$f(-5\tfrac{3}{4})=-5\tfrac{3}{4}-\sqrt{\tfrac{1}{4}}=-6\tfrac{1}{4}$$

$$\left(-5\tfrac{3}{4},-6\tfrac{1}{4}\right)$$

(a) **(i)** Show that $d = 0$.

$$f(x) = 0\cdot0024x^3 + 0\cdot018x^2 + cx + d$$
$$f(0) = 0 + 0 + 0 + d = 0 \quad \Rightarrow \quad d = 0$$

(ii) Using the fact that P is the point $(-5,\ 0\cdot15)$, or otherwise, show that $c = 0$.

$$f(x) = 0\cdot0024x^3 + 0\cdot018x^2 + cx$$
$$f(-5) = 0\cdot0024(-5)^3 + 0\cdot018(-5)^2 + c(-5) = 0\cdot15$$
$$\Rightarrow 0\cdot15 - 5c = 0\cdot15 \quad \Rightarrow \quad c = 0$$

or

The plane lands horizontally at O $\Rightarrow$ $f'(x) = 0$ when $x = 0$

$$f'(x) = 0\cdot0072x^2 + 0\cdot036x + c$$
$$f'(0) = 0 + 0 + c = 0$$
$$\Rightarrow c = 0$$

(b) **(i)** Find the value of $f'(x)$, the derivative of $f(x)$, when $x = -4$.

$$f(x) = 0\cdot0024x^3 + 0\cdot018x^2 + cx + d$$
$$f'(x) = 0\cdot0072x^2 + 0\cdot036x$$
$$f'(-4) = 0\cdot0072(-4)^2 + 0\cdot036(-4)$$
$$= -0\cdot0288$$

(ii) Use your answer to part **(b) (i)** above to find the angle at which the plane is descending when it is 4 km from touchdown. Give your answer correct to the nearest degree.

$\tan\theta = f'(x) = -0\cdot 0288 \Rightarrow \theta = 178\cdot 3503°$
Angle of descent $\alpha = 1\cdot 6497° = 2°$

(c) Show that $(-2\cdot 5,\ 0\cdot 075)$ is the point of inflection of the curve $y = f(x)$.

$$f'(x) = 0\cdot 0072x^2 + 0\cdot 036x$$
$$f''(x) = 0\cdot 0144\,x + 0\cdot 036 = 0$$
$$\Rightarrow x = -2\cdot 5$$
$$f(x) = 0\cdot 0024x^3 + 0\cdot 018x^2$$
$$f(-2\cdot 5) = 0\cdot 0024(-2\cdot 5)^3 + 0\cdot 018(-2\cdot 5)^2$$
$$= -0\cdot 0375 + 0\cdot 1125 = 0\cdot 075$$
$$(-2\cdot 5, 0\cdot 075)$$

(d) (i) If (x, y) is a point on the curve $y = f(x)$, verify that $(-x-5, -y+0\cdot 15)$ is also a point on $y = f(x)$.

$$f(x) = 0\cdot 0024x^3 + 0\cdot 018x^2$$
$$f(-x-5) = 0\cdot 0024(-x-5)^3 + 0\cdot 018(-x-5)^2$$
$$= 0\cdot 0024\left(-x^3 - 15x^2 - 75x - 125\right) + 0\cdot 018\left(x^2 + 10x + 25\right)$$
$$= -0\cdot 0024x^3 - 0\cdot 018x^2 + 0x + 0\cdot 15$$
$$= -y + 0\cdot 15$$

(ii) Find the image of $(-x-5, -y+0\cdot 15)$ under symmetry in the point of inflection.

Point: $(-x-5,\ -y+0\cdot 15)$
Point of inflection: $(-2\cdot 5,\ 0\cdot 075)$

Change in x value: $(-2\cdot 5) - (-x-5) = x + 2\cdot 5$
Change in y value: $0\cdot 075 - (-y + 0\cdot 15) = y - 0\cdot 075$

Image of point of inflection:
x value: $-2\cdot 5 + (x + 2\cdot 5) = x$
y value: $0\cdot 075 + (y - 0\cdot 075) = y$
$\Rightarrow (x, y)$ is image

or

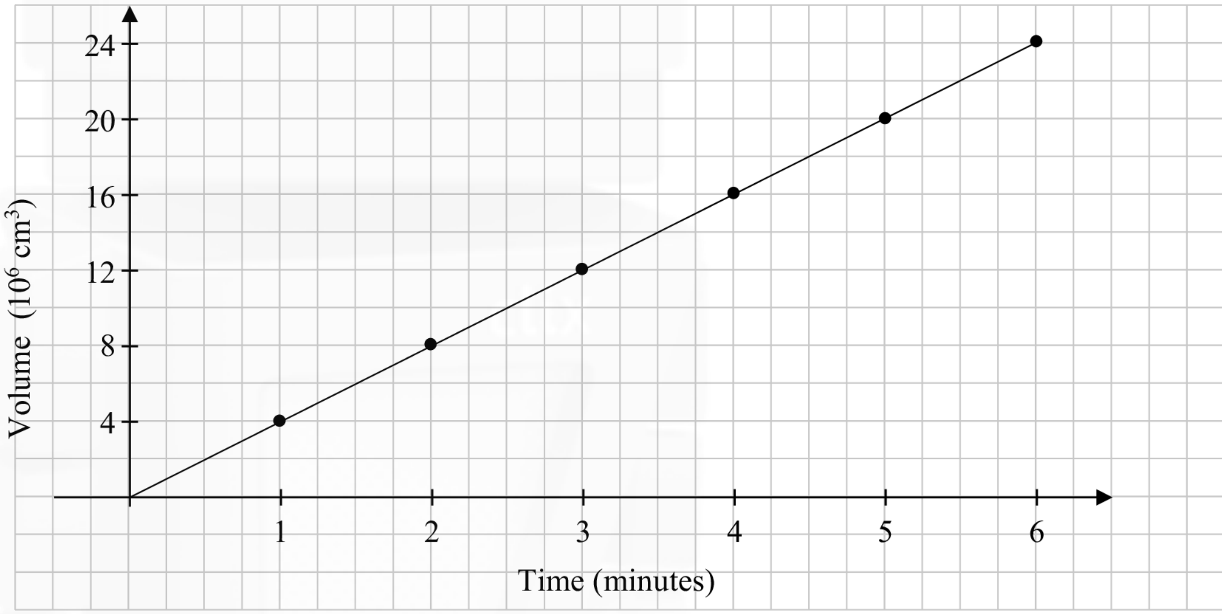
Let (x, y) be the image.

$$\left(\frac{-x-5+x}{2}, \frac{-y+0\cdot 15+y}{2}\right)=(-2\cdot 5, 0\cdot 075)\text{, the point of inflection}$$

(a) **(i)** Complete the table below to show the total volume of oil on the water after each of the first 6 minutes of the oil-spill.

Time (minutes)	1	2	3	4	5	6
Volume (10^6 cm ^{3})	4	8	12	16	20	24

(ii) Draw a graph to show the total volume of oil on the water over the first 6 minutes.



(iii) Write an equation for $V(t)$, the volume of oil on the water, in cm^3, after t minutes.

Line, slope 4×10^6, passing through (0, 0).

$V(t) = \left(4 \times 10^6\right) t$

(b) The spilled oil forms a circular oil slick **1 millimetre** thick.

(i) Write an equation for the volume of oil in the slick, in cm^3, when the radius is r cm.

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi r^2 (0 \cdot 1) \\ &= 0 \cdot 1 \pi r^2 \ \text{cm}^3 \end{aligned}$$

(ii) Find the rate, in cm per minute, at which the radius of the oil slick is increasing when the radius is 50 m.

$$\frac{dV}{dt} = 4\times10^6 \text{ cm}^3 \text{ per minute}$$

$$V = \pi r^2 h \text{ where } h = 0\cdot1 \text{ cm}$$

$$\frac{dV}{dr} = 2\pi rh$$

$$\frac{dV}{dr} = 0\cdot2\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dV}\frac{dV}{dt} = \frac{1}{0\cdot2\pi r}\times4\times10^6$$

$$= \frac{4\times10^6}{0\cdot2\pi(5000)} = 1273\cdot3 \text{ cm per minute}$$

(c) Show that the area of water covered by the oil slick is increasing at a constant rate of 4×10^7 cm^2 per minute.

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r\frac{4\times10^6}{0\cdot2\pi r} = 4\times10^7 \text{ cm}^2 \text{ per minute}$$

or

$$(0\cdot1)\pi r^2 = (4\times10^6)t$$

$$\Rightarrow A = \pi r^2 = (4\times10^7)t$$

$$\frac{dA}{dt} = 4\times10^7$$

(d) The nearest land is 1 km from the point at which the oil-spill began. Find how long it will take for the oil slick to reach land. Give your answer correct to the nearest hour.

$$A = \pi r^2 = \pi(10^5)^2 = \pi10^{10} \text{ cm}^2$$

$$t = \frac{\pi10^{10}}{4\times10^7} = \frac{\pi10^3}{4} = 785\cdot398 \text{ minutes}$$

$$= 13\cdot09 = 13 \text{ hours}$$

(a) Find the length of the day in Galway on June 5th (76 days after March 21st). Give your answer in hours and minutes, correct to the nearest minute.

$$f(t) = 12\cdot25 + 4\cdot75\sin\left(\frac{2\pi}{365}t\right)$$

$$f(76) = 12\cdot25 + 4\cdot75\sin\left(\frac{2\pi}{365}\times76\right)$$

$=12\cdot25+4\cdot587 = 16\cdot837 = 16$ hours 50 minutes

(b) Find a date on which the length of the day in Galway is approximately 15 hours.

$$f(t) = 12\cdot25 + 4\cdot75\sin\left(\frac{2\pi}{365}t\right) = 15$$

$$\Rightarrow \sin\left(\frac{2\pi}{365}t\right) = 0\cdot578947$$

$$\Rightarrow \frac{2\pi}{365}t = 0\cdot6174371$$

$$\Rightarrow t = 35\cdot87$$

36 days after March 21 is April 26.

(c) Find $f'(t)$, the derivative of $f(t)$.

$$f(t) = 12\cdot25 + 4\cdot75\sin\left(\frac{2\pi}{365}t\right)$$

$$f'(t) = 0 + 4\cdot75\times\frac{2\pi}{365}\cos\left(\frac{2\pi}{365}t\right)$$

$$= \frac{9\cdot5\pi}{365}\cos\left(\frac{2\pi}{365}\right)t$$

(d) Hence, or otherwise, find the length of the longest day in Galway.

$f(t)$ is a maximum when $\sin\left(\frac{2\pi}{365}t\right)$ is a maximum of 1.

$t = 12\cdot25 + 4\cdot75 = 17$ hours

or

$$f'(t) = 0 \Rightarrow \frac{9\cdot5\pi}{365}\cos\left(\frac{2\pi}{365}t\right) = 0$$

$$\Rightarrow \cos\left(\frac{2\pi}{365}t\right) = 0$$

$$\Rightarrow \frac{2\pi}{365}t = \frac{\pi}{2}$$

$$\Rightarrow t = \frac{365}{4} = 91\cdot25$$

$$f(91\cdot25) = 12\cdot25 + 4\cdot75\sin\left(\frac{2\pi}{365}\times 91\cdot25\right)$$

$$= 12\cdot25 + 4\cdot75\sin\frac{\pi}{2}$$

$= 17$ hours

(e) Use integration to find the average length of the day in Galway over the six months from March 21st to September 21st (184 days). Give your answer in hours and minutes, correct to the nearest minute.

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{184}\int_0^{184}\left(12\cdot25 + 4\cdot75\sin\left(\frac{2\pi}{365}t\right)\right)dt$$

$$= \frac{1}{184}\left[12\cdot25t - 4\cdot75\times\frac{365}{2\pi}\cos\left(\frac{2\pi}{365}t\right)\right]_0^{184}$$

$$= \frac{1}{184}\left[(2254 + 275\cdot843) - (0 - 275\cdot934)\right]$$

$$= \frac{1}{184}[2805\cdot777]$$

$$= 15\cdot24879$$

$= 15$ hours 15 minutes

(a) Differentiate the function $2x^2 - 3x - 6$ with respect to x from first principles.

$$f(x) = 2x^2 - 3x - 6$$
$$f(x+h) = 2(x+h)^2 - 3(x+h) - 6 = 2x^2 + 4xh + 2h^2 - 3x - 3h - 6$$
$$f(x+h) - f(x) = 4xh + 2h^2 - 3h$$
$$\underset{h\to 0}{Limit}\left(\frac{f(x+h) - f(x)}{h}\right) = \underset{h\to 0}{Limit}\left(\frac{4xh + 2h^2 - 3h}{h}\right) = 4x - 3$$

(b) Let $f(x) = \dfrac{2x}{x+2}$, $x \neq -2$, $x \in \mathbb{R}$. Find the co-ordinates of the points at which the slope of the tangent to the curve $y = f(x)$ is $\dfrac{1}{4}$.

$$f(x) = \frac{2x}{x+2}$$
Let $u(x) = 2x \Rightarrow u'(x) = 2$ and $v(x) = x+2 \Rightarrow v'(x) = 1$
$$f'(x) = \frac{(x+2)(2) - 2x(1)}{(x+2)^2} = \frac{4}{(x+2)^2}$$
$$f'(x) = \frac{1}{4} \Rightarrow \frac{4}{(x+2)^2} = \frac{1}{4}$$
$$\Rightarrow 16 = (x+2)^2$$

$\Rightarrow x+2 = 4$ or $x+2 = -4$
$\Rightarrow x = 2$ or $x = -6$

or

$x^2 + 4x - 12 = 0$
$(x-2)(x+6) = 0$
$\Rightarrow x - 2 = 0$ or $x + 6 = -0$
$\Rightarrow x = 2$ or $x = -6$

$$f(-6) = \frac{-12}{-6+2} = 3 \text{ and } f(2) = \frac{4}{2+2} = 1$$
Points $(-6, 3)$ and $(2, 1)$

(i) Find the value of $f(0.2)$

Substituting 0.2 for x gives

$$f(0.2) = -0.5(0.2)^2 + 5(0.2) - 0.98 = -0.5(0.04) + 1 - 0.98 = 0$$

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(ii) Show that f has a local maximum point at $(5, 11.52)$.

First we calculate the derivative of f:

$$f'(x) = -0.5(2x) + 5(1) - 0 = -x + 5.$$

Now $f'(5) = -5 + 5 = 0$. Therefore $x = 5$ is a stationary point.
Now

$$f''(x) = -1.$$

So $f''(5) = -1 < 0$. That means that $x = -5$ is a local maximum. Finally,

$$f(5) = -0.5(5^2) + 5(5) - 0.98 = 11.52.$$

Therefore the graph of f has a local maximum point at $(5, 11.52)$.

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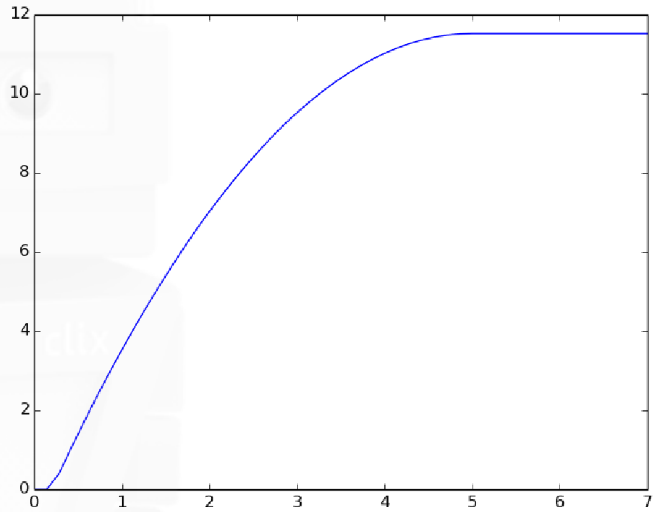

Note that between $t = 0$ and $t = 0.2$ the graph is just a horizontal line along the t-axis. Likewise, for $t \geq 5$ the graph is a horizontal line at height $v = 11.52$. In between $t = 0.2$ and $t = 5$ the function is a quadratic so the graph must be a parabola. We can sketch this by evaluating the function at three or four points. For example $v(1) = 3.52$, $v(2) = 7.02$, $v(3) = 9.52$ and $v(4) = 11.02$. So we plot the points $(1, 3.52)$, $(2, 7.02)$, $(3, 9.52)$ and $(4, 11.02)$ and then join them by a smooth curve. Make sure that this parabolic arc starts at $(0.2, 0)$ and ends at $(5, 11.52)$.

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(ii) Find the distance travelled by the sprinter in the first 5 seconds of the race.

The distance travelled in the first 5 seconds of the race is given by

$$\int_0^5 v(t) \quad dt.$$

Now

$$\begin{aligned}
\int_0^5 v(t) \quad dt &= \int_0^{0.2} v(t) \quad dt + \int_{0.2}^5 v(t) \quad dt \\
&= \int_0^{0.2} 0 \quad dt + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= 0 + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \Big|_{0.2}^5 \\
&= \frac{0.5(5^3)}{3} + \frac{5(5^2)}{2} - 0.98(5) - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2)\right) \\
&= 36.864
\end{aligned}$$

So the sprinter travels 36.864 metres in the first 5 seconds of the race.

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(iii) Find the sprinter's finishing time for the race. Give your answer correct to two decimal places.

We have just seen that the sprinter travels 36.864 metres in the first 5 seconds of the race. So he has 63.136metres left to travel to complete the race at that point. Also after 5 seconds, his velocity is a constant 11.52 metres per second. Therefore it will take him a further $\frac{63.136}{11.52}$ seconds to complete the race. Now $\frac{63.136}{11.52} = 5.48$ correct to two decimal places. So his total time is $5 + 5.48 = 10.48$ seconds, correct to two decimal places.

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After 7 seconds the sprinter's velocity doesn't change. So the first step is to calculate how far he has travelled in the first 7 seconds of the race. As in part (ii) this is equal to the value of the defnite integral

$$\int_{0.2}^{7} (-0.5t^2 + 5t - 0.98) \quad dt.$$

Now

$$\begin{aligned}
\int_{0}^{7} (-0.5t^2 + 5t - 0.98) \quad dt &= \left. \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \right|_{0.2}^{7} \\
&= \frac{0.5(7^3)}{3} + \frac{5(7^2)}{2} - 0.98(7) \\
&\quad - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2) \right) \\
&= 58.571
\end{aligned}$$

So he travels 58.571 metres in 7 seconds. Therefore, he has $100 - 58.571 - 41.429$ metres left to travel at that point. His velocity for rest of the race is 11.52 metres per second. Therefore it will take him another $\frac{41.429}{11.52} = 3.596$ seconds to complete the race. So his total time for the race is $7 + 3.596 = 10.596$. So it takes him 10.60 seconds to finish the race, correct to two decimal places.

(c) A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

(i) Prove that the radius of the snowball is decreasing at a constant rate.

Let t be time. Let r be the radius, A the surface area and V the volume of the snowball. From the Formula and Tables booklet we know that $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$. In particular,

$$\frac{dV}{dr} = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2 = A.$$

Now we are told that the rate of change of volume with respect to time is proportional to the surface area. In other words,

$$\frac{dV}{dt} = kA \tag{1}$$

for some constant k. Clearly $k < 0$ since the volume of the snowball is decreasing as it melts. On the other hand, using the chain rule, we see that

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt} \\ &= A\frac{dr}{dt} \qquad (2)\end{aligned}$$

Therefore by combining (1) and (2) , we see that

$$A\frac{dr}{dt} = kA.$$

Now dividing across by A yields

$$\frac{dr}{dt} = k$$

where k is a constant, as required.

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(ii) If the snowball loses half of its volume in an hour, how long more will it take for it to melt completely?

Give your answer to the nearest minute.

Let r_0 be the initial radius and let r_2 be the radius after 1 hour.
So the initial volume is $\frac{4}{3}\pi r_0^3$. Therefore after one hour, the volume is $\frac{2}{3}\pi r_0^3$. Therefore

$$\frac{4}{3}\pi r_1^3 = \frac{2}{3}\pi r_0^2.$$

Therefore

$$\left(\frac{r_1}{r_0}\right)^3 = \frac{1}{2}$$

or

$$r_1 = \frac{1}{\sqrt[3]{2}}r_0.$$

Now the radius is decreasing at a constant rate and we have found that it takes 1 hour for it to decrease from r_0 to $\frac{1}{\sqrt[3]{2}}r_0$. Therefore the rate of change of the radius is $r_0 - \frac{1}{\sqrt[3]{2}}r_0$ units per hour.
Now the snowball will have melted completely when the radius reaches 0. So we calculate the time required to to change from r_0 to 0. This will be

$$\frac{\text{total change}}{\text{rate of change}} = \frac{r_0 - 0}{r_0 - \frac{1}{\sqrt[3]{2}}r_0} = \frac{1}{1 - \frac{1}{\sqrt[3]{2}}} \text{ hours.}$$

This is equal to 4.8473 hours (correct to four decimal places). So it will take 3.8473 more hours (after the 1 already elapsed) for it to melt completely.
Now 3.8473 hours is equal $3.8473 \times 60 = 230.84$.
So, to the nearest minute, it will take a further 231 minutes for the snowball to melt completely.

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