DifferentiationH

Question 1

Question 3 **(25 marks)**

(a) Differentiate $\frac{1}{3}x^2 - x + 3$ from first principles with respect to x.

(b) $f(x) = \ln(3x^2 + 2)$ and $g(x) = x + 5$, where $x \in \mathbb{R}$.

Find the value of the derivative of $f\left(g(x)\right)$ at $x = \frac{1}{4}$.

Give your answer correct to 3 decimal places.

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Question 5 **(25 marks)**

The function f is such that $f(x) = 2x^3 + 5x^2 - 4x - 3$, where $x \in \mathbb{R}$.

(a) Show that $x = -3$ is a root of $f(x)$ **and** find the other two roots.

(b) Find the co-ordinates of the local maximum point **and** the local minimum point of the function f.

Local maximum point: Local minimum point:

(c) $f(x) + a$, where a is a constant, has only one real root.
Find the range of possible values of a.

Question 3

The height of the water in a port was measured over a period of time. The average height was found to be 1·6 m. The height measured in metres, $h(t)$, was modelled using the function

$$h(t) = 1{\cdot}6 + 1{\cdot}5\cos\left(\frac{\pi}{6}t\right)$$

where t represents the number of hours since the last recorded high tide and $\left(\frac{\pi}{6}t\right)$ is expressed in radians.

(a) Find the period and range of $h(t)$.

Period:

Range:

(b) Find the maximum height of the water in the port.

(c) Find the rate at which the height of the water is changing when $t = 2$, correct to two decimal places. Explain your answer in the context of the question.

Rate:

Explanation:

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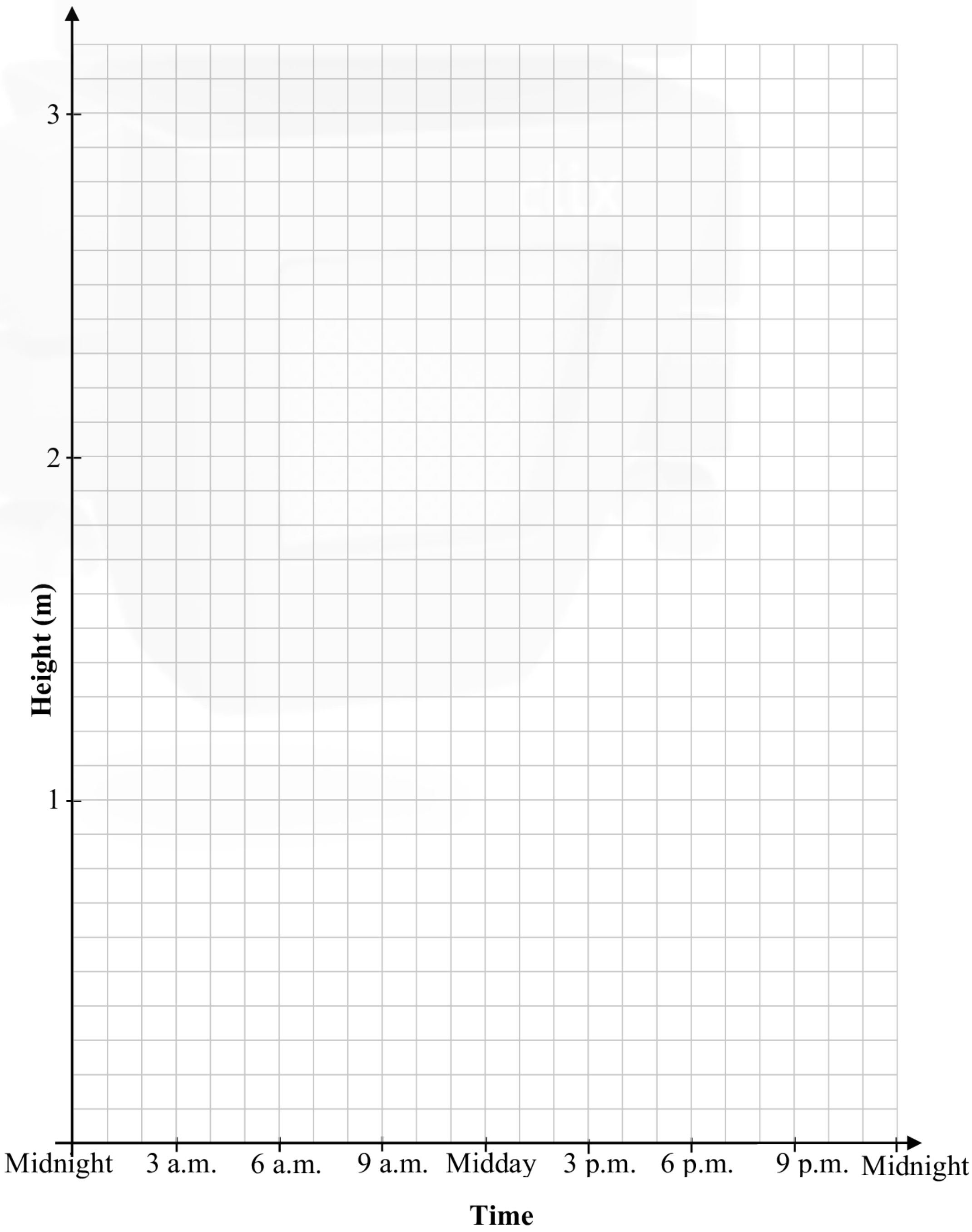
(d) (i) On a particular day the high tide occurred at midnight (i.e. $t = 0$). Use the function to complete the table and show the height, $h(t)$, of the water between midnight and the following midnight.

$h(t) = 1{\cdot}6 + 1{\cdot}5\cos\left(\frac{\pi}{6}t\right)$									
Time	Midnight	3 a.m.	6 a.m.	9 a.m.	12 noon	3 p.m.	6 p.m.	9 p.m.	Midnight
t **(hours)**	0	3							
$h(t)$ **(m)**									

(ii) Sketch the graph of $h(t)$ between midnight and the following midnight.



(e) Find, from your sketch, the difference in water height between low tide and high tide.

(f) A fully loaded barge enters the port, unloads its cargo and departs some time later.
The fully loaded barge requires a minimum water level of 2 m.
When the barge is unloaded it only requires 1·5 m.
Use your graph to estimate the **maximum** amount of time that the barge can spend in port, without resting on the sea-bed.

Question 4

(a) Differentiate the function $(2x+4)^2$ from first principles, with respect to x.

(b) **(i)** If $y = x\sin\left(\frac{1}{x}\right)$, find $\frac{dy}{dx}$.

(ii) Find the slope of the tangent to the curve $y = x\sin\left(\frac{1}{x}\right)$, when $x = \frac{4}{\pi}$.
Give your answer correct to two decimal places.

Question 5

(a) **(i)** Air is pumped into a spherical exercise ball at the rate of 250 cm^3 per second.
Find the rate at which the radius is increasing when the radius of the ball is 20 cm.
Give your answer in terms of π.

(ii) Find the rate at which the surface area of the ball is increasing when the radius of the ball is 20 cm.

(b) The inflated ball is kicked into the air from a point O on the ground. Taking O as the origin, $(x,\ f(x))$ approximately describes the path followed by the ball in the air, where

$$f(x) = -x^2 + 10x$$

and both x and $f(x)$ are measured in metres.

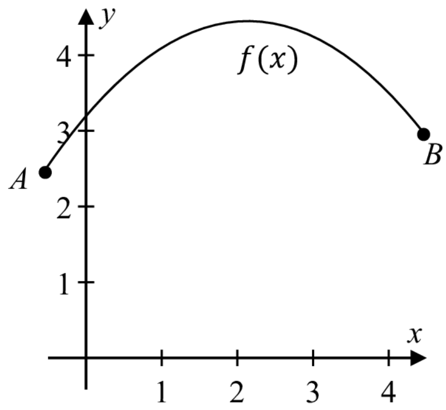
(i) Find the values of x when the ball is on the ground.

(ii) Find the average height of the ball above the ground, during the interval from when it is kicked until it hits the ground again.

(a) The diagram shows Sarah's first throw at the basket in a basketball game. The ball left her hands at A and entered the basket at B. Using the co-ordinate plane with $A(-0{\cdot}5, 2{\cdot}565)$ and $B(4{\cdot}5, 3{\cdot}05)$, the equation of the path of the centre of the ball is

$$f(x) = -0{\cdot}274x^2 + 1{\cdot}193x + 3{\cdot}23,$$

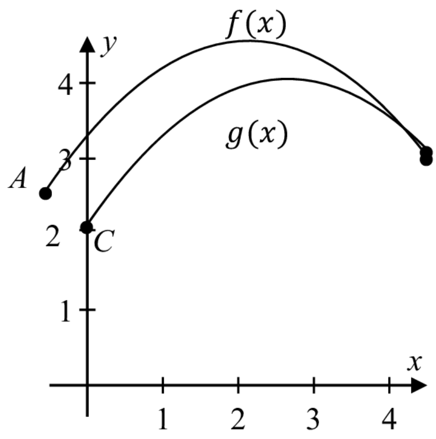
where both x and $f(x)$ are measured in metres.



(i) Find the maximum height reached by the centre of the ball, correct to three decimal places.

(ii) Find the acute angle to the horizontal at which the ball entered the basket. Give your answer correct to the nearest degree.

(iii) Sarah took a second throw. This throw followed the path of the parabola $g(x)$ as shown.
The ball left Sarah's hands at the point $C(0, 2)$.
The graph $y = g(x)$ is the image of the graph $y = f(x)$ under the translation which maps A onto C.
Using your result from part **a(i)**, show that the centre of this ball reached its maximum height at the point $(2{\cdot}677,\ 3{\cdot}964)$, correct to three decimal places.



(iv) Hence, or otherwise, find the equation of the parabola $g(x)$.

(b) Differentiate $x-\sqrt{x+6}$ with respect to x.

(c) Find the co-ordinates of the turning point of the function $y=x-\sqrt{x+6},\ x\geq -6$.

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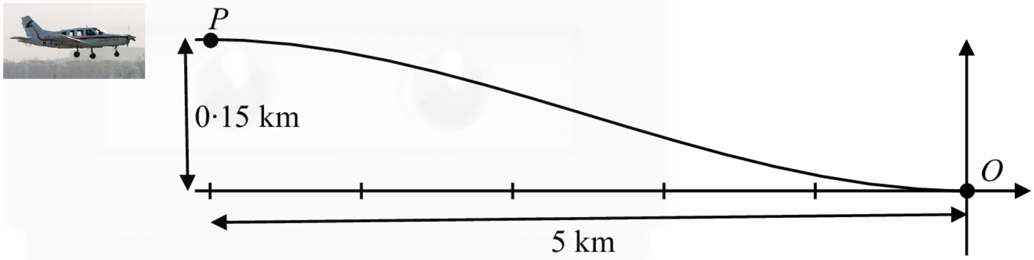
Question 7 **(50 marks)**

A plane is flying horizontally at P at a height of 150 m above level ground when it begins its descent. P is 5 km, horizontally, from the point of touchdown O. The plane lands horizontally at O.



Taking O as the origin, $(x,\ f(x))$ approximately describes the path of the plane's descent where $f(x) = 0\cdot0024x^3 + 0\cdot018x^2 + cx + d,\ -5 \leq x \leq 0,$ and both x and $f(x)$ are measured in km.

(a) **(i)** Show that $d = 0$.

(ii) Using the fact that P is the point $(-5,\ 0\cdot15)$, or otherwise, show that $c = 0$.

(b) **(i)** Find the value of $f'(x)$, the derivative of $f(x)$, when $x = -4$.

(ii) Use your answer to part **(b) (i)** above to find the angle at which the plane is descending when it is 4 km from touchdown. Give your answer correct to the nearest degree.

(c) Show that $(-2{\cdot}5,\ 0{\cdot}075)$ is the point of inflection of the curve $y = f(x)$.

(d) **(i)** If (x, y) is a point on the curve $y = f(x)$, verify that $(-x-5, -y+0{\cdot}15)$ is also a point on $y = f(x)$.

(ii) Find the image of $(-x-5, -y+0{\cdot}15)$ under symmetry in the point of inflection.

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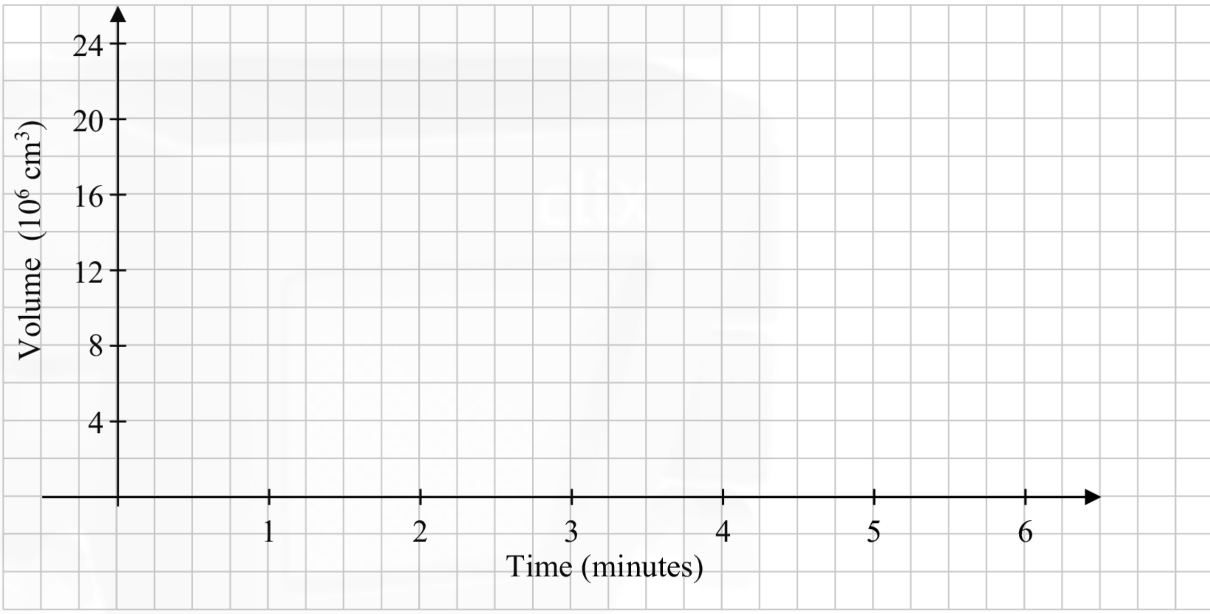
Question 8 **(50 marks)**

An oil-spill occurs off-shore in an area of calm water with no currents. The oil is spilling at a rate of 4×10^{6} cm^{3} per minute. The oil floats on top of the water.

(a) **(i)** Complete the table below to show the total volume of oil on the water after each of the first 6 minutes of the oil-spill.

Time (minutes)	1	2	3	4	5	6
Volume (10^6 cm^3)		8				

(ii) Draw a graph to show the total volume of oil on the water over the first 6 minutes.



(iii) Write an equation for $V(t)$, the volume of oil on the water, in cm^3, after t minutes.

(b) The spilled oil forms a circular oil slick **1 millimetre** thick.

(i) Write an equation for the volume of oil in the slick, in cm^3, when the radius is r cm.

(ii) Find the rate, in cm per minute, at which the radius of the oil slick is increasing when the radius is 50 m.

(c) Show that the area of water covered by the oil slick is increasing at a constant rate of 4×10^7 cm^2 per minute.

(d) The nearest land is 1 km from the point at which the oil-spill began. Find how long it will take for the oil slick to reach land. Give your answer correct to the nearest hour.

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Question 9 **(50 marks)**

The approximate length of the day in Galway, measured in hours from sunrise to sunset, may be calculated using the function

$$f(t) = 12{\cdot}25 + 4{\cdot}75\sin\left(\frac{2\pi}{365}t\right),$$

where t is the number of days after March 21st and $\left(\frac{2\pi}{365}t\right)$ is expressed in radians.

(a) Find the length of the day in Galway on June 5th (76 days after March 21st). Give your answer in hours and minutes, correct to the nearest minute.

(b) Find a date on which the length of the day in Galway is approximately 15 hours.

(c) Find $f'(t)$, the derivative of $f(t)$.

(d) Hence, or otherwise, find the length of the longest day in Galway.

(e) Use integration to find the average length of the day in Galway over the six months from March 21^{st} to September 21^{st} (184 days). Give your answer in hours and minutes, correct to the nearest minute.

Question 11

(a) Differentiate the function $2x^2 - 3x - 6$ with respect to x from first principles.

(b) Let $f(x) = \dfrac{2x}{x+2}$, $x \neq -2$, $x \in \mathbb{R}$. Find the co-ordinates of the points at which the slope of the tangent to the curve $y = f(x)$ is $\dfrac{1}{4}$.

Question 9 **(50 marks)**

(a) Let $f(x) = -0{\cdot}5x^2 + 5x - 0{\cdot}98$, where $x \in \mathbb{R}$.

(i) Find the value of $f(0{\cdot}2)$.

(ii) Show that f has a local maximum point at $(5, 11{\cdot}52)$.

(b) A sprinter's velocity over the course of a particular 100 metre race is approximated by the following model, where v is the velocity in metres per second, and t is the time in seconds from the starting signal:

$$v(t) = \begin{cases} 0, & \text{for } 0 \le t < 0{\cdot}2 \\ -0{\cdot}5t^2 + 5t - 0{\cdot}98, & \text{for } 0{\cdot}2 \le t < 5 \\ 11{\cdot}52, & \text{for } t \ge 5 \end{cases}$$

Note that the function in part **(a)** is relevant to $v(t)$ above.

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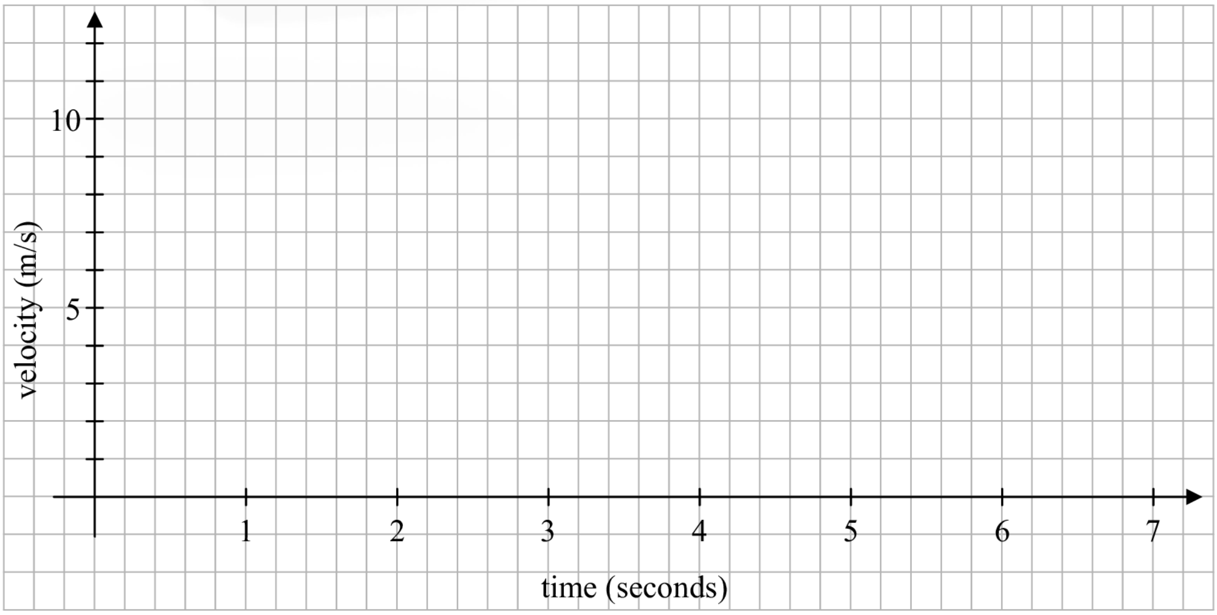
(i) Sketch the graph of v as a function of t for the first 7 seconds of the race.



(ii) Find the distance travelled by the sprinter in the first 5 seconds of the race.

(iii) Find the sprinter's finishing time for the race. Give your answer correct to two decimal places.

(c) A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

(i) Prove that the radius of the snowball is decreasing at a constant rate.

(ii) If the snowball loses half of its volume in an hour, how long more will it take for it to melt completely?

Give your answer correct to the nearest minute.

Question 5 **(25 marks)**

A is the closed interval $[0,5]$. That is, $A = \{x \mid 0 \le x \le 5, x \in \mathbb{R}\}$.

The function f is defined on A by:

$$f : A \to \mathbb{R} : x \mapsto x^3 - 5x^2 + 3x + 5.$$

(a) Find the maximum and minimum values of f.

(b) State whether f is injective. Give a reason for your answer.