Question 1

(ii) Explain what is meant by the indefinite integral of a function f.

The indefinite integral of f is the general form of a function whose derivative is f.

Alternative answer: The indefinite integral of f is $F(x)+C$ where $F' = f$ and C is constant (the constant of integration).

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(iii) Write down the indefinite integral of g, the function in part **(i)**.

Answer: $\int g(x)dx = \frac{1}{4}x^4 - x^3 + 3x + C.$

(b) **(i)** Let $h(x) = x\ln x$, for $x \in \mathbb{R}, x > 0$.
Find $h'(x)$.

Using the product rule we see that

$$h'(x) = (x)'\ln x + x(\ln x)'.$$

But $(x)' = 1$ and $(\ln x)' = \frac{1}{x}$. Therefore

$$\begin{aligned} h'(x) &= (1)\ln x + x\left(\frac{1}{x}\right) \\ &= \ln x + 1. \end{aligned}$$

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(ii) Hence, find $\int \ln x dx$.

We know that $h'(x) = \ln x + 1$. Also, we know that $(x)' = 1$. So if $F(x) = h(x) - x$, then

$$F'(x) = h'(x) - (x)' = \ln x + 1 - 1 = \ln x.$$

Therefore $\int \ln x\, dx = F(x) + c$. But $F(x) = h(x) - x = x\ln x - x$. Therefore

$$\int \ln x\, dx = x\ln x - x + C.$$

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Question 2

Type of function	Function	First derivative	Second derivative
Quadratic	k	B	I
Cubic	f	D	II
Trigonometric	g	A	III
Exponential	h	C	IV

(b) For **one** row in the table, explain your choice of first derivative and second derivative.

A quadratic function differentiates to a line which differentiates to a constant.

Question 3

(a) Differentiate the function $2x^2 - 3x - 6$ with respect to x from first principles.

$$f(x) = 2x^2 - 3x - 6$$
$$f(x+h) = 2(x+h)^2 - 3(x+h) - 6 = 2x^2 + 4xh + 2h^2 - 3x - 3h - 6$$
$$f(x+h) - f(x) = 4xh + 2h^2 - 3h$$
$$\underset{h \to 0}{Limit}\left(\frac{f(x+h) - f(x)}{h}\right) = \underset{h \to 0}{Limit}\left(\frac{4xh + 2h^2 - 3h}{h}\right) = 4x - 3$$

(b) Let $f(x) = \dfrac{2x}{x+2}$, $x \neq -2$, $x \in \mathbb{R}$. Find the co-ordinates of the points at which the slope of the tangent to the curve $y = f(x)$ is $\dfrac{1}{4}$.

$$f(x) = \frac{2x}{x+2}$$
Let $u(x) = 2x \Rightarrow u'(x) = 2$ and $v(x) = x + 2 \Rightarrow v'(x) = 1$
$$f'(x) = \frac{(x+2)(2) - 2x(1)}{(x+2)^2} = \frac{4}{(x+2)^2}$$
$$f'(x) = \frac{1}{4} \Rightarrow \frac{4}{(x+2)^2} = \frac{1}{4}$$
$$\Rightarrow 16 = (x+2)^2$$

$\Rightarrow x + 2 = 4$ or $x + 2 = -4$
$\Rightarrow x = 2$ or $x = -6$

or

$x^2 + 4x - 12 = 0$
$(x-2)(x+6) = 0$
$\Rightarrow x - 2 = 0$ or $x + 6 = -0$
$\Rightarrow x = 2$ or $x = -6$

$$f(-6) = \frac{-12}{-6+2} = 3 \text{ and } f(2) = \frac{4}{2+2} = 1$$
Points $(-6, 3)$ and $(2, 1)$

Question 4

(i) Find the value of $f(0.2)$

Substituting 0.2 for x gives

$$f(0.2) = -0.5(0.2)^2 + 5(0.2) - 0.98 = -0.5(0.04) + 1 - 0.98 = 0$$

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(ii) Show that f has a local maximum point at $(5, 11.52)$.

First we calculate the derivative of f:

$$f'(x) = -0.5(2x) + 5(1) - 0 = -x + 5.$$

Now $f'(5) = -5 + 5 = 0$. Therefore $x = 5$ is a stationary point.
Now

$$f''(x) = -1.$$

So $f''(5) = -1 < 0$. That means that $x = -5$ is a local maximum. Finally,

$$f(5) = -0.5(5^2) + 5(5) - 0.98 = 11.52.$$

Therefore the graph of f has a local maximum point at $(5, 11.52)$.

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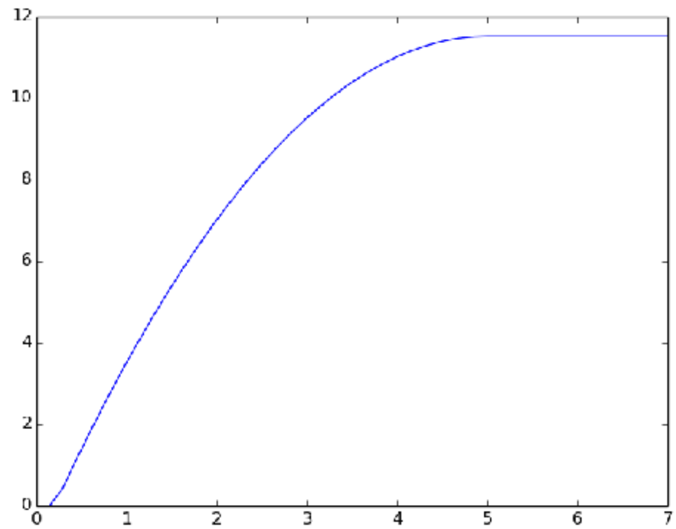

Note that between $t = 0$ and $t = 0.2$ the graph is just a horizontal line along the t-axis. Likewise, for $t \geq 5$ the graph is a horizontal line at height $v = 11.52$. In between $t = 0.2$ and $t = 5$ the function is a quadratic so the graph must be a parabola. We can sketch this by evaluating the function at three or four points. For example $v(1) = 3.52$, $v(2) = 7.02$, $v(3) = 9.52$ and $v(4) = 11.02$. So we plot the points $(1, 3.52)$, $(2, 7.02)$, $(3, 9.52)$ and $(4, 11.02)$ and then join them by a smooth curve. Make sure that this parabolic arc starts at $(0.2, 0)$ and ends at $(5, 11.52)$.

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(ii) Find the distance travelled by the sprinter in the first 5 seconds of the race.

The distance travelled in the first 5 seconds of the race is given by

$$\int_0^5 v(t) \quad dt.$$

Now

$$\begin{aligned}
\int_0^5 v(t) \quad dt &= \int_0^{0.2} v(t) \quad dt + \int_{0.2}^5 v(t) \quad dt \\
&= \int_0^{0.2} 0 \quad dt + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= 0 + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \Big|_{0.2}^5 \\
&= \frac{0.5(5^3)}{3} + \frac{5(5^2)}{2} - 0.98(5) - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2)\right) \\
&= 36.864
\end{aligned}$$

So the sprinter travels 36.864 metres in the first 5 seconds of the race.

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(iii) Find the sprinter's finishing time for the race. Give your answer correct to two decimal places.

We have just seen that the sprinter travels 36.864 metres in the first 5 seconds of the race. So he has 63.136metres left to travel to complete the race at that point. Also after 5 seconds, his velocity is a constant 11.52 metres per second. Therefore it will take him a further $\frac{63.136}{11.52}$ seconds to complete the race. Now $\frac{63.136}{11.52} = 5.48$ correct to two decimal places. So his total time is $5 + 5.48 = 10.48$ seconds, correct to two decimal places.

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Question 5

7 (a)

$$2x + 3y^2 \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1-2x}{3y^2}. \quad \therefore \text{ Slope of tangent at } (3, -2) = \frac{-5}{12}.$$

7 (b) (i)

$$\frac{dx}{dt}=\frac{1(t+1)-1(t-1)}{(t+1)^2}=\frac{2}{(t+1)^2}.$$

$$\frac{dy}{dt}=\frac{-4(t+1)^2+4t(2)(t+1)}{(t+1)^4}=\frac{-4(t+1)+8t}{(t+1)^3}=\frac{4(t-1)}{(t+1)^3}.$$

7 (b) (ii)

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{4(t-1)}{(t+1)^3}\times\frac{(t+1)^2}{2}=\frac{2(t-1)}{t+1}=2x.$$

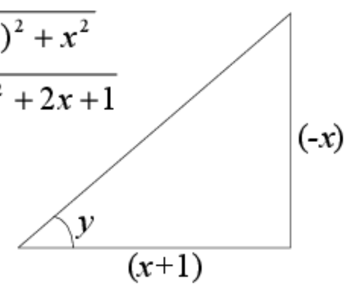
7 (c) (i)

$$f(x): x\rightarrow \tan^{-1}\left(\frac{-x}{x+1}\right)$$

$$f'(x)=\frac{1}{1+\left(\frac{-x}{x+1}\right)^2}\times\frac{-1(x+1)+x(1)}{(x+1)^2}=\frac{(x+1)^2}{x^2+2x+1+x^2}\times\frac{-1}{(x+1)^2}=\frac{-1}{2x^2+2x+1}.$$

OR

$$\sqrt{(x+1)^2+x^2}$$
$$=\sqrt{2x^2+2x+1}$$



$$y=\tan^{-1}\left(\frac{-x}{x+1}\right)$$

$$\tan y=\frac{-x}{x+1}$$

$$\sec^2 y.\frac{dy}{dx}=\frac{(x+1)(-1)-(-x)(1)}{(x+1)^2}$$

$$\frac{1}{\cos^2 y}.\frac{dy}{dx}=\frac{-x-1+x}{(x+1)^2}$$

$$\frac{1}{\cos^2 y}.\frac{dy}{dx}=\frac{-1}{(x+1)^2}$$

$$\cos y=\frac{x+1}{\sqrt{2x^2+2x+1}}$$

$$\cos^2 y=\frac{(x+1)^2}{2x^2+2x+1}$$

(…/)

$$\frac{dy}{dx}=\frac{-\cos^2 y}{(x+1)^2}$$

$$=\frac{-1}{(x+1)^2}.\frac{(x+1)^2}{2x^2+2x+1}$$

$$=\frac{-1}{2x^2+2x+1}$$

7 (c) (ii)

Diagram A is correct.

It cannot be Diagram B, as these curves are not "parallel" (i.e. identical up to a vertical shift, which is necessary because their derivatives are equal for all x).

It cannot be Diagram C as these graphs are increasing, whereas they should be decreasing, because their derivatives are negative for $x > 0$.

OR

Given $f'(x) = g'(x)$

$\Rightarrow m_1 = m_2$ (same slopes)

$\Rightarrow$ parallel curves

$$f'(x) = \frac{-1}{2x^2 + 2x + 1} < 0 \quad \text{when } x > 0$$

$\Rightarrow$ Both $f(x)$ and $g(x)$ are decreasing funtions.

Diagram A: correct
Diagram B: not parallel curves
Diagram C: increasing curves

Question 6

7 (a)

$$x^2 - y^2 = 25$$

$$y^2 = x^2 - 25$$

$$y = \sqrt{x^2 - 25} \qquad \text{OR} \qquad y = -\sqrt{x^2 - 25}$$

$$y = (x^2 - 25)^{\frac{1}{2}} \qquad\qquad y = -(x^2 - 25)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 - 25)^{-\frac{1}{2}}.2x \qquad\qquad \frac{dy}{dx} = -\left[\frac{1}{2}(x^2 - 25)^{-\frac{1}{2}}.2x\right]$$

$$= \frac{x}{\sqrt{x^2 - 25}} \qquad\qquad = -\left[\frac{x}{\sqrt{x^2 - 25}}\right]$$

$$= \frac{x}{y} \qquad\qquad = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

7 (b) (i)

$$x=\frac{3t}{t^2-2} \quad\Rightarrow\quad \frac{dy}{dt}=\frac{3(t^2-2)-3t.2t}{(t^2-2)^2}=\frac{-3t^2-6}{(t^2-2)^2}.$$

$$y=\frac{6}{t^2-2}=6(t^2-2)^{-1} \quad\Rightarrow\quad \frac{dy}{dt}=-6(t^2-2)^{-2}.2t=\frac{-12t}{(t^2-2)^2}.$$

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx} \;=\; \frac{-12t}{(t^2-2)^2}.\frac{(t^2-2)^2}{-3t^2-6} \;=\; \frac{12t}{3t^2+6} \;=\; \frac{4t}{t^2+2}.$$

(b)(ii) Slope, point	**5 marks**	**Att 2**
Equation	**5 marks**	**Att 2**

7 (b) (ii)

$t=2 \;\Rightarrow\; x=\frac{6}{2}=3$ and $t=2 \;\Rightarrow\; y=\frac{6}{2}=3.$ $\therefore$ Point is $(3,3)$.

Slope of tangent at $t=2$ is $\frac{8}{6}=\frac{4}{3}$.

$\therefore$ Equation of tangent: $y-3=\frac{4}{3}(x-3) \;\Rightarrow\; 4x-3y-3=0.$

(c) (i)	**5 marks**	**Att 2**

$f(x)=x^3-3x^2+3x-4$.

$f(2)=8-12+6-4=-2<0.$

$f(3)=27-27+9-4=5>0.$

$\therefore$ root lies between 2 and 3.

(c) (ii)	**5 marks**	**Att 2**

$$\begin{aligned} f(2.5)&=(2.5)^3-3(2.5)^2+3(2.5)-4\\ &=15.625-18.75+7.5-4\\ &=0.375 \end{aligned}$$

$f(2)<0$ and $f(2.5)>0$. $\therefore$ root is between 2 and 2.5.

So, root is closer to 2 than to 3.

(c) (iii) Formula + Differentiation	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

7 (c) (iii)

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$, where $f(x)=x^3-3x^2+3x-4$ and $f'(x)=3x^2-6x+3$

Ann: $f(2)=-2$ and $f'(2)=3$. $x_2=2-\frac{f(2)}{f'(2)}=2-\frac{-2}{3}=2\frac{2}{3}=2.666...$

Barry: $f(3)=5$ and $f'(3)=12$. $x_2=3-\frac{f(3)}{f'3}=3-\frac{5}{12}=2\frac{7}{12}=2.583...$

Both of these are above the root, so the lower one is closer (i.e. Barry's).

QUESTION 7

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

7 (a) Taking 1 as a first approximation of a root of $x^3 + 2x - 4 = 0$, use the Newton Raphson method to calculate a second approximation of this root.

(a) **10 marks** **Att 3**

7 (a)

$f(x) = x^3 + 2x - 4$ $\qquad f'(x) = 3x^2 + 2$

$f(1) = (1)^3 + 2(1) - 4 = -1$ $\qquad f'(1) = 3(1)^2 + 2 = 5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad x_1 = 1$$

$$= 1 - \frac{(-1)}{5} \qquad = 1 + \frac{1}{5} \qquad = \frac{6}{5}$$

Blunder (-3)
B1 Newton-Raphson formula once only
B2 Differentiation
B3 Indices
B4 $x_1 \neq 1$ once only

Slips (-1)
S1 Numerical
S2 Answer not tidied up

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

7 (b) (i) Find the equation of the tangent to the curve $3x^2 + y^2 = 28$ at the point $(2, -4)$.

(ii) $x = e^t \cos t$ and $y = e^t \sin t$. Show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

(i) Differentiation **5 marks** **Att 2**
Equation **5 marks** **Att 2**

7 (b)(i)

$$3x^2 + y^2 = 28$$

$$6x + 2y\frac{dy}{dx} = 0$$

$$2y\left(\frac{dy}{dx}\right) = -6x$$

$$\frac{dy}{dx} = \frac{-6x}{2y} = \frac{-3x}{y}$$

At (2, -4), slope $= \frac{dy}{dx} = \frac{-3(2)}{-4} = \frac{3}{2}$

Tangent is line through (2, -4) with slope $m = \frac{3}{2}$

$$(y - y_1) = m(x - x_1)$$
$$y - (-4) = \frac{3}{2}(x - 2)$$
$$2(y + 4) = 3(x - 2)$$
$$2y + 8 = 3x - 6$$
$$3x - 2y - 14 = 0$$

Blunders (-3)
B1 Differentiation
B2 Incorrect values or no values
B3 Indices
B4 Equation of tangent
B5 Substituting values into formula once only

Slips (-1)
S1 Numerical

Worthless
W1 Integration
W2 No differentiation in 1st 5 marks

(ii) $\frac{dx}{dt}$ **and** $\frac{dy}{dt}$ **5 marks** **Att 2**

$\frac{dy}{dx}$ **5 marks** **Att 2**

7 (b)(ii)

$$x = e^t(\cos t) \qquad y = e^t(\sin t)$$
$$\frac{dx}{dt} = e^t(-\sin t) + \cos t(e^t) \qquad \frac{dy}{dt} = e^t(\cos t) + \sin t(e^t)$$
$$\frac{dx}{dt} = e^t \cos t - e^t \sin t \qquad \frac{dy}{dt} = e^t \cos t + e^t \sin t$$
$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{e^t \cos t + e^t \sin t}{e^t \cos t - e^t \sin t} = \frac{x + y}{x - y}$$

* Note: oversimplified differentiation in first 5 marks leads to Att 2 at most in second marks

Blunders (-3)
B1 Differentiation
B2 Indices
B3 Incorrect $\frac{dy}{dx}$
B4 Answer not in required form

Attempts
A1 Blunder in differentiation formula

Worthless
W1 Integration

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

7 (c) $f(x) = \log_e 3x - 3x$, where $x > 0$.

(i) Show that $(\frac{1}{3}, -1)$ is a local maximum point of $f(x)$.

(ii) Deduce that the graph of $f(x)$ does not intersect the x-axis.

(i) Differentiation	**5 marks**	**Att 2**
Max Value	**5 marks**	**Att 2**
(ii) Only one root for $f'(x) = 0$	**5 marks**	**Att 2**
Absolute max pt.	**5 marks**	**Att 2**

7 (c)(i) $f(x) = \ln(3x) - 3x \qquad x > 0$

$$f'(x) = \frac{1}{3x}(3) - 3 = \frac{1}{x} - 3.$$

$$f''(x) = \frac{-1}{x^2}$$

Local max/min: $f'(x) = 0 \Rightarrow \frac{1}{x} - 3 = 0 \Rightarrow \frac{1}{x} = 3 \Rightarrow x = \frac{1}{3}$.

$$x = \frac{1}{3} \Rightarrow f''(x) = \frac{-1}{x^2} = \frac{-1}{\left(\frac{1}{3}\right)^2} < 0 \Rightarrow \text{local max at } x = \frac{1}{3}$$

$$x = \frac{1}{3} \Rightarrow f(x) = \ln(3x) - (3x) = \ln(1) - (1) = 0 - 1 = -1 \Rightarrow \text{Local max at } \left(\frac{1}{3}, -1\right)$$

or

7(c)(i) $f(x) = \ln 3x - 3x$

$$f'(x) = \frac{1}{x} - 3$$

$$x = \frac{1}{3} \Rightarrow f'(x) = \frac{1}{\left(\frac{1}{3}\right)} - 3 = 3 - 3 = 0 \Rightarrow \text{turning pt at } x = \frac{1}{3}.$$

$$f''(x) = \frac{-1}{x^2} < 0 \text{ for all } x \Rightarrow \text{local max pt at } x = \frac{1}{3}$$

$$x = \frac{1}{3} \Rightarrow y = \ln(3x) - 3x = \ln(1) - 3\left(\frac{1}{3}\right) = -1 \Rightarrow \text{local max is at } \left(\frac{1}{3}, -1\right)$$

(c)(ii)	$f'(x)$ has only one root. This implies that the local max. above is the only turning point. And $f(x)$ is continuous, so the local max pt above is an absolute max. point. Since max pt $\left(\frac{1}{3}, -1\right)$ is below x-axis, the whole graph must lie below x-axis Thus, $f(x)=0$ has no roots, since graph does not cut the x-axis.

* Accept work showing max point to be the only turning point and below x-axis, with or without a diagram.
* No need to mention "absolute" in answer.
* No need to mention continuity

Blunders (-3)
B1 Differentiation
B2 Not testing in $f''(x)$ for max
B3 Incorrect deduction or no deduction from test
B4 Incorrect y value or no y value
B5 Factors once only.

Slips (-1)
S1 $\ln 1 \neq 0$

Worthless
W1 No differentiation

Question 8

7 (a)	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ $x_1 = 2:\quad f(x) = x^3 + x - 9 \Rightarrow f(2) = (2)^3 + 2 - 9 = 1$ $f'(x) = 3x^2 + 1 \Rightarrow f'(2) = 3(2)^2 + 1 = 13$ $x_2 = 2 - \frac{1}{13} = \frac{25}{13}$

7 (b)

$$x = 3\cos\theta - (\cos\theta)^3$$

$$\frac{dx}{d\theta} = -3\sin\theta - 3(\cos\theta)^2.(-\sin\theta)$$
$$= -3\sin\theta + 3\sin\theta\cos^2\theta$$
$$= -3\sin\theta(1-\cos^2\theta)$$
$$= -3\sin^3\theta$$

$$y = 3\sin\theta - (\sin\theta)^3$$

$$\frac{dy}{d\theta} = 3\cos\theta - 3(\sin\theta)^2.\cos\theta$$
$$= 3\cos\theta(1-\sin^2\theta)$$
$$= 3\cos^3\theta$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3\cos^3\theta}{-3\sin^3\theta} = -\frac{1}{\left(\frac{\sin\theta}{\cos\theta}\right)^3} \quad = -\frac{1}{\tan^3\theta}$$

7 (c)

$$y = \ln\left(\frac{3+x}{\sqrt{9-x^2}}\right)$$
$$= \ln(3+x) - \ln\sqrt{9-x^2}$$
$$= \ln(3+x) - \tfrac{1}{2}\ln(9-x^2)$$

$$\frac{dy}{dx} = \frac{1}{3+x} - \frac{1}{2}\left[\frac{1}{9-x^2}(-2x)\right]$$
$$= \frac{1}{3+x} + \frac{x}{9-x^2}$$
$$= \frac{1}{3+x} + \frac{x}{(3-x)(3+x)}$$
$$= \frac{(3-x)+x}{(3-x)(3+x)} = \frac{3}{9-x^2}$$

or

Q7(c)

$$y = \ln\frac{3+x}{\sqrt{(3-x).(3+x)}}$$
$$= \ln\frac{(3+x)^{\frac{1}{2}}}{(3-x)^{\frac{1}{2}}}$$
$$= \tfrac{1}{2}\ln(3+x) - \tfrac{1}{2}\ln(3-x)$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{3+x} - \frac{1}{3-x}(-1)\right]$$
$$= \frac{1}{2}\left[\frac{1}{3+x} + \frac{1}{3-x}\right]$$
$$= \frac{1}{2}\left[\frac{(3-x)+(3+x)}{9-x^2}\right] = \frac{1}{2}\left(\frac{6}{9-x^2}\right) = \frac{3}{9-x^2}$$

Question 9

7(a)

$$f(x) = x^2$$
$$f(x+h) = (x+h)^2$$
$$f(x+h) - f(x) = (x^2 + 2hx + h^2) - x^2$$
$$f(x+h) - f(x) = 2hx + h^2$$
$$\frac{f(x+h) - f(x)}{h} = 2x + h$$
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 2x$$

or

7(a)

$$y = x^2$$
$$y + \Delta y = (x + \Delta x)^2$$
$$\Delta y = (x + \Delta x)^2 - x^2$$
$$= x^2 + 2x\Delta x + \Delta x^2 - x^2$$
$$= 2x.\Delta x + \Delta x^2$$
$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

7 (b) (i)

$$x = 8 + \ln t^2$$
$$x = 8 + 2\ln t$$
$$\frac{dx}{dt} = 2\left(\frac{1}{t}\right) = \frac{2}{t}$$

$$y = \ln(2 + t^2),$$
$$\frac{dy}{dt} = \frac{1}{2 + t^2}.2t$$
$$= \frac{2t}{2 + t^2}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{2t}{2+t^2}\right)}{\left(\frac{2}{t}\right)} = \frac{t^2}{2 + t^2}$$

At $t = \sqrt{2}$: $\quad t^2 = 2 \quad \Rightarrow \frac{dy}{dx} = \frac{t^2}{2 + t^2} = \frac{2}{2+2} = \frac{1}{2}$

7 (b) (ii)

$$xy^2 + y = 6$$
$$\left(x.2y\frac{dy}{dx} + y^2\right) + \frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(2xy + 1) = -y^2$$
$$\frac{dy}{dx} = \frac{-y^2}{2xy + 1}$$

At $p(1, 2)$ $\quad x = 1$ and $y = 2$

$$m = \frac{dy}{dx} = \frac{-(2)^2}{2(1)(2) + 1} = \frac{-4}{5}$$

7(c) (i) Roots $\pm\sqrt{k} \Rightarrow$ Equation: $x^2 - k = 0$.

7(c)(ii) Equation: $x^2 = k$ or $x^2 - k = 0$, so let $f(x) = x^2 - k$.

$$\therefore \quad f(u_n) = u_n^{\,2} - k$$
$$f'(u_n) = 2u_n$$

Newton-Raphson:
$$u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)}$$
$$= u_n - \frac{u_n^{\,2} - k}{2u_n}$$
$$= \frac{2u_n^{\,2} - (u_n^{\,2} - k)}{2u_n}$$
$$u_{n+1} = \frac{u_n^{\,2} + k}{2u_n}$$

Hence the given rule is the Newton-Raphson method applied to $f(x) = x^2 - k$. Thus it can be used with a suitable initial value to find increasingly accurate approximations for $\sqrt{k}$.

7(c)(iii) $u_2 = \dfrac{u_1^{\,2} + k}{2u_1}$ $\qquad k = 3; \quad u_1 = \dfrac{3}{2}$

$$u_2 = \frac{\left(\frac{3}{2}\right)^2 + 3}{2\left(\frac{3}{2}\right)} = \frac{\frac{9}{4} + 3}{3} = \frac{21}{12} = \frac{7}{4}$$
$$u_3 = \frac{(u_2)^2 + k}{2u_2} = \frac{\left(\frac{7}{4}\right)^2 + 3}{2\left(\frac{7}{4}\right)} = \frac{\frac{49}{16} + 3}{\frac{7}{2}} = \frac{\left(\frac{97}{16}\right)}{\left(\frac{7}{2}\right)} = \frac{97}{56}$$

Question 11

7 (a)

$$f(x) = x^2 \Rightarrow f(x+h) = (x+h)^2.$$
$$\frac{dy}{dx} = \underset{h\to 0}{\text{Limit}} \frac{f(x+h) - f(x)}{h} = \underset{h\to 0}{\text{limit}} \frac{(x+h)^2 - x^2}{h} = \underset{h\to 0}{\text{limit}} \frac{2xh + h^2}{h}$$
$$= \underset{h\to 0}{\text{limit}}(2x + h) = 2x.$$

7 (b) (i)

$$y=\frac{\cos x+\sin x}{\cos x-\sin x} \Rightarrow \frac{dy}{dx}=\frac{(\cos x-\sin x)(-\sin x+\cos x)-(\cos x+\sin x)(-\sin x-\cos x)}{(\cos x-\sin x)^2}.$$

$$\frac{dy}{dx}=\frac{(\cos x-\sin x)^2+(\cos x+\sin x)^2}{(\cos x-\sin x)^2}=\frac{2}{(\cos x-\sin x)^2}.$$

7 (b) (ii)

$$\frac{dy}{dx}=\frac{(\cos x-\sin x)^2+(\cos x+\sin x)^2}{(\cos x-\sin x)^2}=1+\frac{(\cos x+\sin x)^2}{(\cos x-\sin x)^2}=1+y^2.$$

OR

7 (b) (i) & 7 (b) (ii)

$$y=\frac{\cos x+\sin x}{\cos x-\sin x}=(\cos x+\sin x).(\cos x-\sin x)^{-1}$$

$$\frac{dy}{dx}=(\cos x+\sin x)\left[-1.(\cos x-\sin x)^{-2}(-\sin x-cox)\right]+(\cos x-\sin x)^{-1}(-\sin x+\cos x)$$

$$=\frac{(\cos x+\sin x)^2}{(\cos x-\sin)^2}+\frac{\cos x-\sin x}{\cos x-\sin x}$$

$$=\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)^2+1$$

$$=y^2+1$$

7 (c) (i)

$$f(x)=(1+x)\log_e(1+x) \quad \Rightarrow \quad f'(x)=(1+x).\left(\frac{1}{1+x}\right)+\log_e(1+x)=1+\log_e(1+x).$$

$$f'(x)=0 \quad \Rightarrow \quad \log_e(1+x)=-1 \quad \Rightarrow \quad 1+x=e^{-1}. \quad \therefore \quad x=\frac{1}{e}-1=\frac{1-e}{e}.$$

$$y=\left(\frac{1}{e}\right)\log_e\left(\frac{1}{e}\right)\Rightarrow y=\frac{1}{e}(-\log_e e)=-\frac{1}{e}.$$ So turning point is $\left(\frac{1-e}{e},\ -\frac{1}{e}\right)$.

OR

7 (c) (i) $f'(x)=[\log_e(1+x)]+1$

At $x=\frac{1-e}{e}$, $f'(x)=\log_e\left(1+\frac{1-e}{e}\right)+1 = \log_e\left(\frac{e+1-e}{e}\right)+1 = \log_e\left(\frac{1}{e}\right)+1$

$$=\left[\log_e(1)-\log_e(e)\right]+1$$

$$=0-1+1 \quad =0.$$

So $f'(x)=0$ at $x=\frac{1-e}{e}$.

Also, at $x=\frac{1-e}{e}$, $y=\left(\frac{1}{e}\right)\log_e\left(\frac{1}{e}\right)\Rightarrow y=\frac{1}{e}(-\log_e e)=-\frac{1}{e}$.

So turning point is $\left(\frac{1-e}{e},\ -\frac{1}{e}\right)$.

7 (c) (ii)

$$f''(x)=\frac{1}{1+x}\Rightarrow f''\left(\frac{1-e}{e}\right)=\frac{1}{1+\frac{1-e}{e}}=\frac{e}{1}=e>0. \quad \therefore \quad \left(\frac{1-e}{e},-\frac{1}{e}\right)$$ is a local minimum.

Question 12

7 (a)

$$\frac{dx}{dt}=6t-6, \quad \frac{dy}{dt}=2-2t.$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{2-2t}{6t-6}=-\frac{1}{3}.$$

7 (b) (i)

$$2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}+4\frac{dy}{dx}=0.$$

$$\therefore \frac{dy}{dx}(2x-6y-4)=2x-2y \quad \Rightarrow \quad \frac{dy}{dx}=\frac{x-y}{x-3y-2}.$$

(ii)

Slope of tangent at $(-3,1)=\frac{-3-1}{-3-3-2}=\frac{1}{2}$.

Slope of tangent at $(1,-3)=\frac{1+3}{1+9-2}=\frac{1}{2}$.

Equal slopes, therefore parallel tangents.

7 (c) (i)

$$f(0) = -1 < 0 \text{ and } f(1) = 32 - 48 + 20 - 1 = 3 > 0.$$

$\therefore f$ has a root between 0 and 1.

(ii)

$$f(x) = 32x^3 - 48x^2 + 20x - 1 \Rightarrow f'(x) = 96x^2 - 96x + 20.$$

$$f(0{\cdot}5) = 1 \text{ and } f'(0{\cdot}5) = -4$$

$$\therefore x_2 = 0{\cdot}5 - \frac{1}{-4} = 0{\cdot}75$$

$$f(0{\cdot}75) = 0{\cdot}5 \text{ and } f'(0{\cdot}75) = 2.$$

$$\therefore x_3 = 0{\cdot}75 - \frac{0{\cdot}5}{2} = 0{\cdot}5.$$

(iii)

All further approximations will continue in the sequence $0{\cdot}5, 0{\cdot}75, 0{\cdot}5, 0{\cdot}75, \ldots$

Question 13

7 (a)

$$f(x) = 2x + \sin 2x \Rightarrow f'(x) = 2 + 2\cos 2x.$$

7 (b) (i)

$$5x^2 + 5y^2 + 6xy = 16.$$

$$\therefore 10x + 10y\frac{dy}{dx} + 6x\frac{dy}{dx} + 6y = 0.$$

$$\therefore \frac{dy}{dx}(10y + 6x) = -10x - 6y \Rightarrow \frac{dy}{dx} = \frac{-5x - 3y}{3x + 5y}.$$

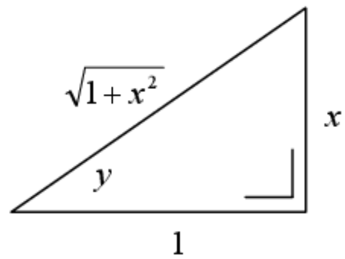
7 (b) (ii)

$$m_1 = \text{slope of tangent at } (1, 1) = \frac{-5 - 3}{3 + 5} = -1.$$

$$m_2 = \text{slope of tangent at } (2, -2) = \frac{-10 + 6}{6 - 10} = 1.$$

But $m_1 . m_2 = -1$, $\therefore$ tangents are perpendicular to each other.

7(c) $y = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) \Rightarrow \sin y = \dfrac{x}{\sqrt{1+x^2}}$



$$\tan y = \frac{x}{1} = x$$

$$y = \tan^{-1} x$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

OR

Differentiation	**15 marks**	**Att 5**
Other work	**5 marks**	**Att 2**

7 (c)

$$y = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$$

$$\sin y = \frac{x}{\sqrt{1+x^2}} = \frac{x}{(1+x^2)^{\frac{1}{2}}}$$

$$\cos y.\frac{dy}{dx} = \frac{(1+x^2)^{\frac{1}{2}}(1) - x\left[\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x\right]}{(1+x^2)}$$

$$= \frac{(1+x^2)^{\frac{1}{2}} - \dfrac{x^2}{(1+x^2)^{\frac{1}{2}}}}{(1+x^2)^{\frac{3}{2}}}$$

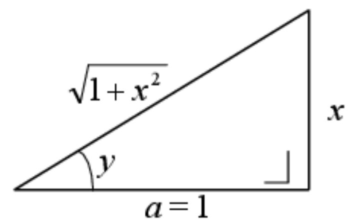
$$= \frac{1+x^2-x^2}{(1+x^2)^{\frac{3}{2}}}$$

$$\cos y.\frac{dy}{dx} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}\cdot\frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$= (1+x^2)^{\frac{1}{2}}.\frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$



$$a^2 + x^2 = \left(\sqrt{1+x^2}\right)^2$$

$$a^2 + x^2 = 1 + x^2$$

$$a^2 = 1$$

$$a = 1$$

$$\sin y = \frac{x}{(1+x^2)^{\frac{1}{2}}}$$

$$\cos y = \frac{1}{(1+x^2)^{\frac{1}{2}}}$$

$$\frac{1}{\cos y} = \frac{(1+x^2)^{\frac{1}{2}}}{1} = (1+x^2)^{\frac{1}{2}}$$