Question 1

Question 6 **(25 marks)**

(a) **(i)** Write down three distinct anti-derivatives of the function

$$g : x \mapsto x^3 - 3x^2 + 3, \qquad x \in \mathbb{R}.$$

1. ______________________________

2. ______________________________

3. ______________________________

(ii) Explain what is meant by the indefinite integral of a function f.

(iii) Write down the indefinite integral of g, the function in part **(i)**.

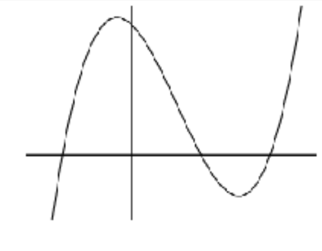
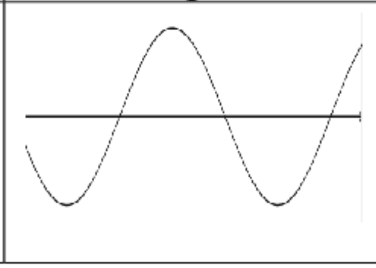
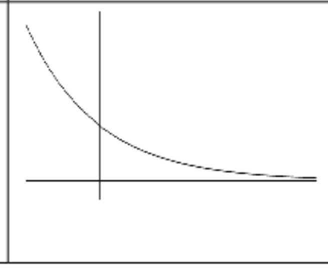
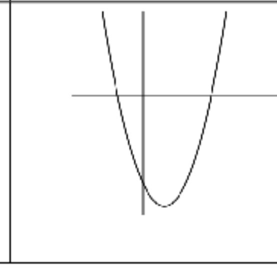
Answer: ______________________________

(b) **(i)** Let $h(x) = x \ln x$, for $x \in \mathbb{R}$, $x > 0$.
Find $h'(x)$.

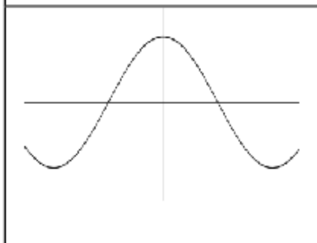
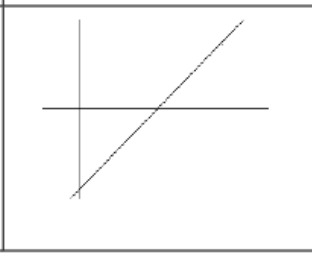
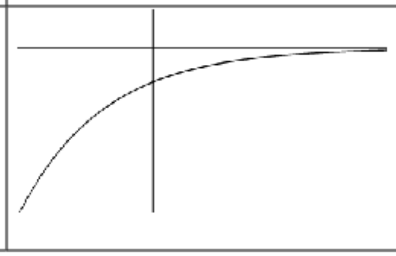
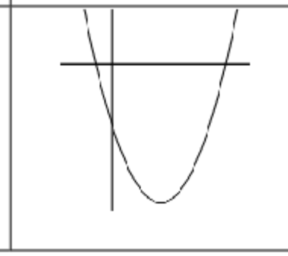
(ii) Hence, find $\int \ln x \, dx$.

Question 2

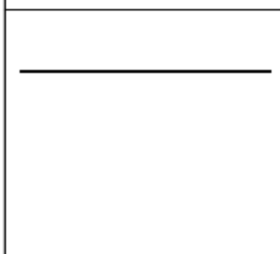
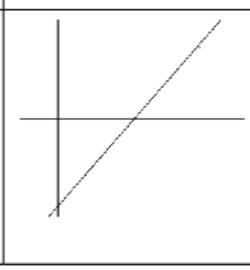
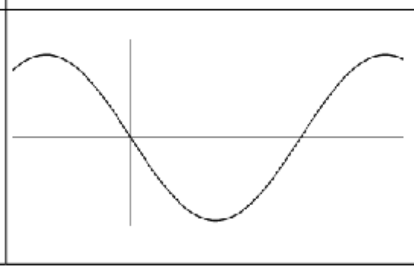
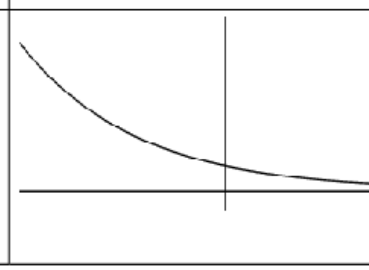
Each diagram below shows part of the graph of a function. Each of these functions is either quadratic or cubic or trigonometric or exponential (not necessarily in that order).

f	*g*	*h*	*k*
			

Each diagram below shows part of the graph of the first derivative of one of the above functions (not necessarily in the same order).

A	B	C	D
			

Each diagram below shows part of the graph of the second derivative of one of the original functions (not necessarily in the same order).

I	II	III	IV
			

(a) Complete the table below by matching the function to its first derivative and its second derivative.

Type of function	Function	First derivative	Second derivative
Quadratic			
Cubic			
Trigonometric			
Exponential			

(b) For **one** row in the table, explain your choice of first derivative and second derivative.

Question 3

(a) Differentiate the function $2x^2 - 3x - 6$ with respect to x from first principles.

(b) Let $f(x) = \frac{2x}{x+2}$, $x \neq -2$, $x \in \mathbb{R}$. Find the co-ordinates of the points at which the slope of the tangent to the curve $y = f(x)$ is $\frac{1}{4}$.

Question 4

Question 9 **(50 marks)**

(a) Let $f(x) = -0{\cdot}5x^2 + 5x - 0{\cdot}98$, where $x \in \mathbb{R}$.

(i) Find the value of $f(0{\cdot}2)$.

(ii) Show that f has a local maximum point at $(5,\ 11{\cdot}52)$.

(b) A sprinter's velocity over the course of a particular 100 metre race is approximated by the following model, where v is the velocity in metres per second, and t is the time in seconds from the starting signal:

$$v(t) = \begin{cases} 0, & \text{for } 0 \le t < 0{\cdot}2 \\ -0{\cdot}5t^2 + 5t - 0{\cdot}98, & \text{for } 0{\cdot}2 \le t < 5 \\ 11{\cdot}52, & \text{for } t \ge 5 \end{cases}$$

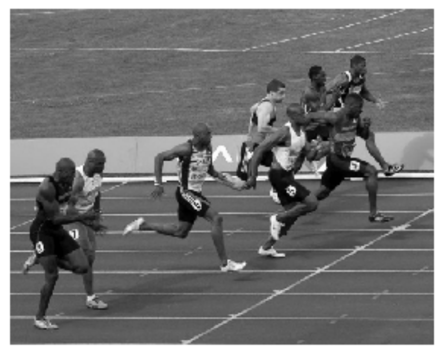
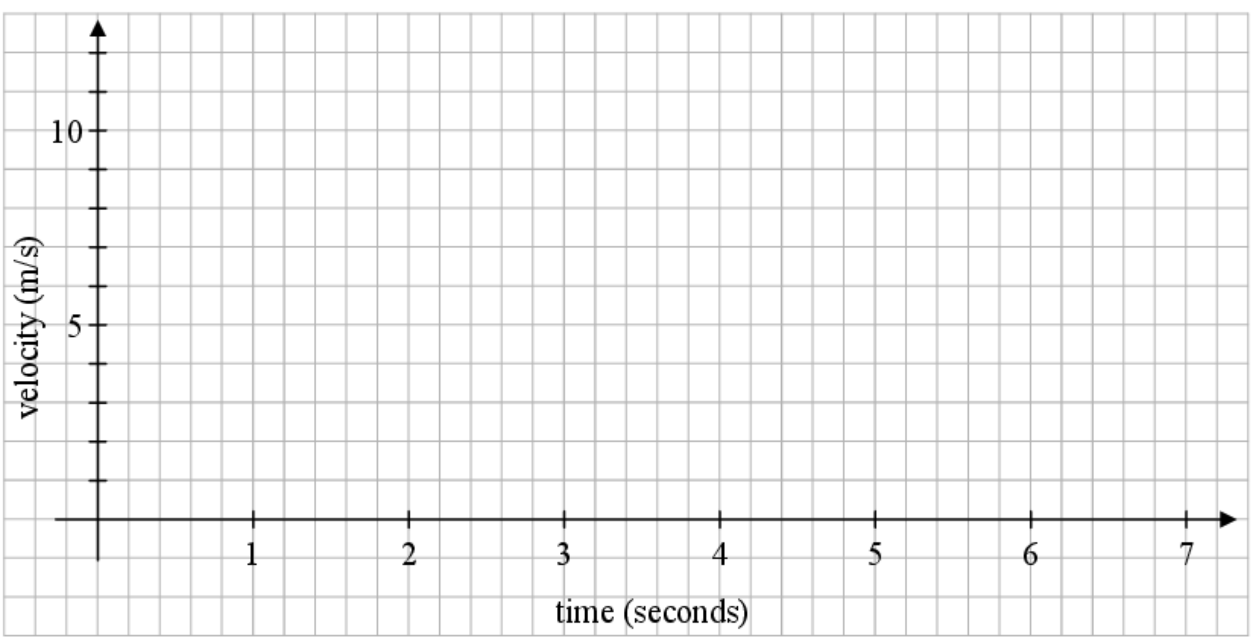


Photo: *William Warby*. Wikimedia Commons. CC BY 2.0

Note that the function in part **(a)** is relevant to $v(t)$ above.

(i) Sketch the graph of v as a function of t for the first 7 seconds of the race.



(ii) Find the distance travelled by the sprinter in the first 5 seconds of the race.

(iii) Find the sprinter's finishing time for the race. Give your answer correct to two decimal places.

(c) A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

(i) Prove that the radius of the snowball is decreasing at a constant rate.

(ii) If the snowball loses half of its volume in an hour, how long more will it take for it to melt completely?

Give your answer correct to the nearest minute.

Question 5

7. **(a)** Find the slope of the tangent to the curve $x^2 + y^3 = x - 2$ at the point $(3, -2)$.

(b) A curve is defined by the parametric equations

$$x = \frac{t-1}{t+1} \quad \text{and} \quad y = \frac{-4t}{(t+1)^2}, \quad \text{where } t \neq -1.$$

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

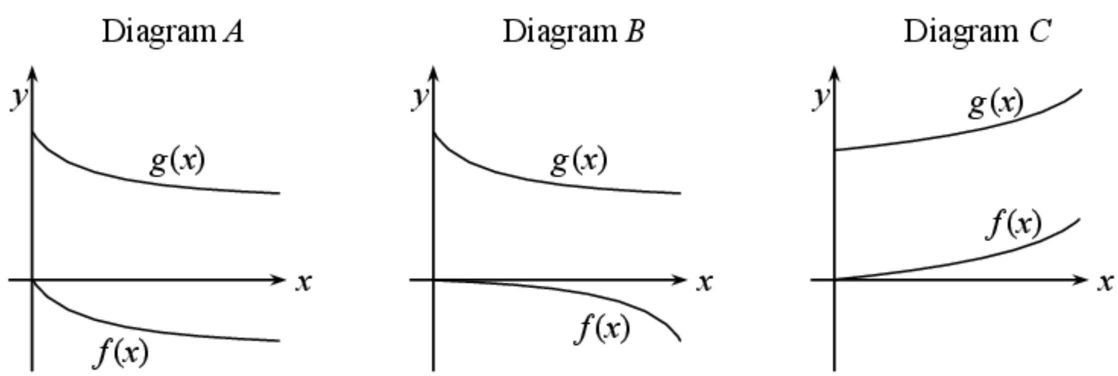
(ii) Hence find $\frac{dy}{dx}$, and express your answer in terms of x.

(c) The functions f and g are defined on the domain $x \in \mathbb{R} \setminus \{-1, 0\}$ as follows:

$$f : x \to \tan^{-1}\left(\frac{-x}{x+1}\right) \quad \text{and} \quad g : x \to \tan^{-1}\left(\frac{x+1}{x}\right).$$

(i) Show that $f'(x) = \dfrac{-1}{2x^2 + 2x + 1}$.

(ii) It can be shown that $f'(x) = g'(x)$.
One of the three diagrams A, B, or C below represents parts of the graphs of f and g. Based only on the derivatives, state which diagram is the correct one, and state also why each of the other two diagrams is incorrect.



Question 6

7. **(a)** The equation of a curve is $x^2 - y^2 = 25$. Find $\frac{dy}{dx}$ in terms of x and y.

(b) A curve is defined by the parametric equations

$$x = \frac{3t}{t^2 - 2} \text{ and } y = \frac{6}{t^2 - 2}, \text{ where } t \neq \pm\sqrt{2}.$$

(i) Find $\frac{dy}{dx}$ in terms of t.

(ii) Find the equation of the tangent to the curve at the point given by $t = 2$.

(c) The function $f(x) = x^3 - 3x^2 + 3x - 4$ has only one real root.

(i) Show that the root lies between 2 and 3.

Anne and Barry are each using the Newton-Raphson method to approximate the root. Anne is starting with 2 as a first approximation and Barry is starting with 3.

(ii) Show that Anne's starting approximation is closer to the root than Barry's. (That is, show that the root is less than 2·5.)

(iii) Show, however, that Barry's next approximation is closer to the root than Anne's.

Question 7

7. **(a)** Taking 1 as the first approximation of a root of $x^3 + 2x - 4 = 0$, use the Newton-Raphson method to calculate the second approximation of this root.

(b) **(i)** Find the equation of the tangent to the curve

$$3x^2 + y^2 = 28 \text{ at the point } (2, -4).$$

(ii) $x = e^t \cos t$ and $y = e^t \sin t$. Show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

(c) $f(x) = \log_e 3x - 3x$, where $x > 0$.

(i) Show that $(\frac{1}{3}, -1)$ is a local maximum point of $f(x)$.

(ii) Deduce that the graph of $f(x)$ does not intersect the x-axis.

Question 8

7. **(a)** Taking $x_1 = 2$ as the first approximation to the real root of the equation

$$x^3 + x - 9 = 0,$$

use the Newton-Raphson method to find x_2, the second approximation.

(b) The parametric equations of a curve are:

$$x = 3\cos\theta - \cos^3\theta$$

$$y = 3\sin\theta - \sin^3\theta, \text{ where } 0 < \theta < \frac{\pi}{2}.$$

(i) Find $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$.

(ii) Hence show that $\frac{dy}{dx} = \frac{-1}{\tan^3\theta}$.

(c) Given $y = \ln\left(\frac{3+x}{\sqrt{9-x^2}}\right)$, find $\frac{dy}{dx}$ and express it in the form $\frac{a}{b-x^n}$.

Question 9

7. **(a)** Find from first principles the derivative of x^2 with respect to x.

(b) **(i)** The parametric equations of a curve are:

$$x = 8 + \ln t^2$$
$$y = \ln(2 + t^2), \text{ where } t > 0.$$

Find $\frac{dy}{dx}$ in terms of t and calculate its value at $t = \sqrt{2}$.

(ii) Find the slope of the tangent to the curve $xy^2 + y = 6$ at the point (1, 2).

(c) **(i)** Write down a quadratic equation whose roots are $\pm\sqrt{k}$.

(ii) Hence use the Newton-Raphson method to show that the rule

$$u_{n+1} = \frac{(u_n)^2 + k}{2u_n}$$

can be used to find increasingly accurate approximations for $\sqrt{k}$.

(iii) Using the above rule and taking $\frac{3}{2}$ as the first approximation for $\sqrt{3}$, find the third approximation, as a fraction.

Question 10

Question 5 **(25 marks)**

A is the closed interval $[0,5]$. That is, $A=\{x \mid 0 \le x \le 5, x \in \mathbb{R}\}$.

The function f is defined on A by:

$$f : A \to \mathbb{R} : x \mapsto x^3 - 5x^2 + 3x + 5 .$$

(a) Find the maximum and minimum values of f.

(b) State whether f is injective. Give a reason for your answer.

Question 11

7. **(a)** Differentiate x^2 with respect to x from first principles.

(b) Let $y = \dfrac{\cos x + \sin x}{\cos x - \sin x}$.

(i) Find $\dfrac{dy}{dx}$.

(ii) Show that $\dfrac{dy}{dx} = 1 + y^2$.

(c) The function $f(x) = (1+x)\log_e(1+x)$ is defined for $x > -1$.

(i) Show that the curve $y = f(x)$ has a turning point at $\left(\dfrac{1-e}{e}, -\dfrac{1}{e}\right)$.

(ii) Determine whether the turning point is a local maximum or a local minimum.

Question 12

7. **(a)** Given that $x = 3t^2 - 6t$ and $y = 2t - t^2$, for $t \in \mathbb{R}$, show that $\dfrac{dy}{dx}$ is constant.

(b) A curve is defined by the equation $x^2 - 2xy + 3y^2 + 4y = 22$.

(i) Find $\dfrac{dy}{dx}$ in terms of x and y.

(ii) The points $(-3, 1)$ and $(1, -3)$ are both on this curve.
Show that the tangents at these two points are parallel to each other.

(c) Let $f(x) = 32x^3 - 48x^2 + 20x - 1$, where $x \in \mathbb{R}$.

(i) Show that f has a root between 0 and 1.

(ii) Take $x_1 = 0{\cdot}5$ as a first approximation to this root. Use the Newton-Raphson method to find x_2 and x_3, the second and third approximations.

(iii) What can you conclude about all further approximations?

Question 13

7. **(a)** Differentiate $2x + \sin 2x$ with respect to x.

(b) The equation of a curve is $5x^2 + 5y^2 + 6xy = 16$.

(i) Find $\frac{dy}{dx}$ in terms of x and y.

(ii) $(1,1)$ and $(2, -2)$ are two points on the curve.
Show that the tangents at these points are perpendicular to each other.

(c) Let $y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$.

Find $\frac{dy}{dx}$ and express it in the form $\frac{a}{a+x^b}$, where $a, b \in \mathbf{N}$.