MarkingScheme

CoordGeomCircleH

Question 1 (2017)

(a)	$x^2+y^2+2gx+2fy+c=0$ $(0,0)\Rightarrow 0+0+0+0+c=0$ $\Rightarrow c=0$ $(6{\cdot}5,0)\Rightarrow 42{\cdot}25+0+13g+0+0=0$ $\Rightarrow g=-3{\cdot}25$ $(10,7)\Rightarrow 100+49+2(-3{\cdot}25)(10)+2f(7)=0$ $14f=-84$ $f=-6$ $x^2+y^2-6{\cdot}5x-12y=0$	**Scale 10D (0, 3, 5, 8, 10)** *Low Partial Credit:* • $c=0$ • One relevant equation in *g* and/or *f* *Mid Partial Credit:* • 2 of *g, f, c* found *High Partial Credit:* • *g*, *f*, and *c* found or equivalent
	or ⊥ Bisector of [*AB*] $x=\frac{13}{4}$ (l_1) ⊥ Bisector of [*AC*] Midpoint [*AC*] $=\left(5,\ \frac{7}{2}\right)$, Slope [*AC*] $=\frac{7}{10}$ Eq. of mediator [*AC*] $y-\frac{7}{2}=-\frac{10}{7}(x-5)$ $10x+7y=\frac{149}{2}$ (l_2) $l_1\cap l_2=\left(\frac{13}{4},6\right)$ $r=\sqrt{\left(\frac{13}{4}-0\right)^2+(6-0)^2}=\frac{\sqrt{745}}{4}$ $\left(x-\frac{13}{4}\right)^2+(y-6)^2=\frac{745}{16}$	*Low Partial Credit:* • Effort at formulating equation of 1 ⊥ bisector *Mid Partial Credit:* • Point t of intersection of 2 ⊥ bisectors found *High Partial Credit:* • Point of intersection of 2 ⊥ bisectors and radius
	or $(-g,-f)\in$ mediator $(0,0)$and $(6{\cdot}5,0)$. $\therefore -g=3{\cdot}25$ Centre $(3{\cdot}25,-f)$. Since $(0,0)\in$ of circle $\therefore c=0$. Equation of circle $x^2+y^2-6{\cdot}5x+2fy+0=0$ $(10,7)$on circle: $100+49-65+14f=0$ $84+14f=0$ $f=-6$ $x^2+y^2-6{\cdot}5x-12y=0$	*Low Partial Credit:* • $c=0$ • One point substituted into equation of circle • Midpoint (0,0) and (6·5, 0) formulated *Mid Partial Credit:* • 2 of *g, f, c* found *High Partial Credit:* • *g*, *f*, and c found or equivalent

(b)	Slope $AC = \frac{7}{10}$ Slope $CB = \frac{0-7}{6{\cdot}5-10} = 2$ $\tan\theta = \pm\frac{\frac{7}{10}-2}{1+\frac{7}{5}} = \pm\frac{-13}{24}$ $\theta = 28{\cdot}44$ **or** Cosine rule $\lvert AB\rvert^2 = 42{\cdot}25,$ $\lvert AC\rvert^2 = 149$ $\lvert BC\rvert^2 = 61{\cdot}25$ $\cos\theta = \frac{149+61{\cdot}25-42{\cdot}25}{2\times\sqrt{149}\times\sqrt{61{\cdot}25}} = 0{\cdot}8793$ $\Rightarrow \theta = 28{\cdot}44$	**Scale 15C (0, 6, 9, 15)** *Low Partial Credit:* • one relevant slope *High Partial Credit:* • $\tan\theta$ fully substituted *Low Partial Credit:* • one relevant length *High Partial Credit:* • $\cos\theta$ fully substituted

Q2	Model Solution – 25 Marks	Marking Notes
(a)	$y - 6 = \frac{1}{7}(x + 1)$ $x - 7y + 43 = 0$	Scale 10C (0, 3, 7, 10) *Low Partial Credit:* • equation of line formula with some relevant substitution *High Partial Credit:* • equation of line not in required form
(b)	$D = \frac{\lvert ax_1 + by_1 + c\rvert}{\sqrt{a^2 + b^2}}$ $D = \frac{\lvert 3(-g) + 4(-f) - 21\rvert}{\sqrt{3^2 + 4^2}}$ $25 = \lvert -3g - 4f - 21\rvert$ $-3g - 4f - 21 = \pm 25$ $\Longrightarrow 3g + 4f = -46$ ... (i) and $3g + 4f = 4$ ... (ii) But $(-g, -f) \in x - 7y + 43 = 0$ $\Rightarrow -g + 7f + 43 = 0$ ... (iii) $\Rightarrow g = 7f + 43$ Solving : $g = 7f + 43$ and $3g + 4f = -46$ $f = -7$ and $g = -6$ Centre (6, 7) $(x - 6)^2 + (y - 7)^2 = 25$ **or** Solving: $g = 7f + 43$ and $3g + 4f = 4$ $f = -5$ and $g = 8$ Centre (- 8, 5) $(x + 8)^2 + (y - 5)^2 = 25$	Scale 15D (0, 4, 7,11,1 5) *Low Partial Credit* • some correct substitution into relevant formula (line, circle, perpendicular distance). *Mid Partial Credit* • one relevant equation in *g* and *f* • (either(i) or (ii) or (iii)) *High Partial Credit* • two relevant equations (either (i) and (iii) or (ii) and (iii))

<table>
<tr><th>Q4</th><th>Model Solution – 25 Marks</th><th>Marking Notes</th></tr>
<tr><td>(a)
(i)</td><td>$|\angle ABD| = |\angle CBD| = 90°$.......(i)
$|\angle BDC| + |\angle BCD| = 90°$ ...angles in triangle sum to $180°$
$|\angle ADB| + |\angle BDC| = 90°$ .... angle in semicircle
$|\angle ADB| + |\angle BDC| = |\angle BDC| + |\angle BCD|$
$|\angle ADB| = |\angle BCD|$........(ii)
$\therefore$ Triangles are equiangular (or similar)
or

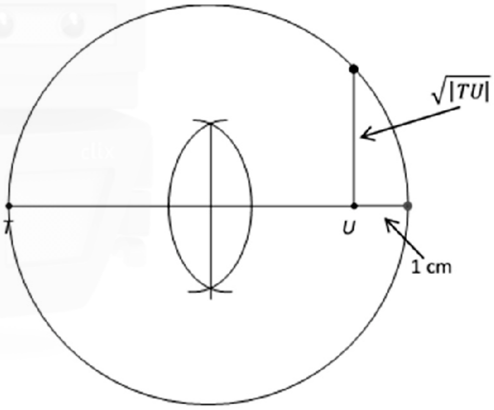
$|\angle ABD| = |\angle CBD| = 90°$.......(i)
$|\angle DAB| = |\angle DAC|$ same angle $\Rightarrow |\angle ADB| = |\angle DCA|$ (reasons as above) which is also $\angle DCB$.........(ii)</td><td>Scale 15C (0, 5, 10, 15)
Low Partial Credit
• identifies one angle of same size in each triangle

High Partial Credit
• identifies second angle of same size in each triangle
• implies triangles are similar without justifying (ii) in model solution or equivalent</td></tr>
<tr><td>(a)
(ii)</td><td>$\frac{y}{1} = \frac{x}{y}$
$\Rightarrow y^2 = x$
$y = \sqrt{x}$
or
$|AD|^2 + |DC|^2 = |AC|^2$
$|AD| = \sqrt{x^2 + y^2}$
$|DC| = \sqrt{y^2 + 1}$
$x^2 + y^2 + y^2 + 1 = (x+1)^2$
$2y^2 = 2x$
$y = \sqrt{x}$

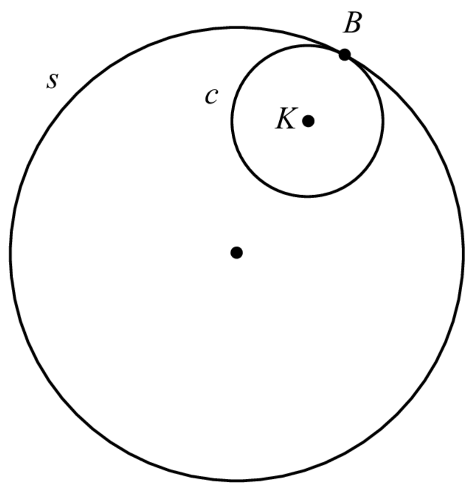
Or

$\frac{\sqrt{x^2 + y^2}}{\sqrt{y^2 + 1}} = \frac{y}{1} \Rightarrow x^2 + y^2 = y^2(y^2 + 1)$
$y^4 = x^2 \Rightarrow y^2 = x \Rightarrow y = \sqrt{x}$</td><td>Scale 5C (0, 2, 4, 5)
Low Partial Credit
• one set of corresponding sides identified
• indicates relevant use of Pythagoras

High Partial Credit
• corresponding sides fully substituted
• expression in y^2 or y^4, i.e. fails to finish</td></tr>
</table>

(b)	Construction  	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • perpendicular line drawn at *U* or *T* • relevant use of 1 cm length • mid point of incorrect extended segment constructed *High Partial Credit* • correct mid-point constructed

Two circles s and c touch internally at B, as shown.



(a) The equation of the circle s is
$(x-1)^2+(y+6)^2=360.$
Write down the co-ordinates of the centre of s.

Centre: $(1,-6)$

Write down the radius of s in the form $a\sqrt{10}$, where $a\in\mathbb{N}$.

Radius: $\sqrt{360}=6\sqrt{10}$

(b) **(i)** The point K is the centre of circle c.
The radius of c is one-third the radius of s.
The co-ordinates of B are (7, 12).
Find the co-ordinates of K.

	or
$\lvert AK\rvert:\lvert KB\rvert=2:1$ $K\left(\frac{2\times7+1\times1}{2+1},\frac{2\times12+1\times-6}{2+1}\right)=(5,6)$	Centre of s to B (translation) X ordinate goes up by 6 Y ordinate goes up by 18 $\frac{2}{3}(6)+1=5$ $\frac{2}{3}(18)-6=6$

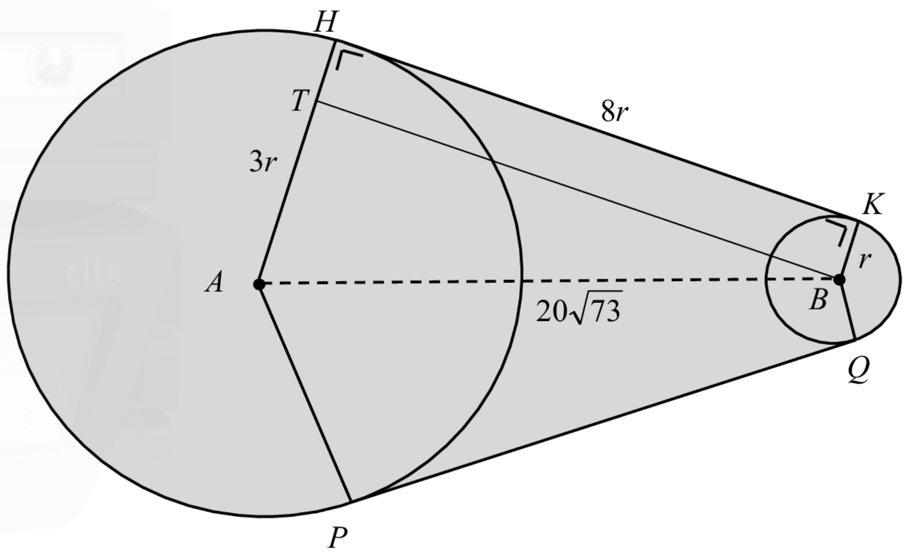
(ii) Find the equation of c.

$(x-5)^2+(y-6)^2=\left(2\sqrt{10}\right)^2=40$

(c) Find the equation of the common tangent at B.
Give your answer in the form $ax+by+c=0$, where $a, b, c\in Z$.

Slope $AB=\frac{12+6}{7-1}=\frac{18}{6}=3$ Slope of tangent $=-\frac{1}{3}$ Equation: $y-12=-\frac{1}{3}(x-7)\;\Rightarrow\; x+3y-43=0$

(a) Find r, the radius of the smaller circle. (Hint: Draw $BT \parallel KH$, $T \in AH$.)



$$|AT|^2 + |BT|^2 = |AB|^2 \Rightarrow (3r)^2 + (8r)^2 = (20\sqrt{73})^2$$

$$\Rightarrow 9r^2 + 64r^2 = 29200$$

$$\Rightarrow r^2 = 400 \Rightarrow r = 20 \text{ cm}$$

(b) Find the area of the quadrilateral *ABKH*.

$$|ABKH| = |BKHT| + |\Delta ABT|$$
$$= 20\times 160 + \tfrac{1}{2}(60)(160)$$
$$= 8000 \text{ cm}^2$$

(c) **(i)** Find $|\angle HAP|$, in degrees, correct to one decimal place.

$$\tan|\angle HAB| = \frac{160}{60} \Rightarrow |\angle HAB| = 69\cdot 44°$$
$$\Rightarrow |\angle HAP| = 138\cdot 9°$$

(ii) Find the area of the machine part, correct to the nearest cm^2.

Area large sector *HAP* + 2 area *HABK* + area sector *KBQ*

$$= \pi(80)^2\left(\frac{221\cdot 1}{360}\right) + 2\times 8000 + \pi(20)^2\left(\frac{138\cdot 9}{360}\right)$$
$$= 12348\cdot 55 + 16000 + 484\cdot 85$$
$$= 28833\cdot 4$$
$$= 28833$$

(a) Name two similar triangles in the diagram above and give reasons for your answer.

ΔADE and ΔBCE are similar

$|\angle EAD| = |\angle BCE|$, on arc BD

$|\angle DEA| = |\angle CEB|$, same angle

$|\angle ADE| = |\angle EBC|$, third angle

Also (i) ΔAXB and ΔDXC are similar, where $AD \cap CB = \{X\}$

and (ii) ΔAXC and ΔBXD are similar, where $AD \cap CB = \{X\}$

(b) Prove that $|EA|.|EB| = |EC|.|ED|$.

ΔADE and ΔBCE are similar.

Hence, $\dfrac{|EA|}{|EC|} = \dfrac{|ED|}{|EB|}$

$\Rightarrow |EA|.|EB| = |EC|.|ED|$

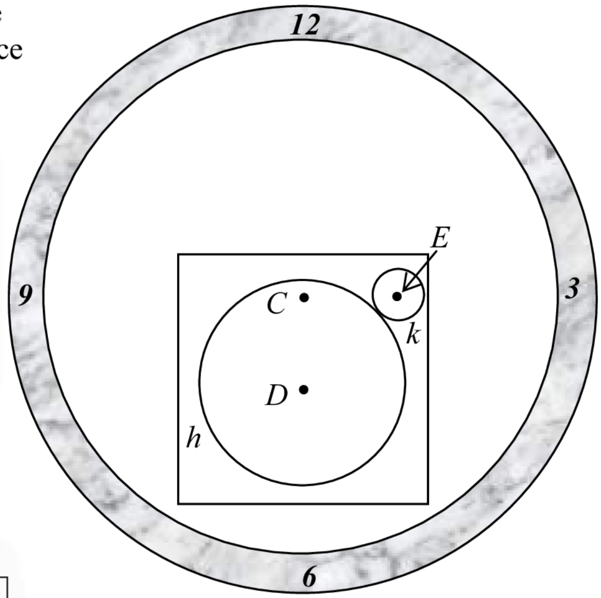
(c) Given that $|EB| = 6\cdot 25$, $|ED| = 5\cdot 94$ and $|CB| = 10$, find $|AD|$.

$$\frac{|ED|}{|EB|} = \frac{|AD|}{|CB|} \Rightarrow \frac{5\cdot 94}{6\cdot 25} = \frac{|AD|}{10}$$

$$\Rightarrow |AD| = \frac{5\cdot 94 \times 10}{6\cdot 25} = 9\cdot 504$$

(a) The diagram shows a circular clock face, with the hands not shown. The square part of the clock face is glass so that the mechanism is visible. Two circular cogs, h and k, which touch externally are shown.

The point C is the centre of the clock face. The point D is the centre of the larger cog, h, and the point E is the centre of the smaller cog, k.



(i) In suitable co-ordinates the equation of the circle h is

$$x^2 + y^2 + 4x + 6y - 19 = 0.$$

Find the radius of h and the co-ordinates of its centre, D.

$r_1 = \sqrt{4+9+19} = \sqrt{32} = 4\sqrt{2}$

$D(-2, -3)$

(ii) The point E has co-ordinates $(3, 2)$. Find the radius of the circle k.

$|DE| = \sqrt{(3+2)^2 + (2+3)^2} = \sqrt{50} = 5\sqrt{2}$

$r_1 + r_2 = |DE| \Rightarrow 4\sqrt{2} + r_2 = 5\sqrt{2} \Rightarrow r_2 = \sqrt{2}$

(iii) Show that the distance from $C(-2, 2)$ to the line DE is half the length of $[DE]$.

Slope $DE = \dfrac{2+3}{3+2} = 1$

Equation $DE : y + 3 = 1(x+2) \Rightarrow x - y - 1 = 0$

Distance from C to DE: $p = \left|\dfrac{-2-2-1}{\sqrt{1+1}}\right| = \left|\dfrac{5}{\sqrt{2}}\right| = \dfrac{5\sqrt{2}}{2} = \dfrac{1}{2}|DE|$

(iv) The translation which maps the midpoint of $[DE]$ to the point C maps the circle k to the circle j. Find the equation of the circle j.

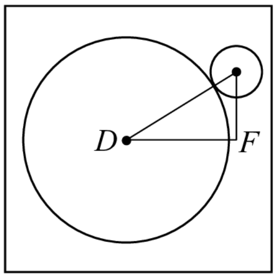
Midpoint $[DE] = \left(\frac{-2+3}{2}, \frac{-3+2}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right)$

$\left(\frac{1}{2}, -\frac{1}{2}\right) \to (-2, 2)$ maps $(3,\ 2) \to \left(\frac{1}{2}, \frac{9}{2}\right)$

$j: \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{9}{2}\right)^2 = \left(\sqrt{2}\right)^2 = 2$

$4x^2 + 4y^2 - 4x - 36y + 74 = 0$

(v) The glass square is of side length l. Find the smallest whole number l such that the two cogs, h and k, are fully visible through the glass.



$D(-2, -3),\quad F(3, -3)$

$|DF| = 5$

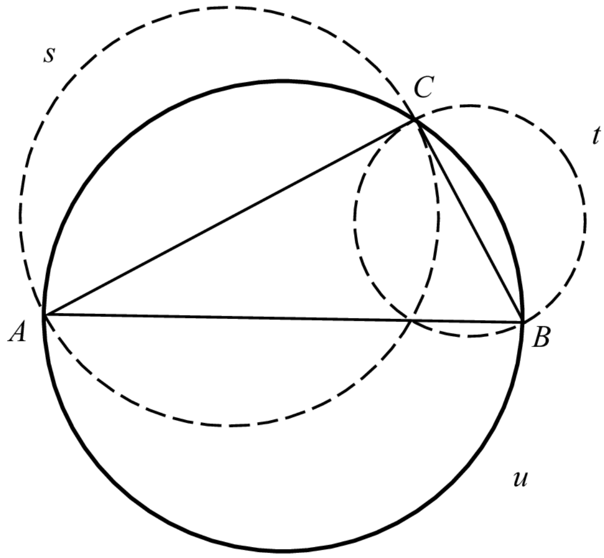
Length: $r_1 + |DF| + r_2 = 4\sqrt{2} + 5 + \sqrt{2} = 5\sqrt{2} + 5 = 12 \cdot 07$

$l = 13$

(b) The triangle ABC is right-angled at C.

The circle s has diameter $[AC]$ and the circle t has diameter $[CB]$.

(i) Draw the circle u which has diameter $[AB]$.



(ii) Prove that in any right-angles triangle *ABC*, the area of the circle *u* equals the sum of the areas of the circles *s* and *t*.

Triangle *ABC* is right-angled:

$$|AB|^2 = |AC|^2 + |CB|^2$$

$$\Rightarrow \frac{\pi}{4}\left(|AB|^2\right) = \frac{\pi}{4}\left(|AC|^2 + |CB|^2\right)$$

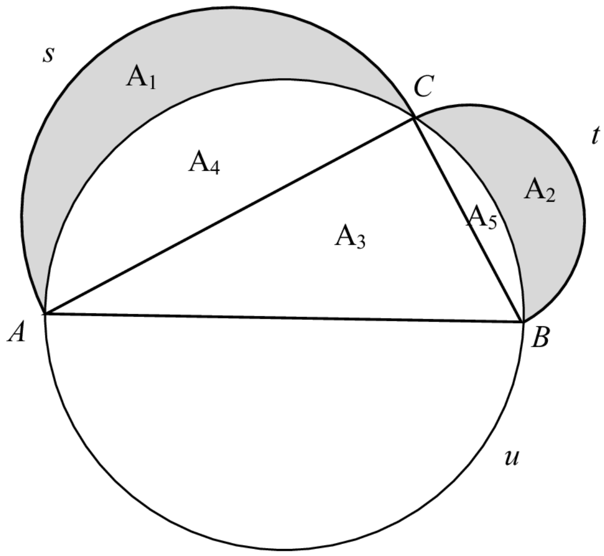
$$\Rightarrow \pi\left(\frac{|AB|}{2}\right)^2 = \pi\left(\frac{|AC|}{2}\right)^2 + \pi\left(\frac{|CB|}{2}\right)^2$$

Thus, area of *u* = area of *s* + area of *t*.

(iii) The diagram shows the right-angled triangle *ABC* and arcs of the circles *s*, *t* and *u*.

Each of the shaded areas in the diagram is called a lune, a crescent-shaped area bounded by arcs of the circles.

Prove that the sum of the areas of the two shaded lunes is equal to the area of the triangle *ABC*.

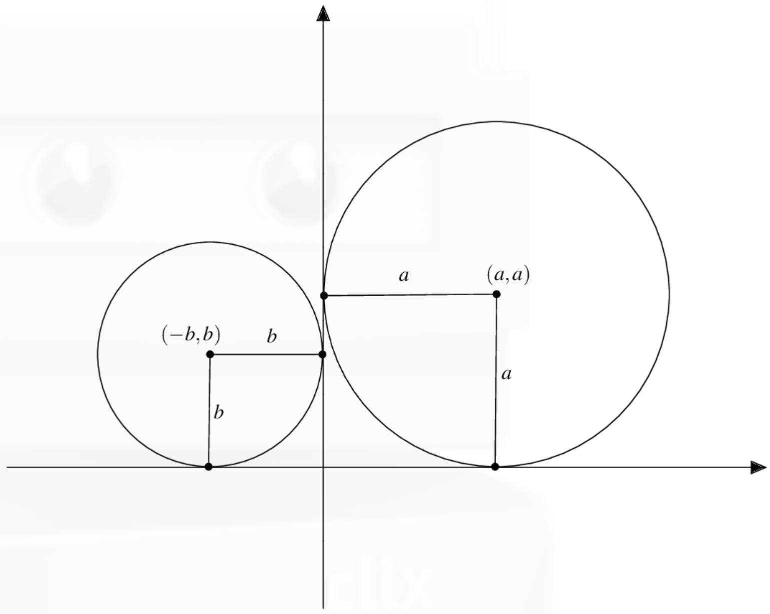


$$\frac{1}{2} \text{ area of } u = \frac{1}{2}(\text{ area of } s + \text{area of } t)$$

$$\Rightarrow A_3 + A_4 + A_5 = (A_1 + A_4) + (A_2 + A_5)$$

$$\Rightarrow A_3 = A_1 + A_2$$

Consider the diagram below:



We can see from this diagram that if (x,y) is the centre of a circle that has both the x-axis and the y-axis as tangents, then either

- Case 1: $y = x$
- Case 2: $y = -x$

In either case the radius r is $\pm x$. Since the radius is positive, we must have $r = |x|$.
Case 1: $y = x$. We are also told that $x + 2y - 6 = 0$. Subsituting x for y in the latter equation gives

$$x + 2x - 6 = 0 \Leftrightarrow 3x - 6 = 0 \Leftrightarrow x = 2.$$

Now $y = x$ so $y = 2$. Therefore the centre of the circle has co-ordinates $(2,2)$ and the radius is 2. Therefore in this case the circle has equation

$$(x-2)^2 + (y-2)^2 = 4.$$

Case 2: $y = -x$. As before we use this to substitute $-x$ for y in the equation $x + 2y - 6 = 0$. This gives

$$x + 2(-x) - 6 = 0 \Leftrightarrow -x - 6 = 0 \Leftrightarrow x = -6.$$

It follows that $y = -(-6) = 6$. So in this case the centre has co-ordinates $(-6,6)$ and the radius is 6. So this circle has equation

$$(x+6)^2 + (y-6)^2 = 36.$$

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Circle	Centre	Radius	Equation
c_1	$(-3, -2)$	2	$(x+3)^2+(y+2)^2=4$ **OR** $x^2+y^2+6x+4y+9=0$
c_2	$(1, 1)$	3	$x^2+y^2-2x-2y-7=0$

(b) (i) Find the co-ordinates of the point of contact of c_1 and c_2.

Divide line segment joining $(-3, -2)$ and $(1, 1)$ in ratio 2 : 3

$$\left(\frac{2(1)+3(-3)}{2+3}, \frac{2(1)+3(-2)}{2+3}\right) = \left(-\frac{7}{5}, -\frac{4}{5}\right)$$

OR

Slope line of centres $= \frac{3}{4}$.

Equation line of centres: $y-1=\frac{3}{4}(x-1) \Rightarrow 3x-4y+1=0$

$c_1 - c_2 = 4x+3y+8=0$

$4x+3y+8=0 \cap 3x-4y+1=0 \Rightarrow x=-\frac{7}{5}, y=-\frac{4}{5}$

(ii) Hence, or otherwise, find the equation of the tangent, t, common to c_1 and c_2.

Slope of line of centres: $\frac{1+2}{1+3}=\frac{3}{4}$

Slope of tangent: $m=-\frac{4}{3}$

Equation of tangent: $y+\frac{4}{5}=-\frac{4}{3}\left(x+\frac{7}{5}\right)$

$\Rightarrow\ 3y+\frac{12}{5}=-4x-\frac{28}{5}$

$\Rightarrow\ 4x+3y+8=0$

OR

$c_1-c_2=x^2+y^2+6x+4y+9-(x^2+y^2-2x-2y-7)=0$

$\Rightarrow 6x+4y+9-(-2x-2y-7)=0$

$\Rightarrow 8x+6y+16=0 \Rightarrow 4x+3y+8=0$

OR

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$x(-\frac{7}{5})+y(-\frac{4}{5})+3(x+(-\frac{7}{5}))+2(y+(-\frac{4}{5})+9=0$

$\Rightarrow 4x+3y+8=0$

(i) Show that, for fixed r_1, the perimeter of the arbelos is independent of the values of r_2 and r_3.

Perimeter $= \pi r_1 + (\pi r_2 + \pi r_3) = \pi(r_1 + (r_2 + r_3) = \pi(r_1 + r_1) = 2\pi r_1$

which is independent of r_2 and r_3

(ii) If $r_2 = 2$ and $r_3 = 4$, show that the area of the arbelos is the same as the area of the circle of diameter *k*.

$$\begin{aligned}\text{Area of arbelos } &= \tfrac{1}{2}\pi r_1^2 - \tfrac{1}{2}\pi(r_2^2 + r_3^2) \\ &= \tfrac{1}{2}\pi(6^2) - \tfrac{1}{2}\pi(2^2 + 4^2) \\ &= \tfrac{1}{2}\pi(36 - 20) \\ &= 8\pi\end{aligned}$$

$k^2 + 4 = 36$

$k = \sqrt{32}$

$$\text{Area of circle } = \pi\left(\frac{k}{2}\right)^2 = \pi\left(\frac{\sqrt{32}}{2}\right)^2 = \frac{\pi\left(\sqrt{32}\right)^2}{4} = 8\pi$$

(c) To investigate the area of an arbelos, a student fixed the value of r_1 at 6 cm and completed the following table for different values of r_2 and r_3.

(i) Complete the table.

r_1	r_2	r_3	Area of arbelos
6	1	5	$\frac{1}{2}\pi\left(6^2-(1^2+5^2)\right)=5\pi\text{ cm}^2$
6	2	4	$\frac{1}{2}\pi\left(6^2-(2^2+4^2)\right)=8\pi\text{ cm}^2$
6	3	3	$\frac{1}{2}\pi\left(6^2-(3^2+3^2)\right)=9\pi\text{ cm}^2$
6	4	2	$\frac{1}{2}\pi\left(6^2-(4^2+1^2)\right)=8\pi\text{ cm}^2$
6	5	1	$\frac{1}{2}\pi\left(6^2-(5^2+1^2)\right)=5\pi\text{ cm}^2$

(ii) In general for $r_1 = 6$ cm and $r_2 = x$, $0 < x < 6, x \in \mathbb{R}$, find an expression in x for the area of the arbelos.

$$\begin{aligned}\text{Area of arbelos } &= \tfrac{1}{2}\pi r_1^2 - \tfrac{1}{2}\pi\left(r_2^2 + r_3^2\right)\\ &= \tfrac{1}{2}\pi\left(r_1^2 - (r_2^2 + r_3^2\right)\\ &= \tfrac{1}{2}\pi\left(36 - \left(x^2 + (6-x)^2\right)\right)\\ &= \pi\left(6x - x^2\right)cm^2\end{aligned}$$

(iii) Hence, or otherwise, find the maximum area of an arbelos that can be formed in a semi circle of radius 6 cm.

$$A = \pi\left(6x - x^2\right) \Rightarrow \frac{dA}{dx} = \pi(6-2x)$$

$$\pi(6-2x) = 0 \Rightarrow x = 3$$

$$\frac{dA}{dx} = \pi(6-2x) \Rightarrow \frac{d^2A}{dx^2} = -2\pi < 0 \Rightarrow \text{maximum}$$

Maximum area when $x = 3$, giving area = 9π cm^2

$|\angle TSR| = 90°$ ..... Angle in a semicircle
$|\angle CTA| = 90°$ ..... Angle in a semicircle
Hence, $|\angle STC| = 90°$
$|\angle FRC| = 90°$ ..... Angle in a semicircle
Hence, $|\angle CRS| = 90°$

Hence, the angles in *RSTC* are right angles and so *RSTC* is a rectangle.

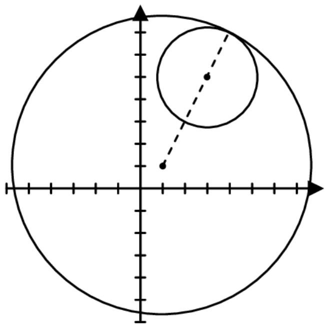
(a) Write down the centre and radius-length of each circle.

$c_1 : (x-3)^2 + (y-5)^2 = 5$	$c_2 : (x-1)^2 + (y-1)^2 = 45$
$\therefore$ centre (3, 5); radius $\sqrt{5}$.	$\therefore$ centre (1, 1); radius $\sqrt{45} = 3\sqrt{5}$.

(b) Prove that the circles are touching.

Distance between centres: $\sqrt{(3-1)^2 + (5-1)^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

The distance between the centres is the difference of the radii $\Rightarrow$ circles touch (internally).



(c) Verify that (4, 7) is the point that they have in common.

$4^2 + 7^2 - 6(4) - 10(7) + 29 = 0 \quad \Rightarrow \quad (4,7) \in c_1$

$4^2 + 7^2 - 2(4) - 2(7) - 43 = 0 \quad \Rightarrow \quad (4,7) \in c_2$

OR

$c_1 - c_2 : \quad x + 2y - 18 = 0 \Rightarrow x = -2y + 18$

$(-2y+18)^2 + y^2 - 6(-2y+18) - 10y + 29 = 0$

$(y-7)^2 = 0$

$y = 7$

$x = 4$

$\therefore$ (4, 7) common

(d) Find the equation of the common tangent.

Slope from (3, 5) to (4, 7) is: $\dfrac{7-5}{4-3}=2$

$\therefore$ slope of tangent $=-\frac{1}{2}$.

Equation of tangent:

$$y-7=-\tfrac{1}{2}(x-4)$$
$$2y-14=-x+4$$
$$x+2y-18=0$$

OR

Equation of Tangent: $c_1-c_2 : x+2y-18=0$

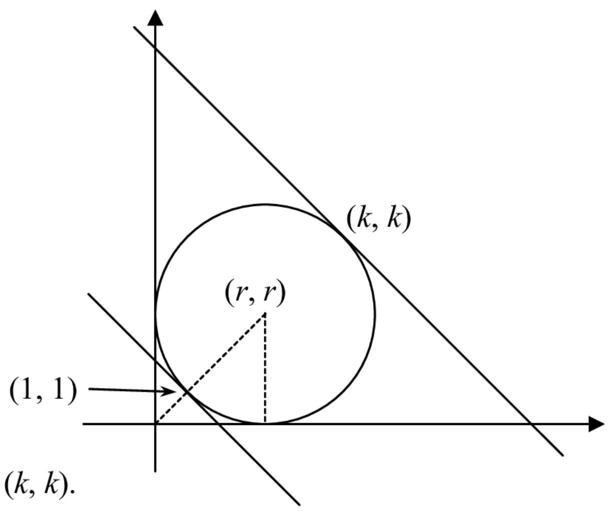
OR

$$(x-h)(x_1-h)+(y-k)(y_1-k)=r^2$$
$$(x-3)(4-3)+(y-5)(7-5)=\left(\sqrt{5}\right)^2$$
$$(x-3)+(y-5)(2)=5$$
$$x+2y-18=0$$

OR

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$
$$4x+7y-3(x+4)-5(y+7)+29=0$$
$$x+2y-18=0$$

Question 12 (2012)



$$r^2+r^2=\left(r+\sqrt{2}\right)^2$$
$$2r^2=r^2+2\sqrt{2}r+2$$
$$r^2-2\sqrt{2}r-2=0$$
$$\left(r-\sqrt{2}\right)^2=4$$
$$r=\sqrt{2}+2, \quad (r>0)$$

(r, r) is midpoint of segment from $(1, 1)$ to (k, k).

$$\frac{k+1}{2}=r$$
$$k=2r-1$$
$$k=3+2\sqrt{2}$$

Question 4 **(25 marks)**

(a) Write down the equation of the circle with centre $(-3,2)$ and radius 4.

Let the centre of the circle $(h,k) = (-3,2)$ and $r = 4$. So the equation of the circle is

$$\begin{aligned} (x-h)^2+(y-k)^2 &= r^2 \\ (x+3)^2+(y-2)^2 &= 4^2 \\ x^2+6x+9+y^2-4y+4 &= 16 \\ x^2+y^2+6x-4y-3 &= 0 \end{aligned}$$

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(b) A circle has equation $x^2+y^2-2x+4y-15=0$. Find the values of m for which the line $mx+2y-7=0$ is a tangent line.

Re-write this equation as $x^2+y^2+2(-1)x+2(2)y-15=0$ which matches the equation of a circle with $g=-1, f=2, c=-15$. So this circle has centre $(-g,-f)=(1,-2)$ and radius $r=\sqrt{g^2+f^2-c}=\sqrt{1+4+15}=\sqrt{20}$.

For a line to be tangent to this circle, the perpendicular distance from that line to the centre $(1,-2)$ must be equal to the radius. The distance from the line $mx+2y-7=0$ to $(1,-2)$ is

$$\frac{|m(1)+2(-2)-7|}{\sqrt{m^2+2^2}}$$

This must be equal to the radius in order to be tangent which means

$$\begin{aligned} \frac{|m(1)+2(-2)-7|}{\sqrt{m^2+2^2}} &= \sqrt{20} \\ |m-4-7| &= \sqrt{20}\sqrt{m^2+4} \\ |m-11| &= \sqrt{20m^2+80} \\ (m-11)^2 &= 20m^2+80 \\ m^2-22m+121 &= 20m^2+80 \\ 0 &= 19m^2+22m-41 \end{aligned}$$

We can solve this quadratic to get solutions $m=1$ and $m=-\frac{41}{19}$

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Line $\Rightarrow$ $x = 20 - 3y$

$$\therefore (20-3y)^2 + y^2 - 6(20-3y) - 8y = 0$$

$$9y^2 - 120y + 400 + y^2 - 120 + 18y - 8y = 0$$
$$10y^2 - 110y + 280 = 0$$
$$y^2 - 11y + 28 = 0$$
$$(y-7)(y-4) = 0$$
$$y = 7 \quad \text{or} \quad y = 4$$
$$x = -1 \quad \text{or} \quad x = 8$$

$P(-1, 7)$ and $Q(8, 4)$

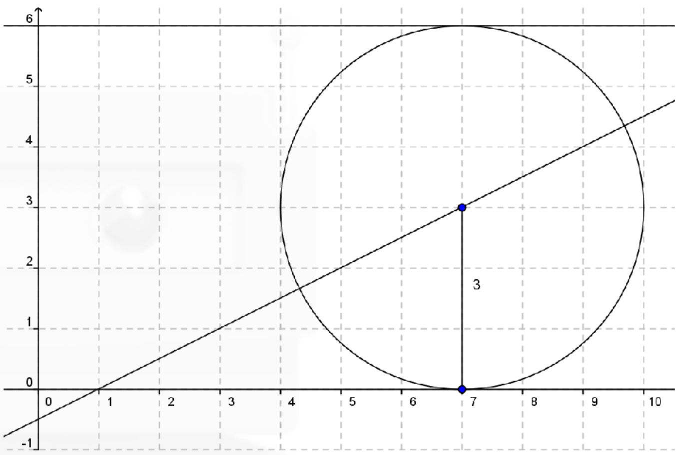
Centre is midpoint of $[PQ]$: $C\left(\frac{7}{2}, \frac{11}{2}\right)$

$$r = \sqrt{\left(\frac{7}{2}+1\right)^2 + \left(\frac{11}{2}-7\right)^2}$$

$$= \sqrt{20{\cdot}25 + 2{\cdot}25}$$

$$= \sqrt{22{\cdot}5} \quad \text{or} \quad \sqrt{\frac{45}{2}}$$

Equation: $\left(x - \frac{7}{2}\right)^2 + \left(y - \frac{11}{2}\right)^2 = \frac{45}{2}$



$$r = 3$$
$$\text{centre: } (h, 3)$$
$$h - 2(3) - 1 = 0$$
$$h = 7$$

Equation of circle:

$$(x-7)^2 + (y-3)^2 = 3^2$$
$$(x-7)^2 + (y-3)^2 = 9$$

or

$$x^2 + y^2 - 14x - 6y + 49 = 0$$

(b) A different circle has equation $x^2 + y^2 - 6x - 12y + 41 = 0$. Show that this circle and the circle in part **(a)** touch externally.

$x^2 + y^2 - 6x - 12y + 41 = 0$.

centre: (3, 6); radius= $\sqrt{9+36-41} = \sqrt{4} = 2$.

Distance between centres: $\sqrt{(7-3)^2 + (3-6)^2} = \sqrt{25} = 5$

Sum of radii: 3 + 2 = 5 = distance between centres.
$\therefore$ circles touch externally.