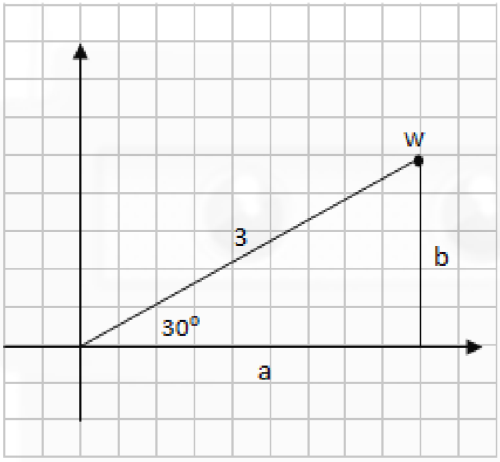
MarkingScheme

ComplexNumH

Question 1 (2017)

(a)	$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ $z^4 = \left(2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right)^4$ $z^4 = 16\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$ $= -8 - 8\sqrt{3}i$	**Scale 15D (0, 5, 8, 12, 15)** *Low Partial Credit:* • θ or $\lvert z \rvert$ found *Mid Partial Credit:* • z written in polar form *High Partial Credit:* • De Moivre's Theorem applied correctly **Note***:* Not using De Moivre: Low partial credit for fully correct work
(b)	$w = 3(\cos 30 + i\sin 30)$ $zw = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \times 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ $zw = 6(\cos\pi + i\sin\pi)$ $= 6(-1 + 0i)$ $= -6$ **OR (contd)**	**Scale 10D (0, 4, 7, 8, 10)** *Low Partial Credit:* • Work towards w in Cartesian or polar form *Mid Partial Credit* • zw expressed as a product *High Partial Credit:* • zw in Cartesian or polar form



$$w = a + bi$$

$$a^2 + b^2 = 9$$

$$\frac{b}{3} = \sin 30° = \frac{1}{2}$$

$$b = \frac{3}{2}$$

$$a^2 + \left(\frac{3}{2}\right)^2 = 9$$

$$a^2 = \frac{27}{4}$$

$$a = \sqrt{\frac{27}{4}} = \frac{3\sqrt{3}}{2}$$

$$w = a + bi = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

$$z = -\sqrt{3} + i$$

$$zw = \left(-\sqrt{3} + i\right)\left(\frac{3\sqrt{3}}{2} + \frac{3}{2}i\right)$$

$$= -\frac{9}{2} - \frac{3\sqrt{3}i}{2} + \frac{3\sqrt{3}i}{2} - \frac{3}{2}$$

$$= -6$$

(a)	$-4-3i$	Scale 5B (0, 2, 5) *Partial Credit:* • real or imaginary part correct
(b)	$r=\sqrt{1^2+1^2}=\sqrt{2} \quad \theta=\frac{\pi}{4}$ $(1+i)^8=\left\{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right\}^8$ $(1+i)^8=\{16(\cos 2\pi+i\sin 2\pi)\}$ $(1+i)^8$= 16(1) = 16	Scale 10C (0, 3, 7, 10) *Low Partial Credit:* • correct answer without use of De Moivre's • modulus **or** argument correct • formula • statement of De Moivre's *High Partial Credit:* • $16(\cos 2\pi+i\sin 2\pi)$ **Note:** *not De Moivre and incorrect answer merits 0 marks*
(c)	$z=\frac{(2-i)\pm\sqrt{(-2+i)^2-4(3-i)}}{2}$ $=\frac{(2-i)\pm\sqrt{4-4i-1-12+4i}}{2}$ $=\frac{2-i\pm\sqrt{-9}}{2}$ $=\frac{2-i\pm 3i}{2}$ $=1-2i$ **or** $1+i$ **Or** $ax^2+bx+c=0$ $x^2-\left(\frac{-b}{a}\right)x+\frac{c}{a}=0$ Sum of roots $=-\frac{b}{a}$ $1+i+z_1=2-i$ $z_1=1-2i$	Scale 10C (0, 3, 7, 10) *Low Partial Credit:* • root formula with some substitution *High Partial Credit* • formula fully substituted **Or** Scale 10C (0, 3, 7, 10) *Low Partial Credit:* • equation rearranged • $-\frac{b}{a}$ *High Partial Credit* • correct substitution

	Or	**Or**
	$(z-1-i)(z-z_1)$ $= z^2 - z - zi - z.z_1 + z_1 + z_1 i$ $= z^2 - (1+i+z_1)z + z_1(1+i)$ $= z^2 + (-2+i)z + (3-i)$ $\Longrightarrow z_1(1+i) = 3-i$ $z_1 = \frac{3-i}{1+i}.\frac{1-i}{1-i} = 1-2i$	Scale 10C (0, 3, 7, 10) *Low Partial Credit:* • correct factor(s) *High Partial Credit* • identification of equal terms
		Or
	Or $z-1-i \overline{\smash{\big)} z^2 - 2z + iz + 3 - i}$ quotient $z-1+2i$ $\underline{z^2 - z - iz}$ $-z + 2iz + 3 - i$ $\underline{-z + 1 \qquad + i}$ $2iz + 2 - 2i$ $\underline{2iz + 2 - 2i}$ $z - 1 + 2i = 0$ $z = 1 - 2i$	Scale 10C (0, 3, 7, 10) *Low Partial Credit:* • long division formulated correctly *High Partial Credit* • two correct lines in division **Or**
	Or $(1+i)(m+ni) = 3-i$ $(m-n) + (m+n)i = 3 + (-1)i$ $m - n = 3 \quad and \quad m + n = -1$ Solving $\quad m = 1 \quad and \quad n = -2$	Scale 10C (0, 3, 7, 10) *Low Partial Credit:* • correct multiplication • substitution of $(m+ni)$ into quadratic and stops *High Partial Credit* • identification of equal terms **Note:** *substitution of* $(1+i)$ *merits 0 marks*

(a) The complex numbers z_1, z_2 and z_3 are such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, $z_2 = 2 + 3i$ and $z_3 = 3 - 2i$, where $i^2 = -1$. Write z_1 in the form $a + bi$, where $a, b \in \mathbb{Z}$.

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} = \frac{1}{2+3i} + \frac{1}{3-2i}$$

$$= \frac{3-2i+2+3i}{(2+3i)(3-2i)} = \frac{5+i}{12+5i}$$

$$\Rightarrow \frac{z_1}{2} = \frac{12+5i}{5+i}$$

$$= \frac{12+5i}{5+i} \times \frac{5-i}{5-i}$$

$$= \frac{65+13i}{26}$$

$$\Rightarrow z_1 = 5+i$$

or

$$\frac{1}{2+3i} = \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{4+9} = \frac{2-3i}{13}$$

$$\frac{1}{3-2i} = \frac{1}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{3+2i}{4+9} = \frac{3+2i}{13}$$

$$\frac{1}{2+3i} + \frac{1}{3-2i} = \frac{2-3i}{13} + \frac{3+2i}{13} = \frac{5-i}{13}$$

$$\frac{2}{z_1} = \frac{5-i}{13}$$

Let $z_1 = a + bi$

$$\frac{2}{a+bi} = \frac{5-i}{13}$$

$$26 = (5-i)(a+bi)$$

$$26 + (0)i = 5a + 5bi - ai + b$$

$$26 + (0)i = (5a+b) + (-a+5b)i$$

$\Rightarrow 5a + b = 26$ …(i) and $-a + 5b = 0$ …(ii)

(i): $5a + b = 26$
(ii): $-5a + 25b = 0$

$26b = 26$

$b = 1$

From (ii): $5b = a$

$\Rightarrow a = 5$

$z_1 = 5 + i$

Question 4 (2014)

Let $z_1 = 1 - 2i$, where $i^2 = -1$.

(a) The complex number z_1 is a root of the equation $2z^3 - 7z^2 + 16z - 15 = 0$.
Find the other two roots of the equation.

$z_1 = 1 - 2i$ a root $\Rightarrow \bar{z}_1 = 1 + 2i$ a root.

$(z - 1 + 2i)(z - 1 - 2i) = z^2 - 2z + 5$, a factor

Hence, $(z^2 - 2z + 5)(az + b) = 2z^3 - 7z^2 + 16z - 15$

Equate coefficients: $a = 2$ and $b - 2a = -7 \Rightarrow b = -3$

Third factor: $2z - 3 \Rightarrow z = \frac{3}{2}$

Or

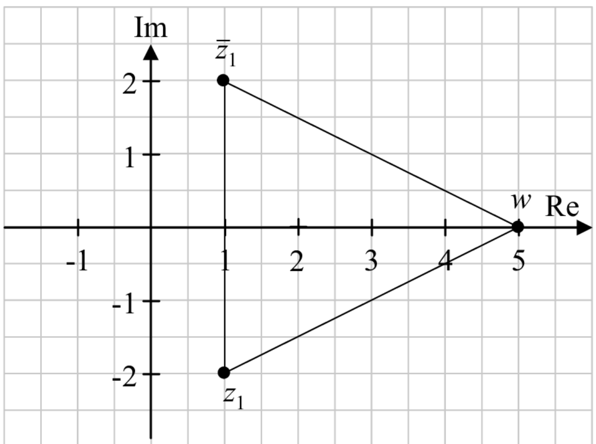
$(2z^3 - 7z^2 + 16z - 15) \div (z^2 - 2z + 5) = 2z - 3$

Third factor: $2z - 3 \Rightarrow z = \frac{3}{2}$

Other roots: $z_2 = 1 + 2i,\ z_3 = \frac{3}{2}$

(b) **(i)** Let $w = z_1 . \bar{z}_1$, where $\bar{z}_1$ is the conjugate of z_1. Plot z_1, $\bar{z}_1$ and w on the Argand diagram and label each point.

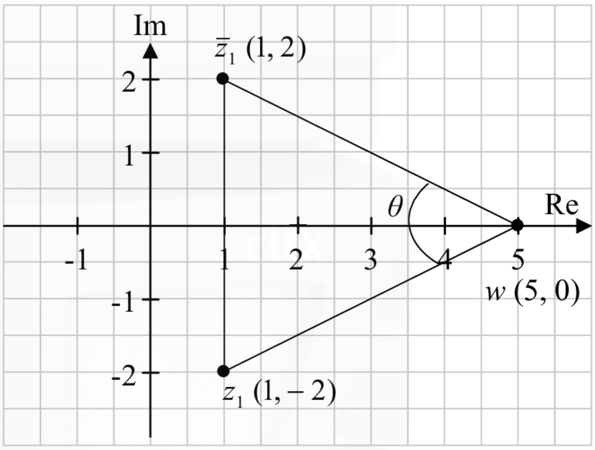
$w = (1 - 2i)(1 + 2i)$
$= 5$



(ii) Find the measure of the acute angle, $\bar{z}_1 w z_1$, formed by joining $\bar{z}_1$ to w to z_1 on the diagram above. Give your answer correct to the nearest degree.

$$\tan\tfrac{1}{2}\angle\bar{z}_1 w z_1 = \tfrac{2}{4} \quad\Rightarrow\quad \tfrac{1}{2}|\angle\bar{z}_1 w z_1| = 26\cdot57 \quad\Rightarrow\quad |\angle\bar{z}_1 w z_1| = 53\cdot14 \approx 53°$$

OR



$$|z_1 w| = \sqrt{(0+2)^2 + (5-1)^2} = \sqrt{16+4} = \sqrt{20}$$

$|z_1 w| = \sqrt{20}$ $\quad |\bar{z}_1 w| = \sqrt{20}$ $\quad |\bar{z}_1 z_1| = 4$

Cosine rule:

$$4^2 = (\sqrt{20})^2 + (\sqrt{20})^2 - 2(\sqrt{20})(\sqrt{20})\cos\theta$$

$$40\cos\theta = 24$$

$$\cos\theta = \frac{24}{40} = 0\cdot6$$

$$|\theta| = 53\cdot13 \approx 53°$$

Question 5 (2014)

(a) $w = -1 + \sqrt{3}i$ is a complex number, where $i^2 = -1$.

(i) Write w in polar form.

We have $|w| = \sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{4} = 2$. Also, if $\arg(w) = \theta$, then $\tan(\theta) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$ and θ lies in the second quadrant (from the diagram). Therefore $\theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$ radians. So

$$w = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

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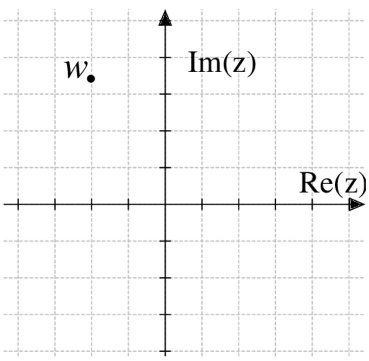

(ii) Use De Moivre's Theorem to solve the equation $z^2 = -1 + \sqrt{3}i$. Give your answer(s) in rectangular form.

Suppose that the polar form of z is given by $z = r(\cos\theta + i\sin\theta)$. Then

$$[r(\cos\theta + i\sin\theta)]^2 = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$$

By De Moivre's Theorem this is equivalent to

$$r^2(\cos(2\theta) + i\sin(2\theta)) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Therefore $r^2 = 2$ and $2\theta = \frac{2\pi}{3} + 2n\pi, n \in \mathbb{Z}$. So $r = \sqrt{2}$ and $\theta = \frac{\pi}{3} + n\pi$. We get two distinct solutions (corresponding to $n = 0$ and $n = 1$).

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}$$

and

$$z_2 = \sqrt{2}\left(\cos(\frac{\pi}{3} + \pi) + i\sin(\frac{\pi}{3} + \pi)\right) = \sqrt{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{\sqrt{2}} - i\sqrt{\frac{3}{2}}$$

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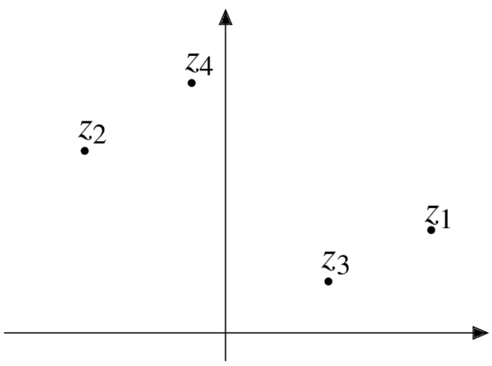
(b) Four complex numbers z_1, z_2, z_3 and z_4 are shown on the Argand diagram. They satisfy the following conditions:

$$\begin{aligned} z_2 &= iz_1 \\ z_3 &= kz_1, \text{ where } k \in \mathbb{R} \\ z_4 &= z_2 + z_3 \end{aligned}$$

The same scale is used on both axes.

(i) Identify which number is which by labelling the points on the diagram.

(ii) Write down the approximate value of k.



Answer: $\frac{1}{2}$

Explanation: Multiplication by i rotates a complex number by 90° anticlockwise about the origin - so z_2 is obtained by rotating z_1 through 90° about the origin.

Since $z_3 = kz_1$, we must have 0, z_1 and z_3 being collinear.

Since $z_4 = z_2 + z_3$, we must have 0, z_2, z_4 and z_3 forming a parallelogram.

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(a) Verify that z can be written as $1-\sqrt{3}i$.

$$z=\frac{4}{1+\sqrt{3}i}=\frac{4}{1+\sqrt{3}i}\times\frac{1-\sqrt{3}i}{1-\sqrt{3}i}=\frac{4-4\sqrt{3}i}{1+3}=1-\sqrt{3}i$$

OR

If $z=\dfrac{4}{1+\sqrt{3}i}=1-\sqrt{3}i$

then $4=(1+\sqrt{3}i)(1-\sqrt{3}i)=(1)^2+(\sqrt{3})^2=4$

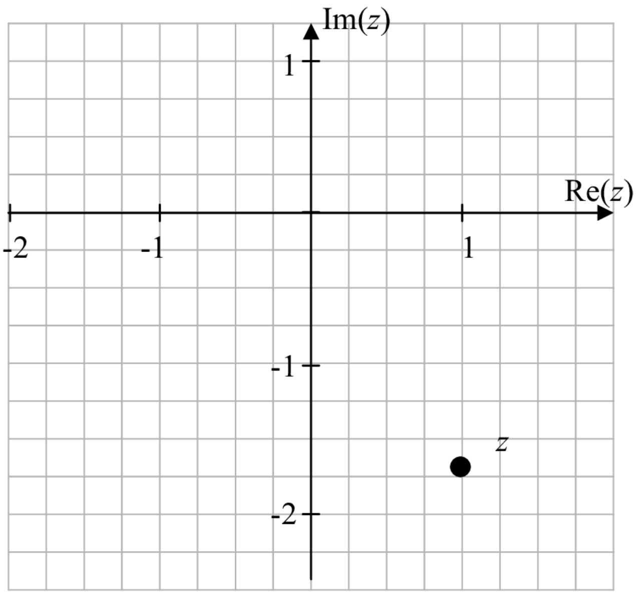
$\Rightarrow$ True

(b) Plot z on an Argand diagram and write z in polar form.

$$\tan\alpha=\frac{\sqrt{3}}{1}\Rightarrow\alpha=\frac{\pi}{3}\Rightarrow\theta=\frac{5\pi}{3}$$

$$r=|1-\sqrt{3}i|=\sqrt{1+3}=\sqrt{4}=2$$

$$z=2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$$



(c) Use De Moivre's theorem to show that $z^{10}=-2^9\left(1-\sqrt{3}i\right)$.

$$z^{10}=\left[2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)\right]^{10}$$

$$=2^{10}\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)^{10}=2^{10}\left(\cos\frac{50\pi}{3}+i\sin\frac{50\pi}{3}\right)$$

$$=2^{10}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=2^{10}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-2^9\left(1-\sqrt{3}i\right)$$

Question 7 (2012)

$$w^4 = \frac{81}{16}\left(\cos\left(\frac{4\pi}{9} + 2n\pi\right) + i\sin\left(\frac{4\pi}{9} + 2n\pi\right)\right)$$

$$w = \frac{3}{2}\left(\cos\left(\frac{\pi}{9} + \frac{n\pi}{2}\right) + i\sin\left(\frac{\pi}{9} + \frac{n\pi}{2}\right)\right), \quad n = 0, 1, 2, 3.$$

$$w = \frac{3}{2}\left(\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right), \quad \frac{3}{2}\left(\cos\left(\frac{11\pi}{18}\right) + i\sin\left(\frac{11\pi}{18}\right)\right),$$

$$\frac{3}{2}\left(\cos\left(\frac{10\pi}{9}\right) + i\sin\left(\frac{10\pi}{9}\right)\right), \quad \frac{3}{2}\left(\cos\left(\frac{29\pi}{18}\right) + i\sin\left(\frac{29\pi}{18}\right)\right)$$