MarkingScheme

AreaVolumeH

Question 1 (2017)

(a)	$\lvert AJ\rvert = 6371 + 0{\cdot}214$ $\lvert JH\rvert^2 = \lvert AJ\rvert^2 - \lvert AH\rvert^2$ $\lvert JH\rvert = \sqrt{(6371 + 0{\cdot}214)^2 - 6371^2}$ $= 52{\cdot}21 = 52$	**Scale 15C (0, 6, 9, 15)** *Low Partial Credit:* • |AJ| formulated • indication of Pythagoras *High Partial Credit:* • Pythagoras fully substituted
(b)	$\cos 53° = \frac{r}{6371}$ or $\sin 37^o = \frac{r}{6371}$ $r_{S_1} = 6371 \times \cos 53 = 3834{\cdot}1635$ $l_{S_1} = 2\pi r_{S_1} = 2\pi(3834{\cdot}1635) = 24091$	**Scale 10C (0, 4, 5, 10)** *Low Partial Credit:* • $\cos 53°$ **or** $\sin 47°$ *High Partial Credit:* • radius of s_1 calculated and stops • length of circle formula fully substituted

(a)	$\text{Vol of space} = \text{Cylinder} - 2\times\text{Cone}$ $= \pi R^2(2R) - \frac{2}{3}\pi R^2(R)$ $= 2\pi R^3 - \frac{2}{3}\pi R^3$ $= \frac{4}{3}\pi R^3$	**Scale 10C (0, 4, 5, 10)** *Low Partial Credit:* • A relevant volume formulated *High Partial Credit:* • Vol of space formulated in terms of π and R
(b) (i)	$12^2 = 6^2 + \lvert AB\rvert^2$ $\lvert AB\rvert = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3}$	**Scale 10B (0, 5, 10)** *Partial Credit:* • indication of Pythagoras
(b) (ii)	$\frac{h_1}{h_2} = \frac{6}{12} = \frac{r}{12}$ $r = 6 \text{ cm}$	**Scale 10C (0, 4, 5, 10)** *Low Partial Credit:* • indication of similar triangles • indication of a relevant ratio *High Partial Credit:* • corresponding ratios identified but fails to finish **Note**: Accept correct answer without work
(b) (iii)	$\text{Cylinder} = \pi 12^2 - \pi 6^2 = 108\pi$ $\text{Sphere} = \pi\left(6\sqrt{3}\right)^2 = 108\pi$	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* • Surface Area in Fig. 3 substituted • Surface Area in Fig 4 substituted *High Partial Credit:* • One Surface Area found
(c)	$\text{Vol} = \pi(12^2)(6)$ $-\left(\frac{1}{3}\pi 12^2 \times 12 - \frac{1}{3}\pi 6^2 \times 6\right)$ $\text{Vol} = 360\pi \text{ cm}^3$	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* • Vol of cylinder found • Vol of truncated cone substituted • Vol of one cone found (12 or 6) *High Partial Credit:* • Volume fully substituted but fails to finish • Volume of truncated cone found

(a)	$\tan 60° = \dfrac{\lvert TE\rvert}{\lvert CT\rvert}$ $\sqrt{3}\lvert CT\rvert = \lvert TE\rvert$	**Scale 10B (0, 5, 10)** *Partial Credit:* • $\tan 60°$ • effort to express $\lvert TE\rvert$ in terms of another side of the triangle
(b)	$\tan 30° = \dfrac{\lvert TE\rvert}{\lvert DT\rvert}$ $\lvert TE\rvert = \lvert DT\rvert\dfrac{1}{\sqrt{3}}$ $\lvert TE\rvert = \dfrac{\sqrt{225+\lvert CT^2\rvert}}{\sqrt{3}}$ $\lvert TE\rvert = \sqrt{\dfrac{225+\lvert CT\rvert^2}{3}}$	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* • $\tan 30°$ • Use of Pythagoras for $\lvert DT\rvert$ • Effort at expressing $\lvert DT\rvert$ in terms of another side of ΔDET *High Partial Credit:* • $\lvert TE\rvert = \lvert DT\rvert\frac{1}{\sqrt{3}}$
(c)	$\sqrt{3}\lvert CT\rvert = \sqrt{\dfrac{225+\lvert CT\rvert^2}{3}}$ $\lvert CT\rvert = \sqrt{\dfrac{225}{8}}$ $= 5{\cdot}3033$ m $= 5{\cdot}3$ m	**Scale 10C (0, 4, 5, 10)** *Low Partial Credit:* • equates both expressions *High Partial Credit:* • Isolate $\lvert CT\rvert$ in equation

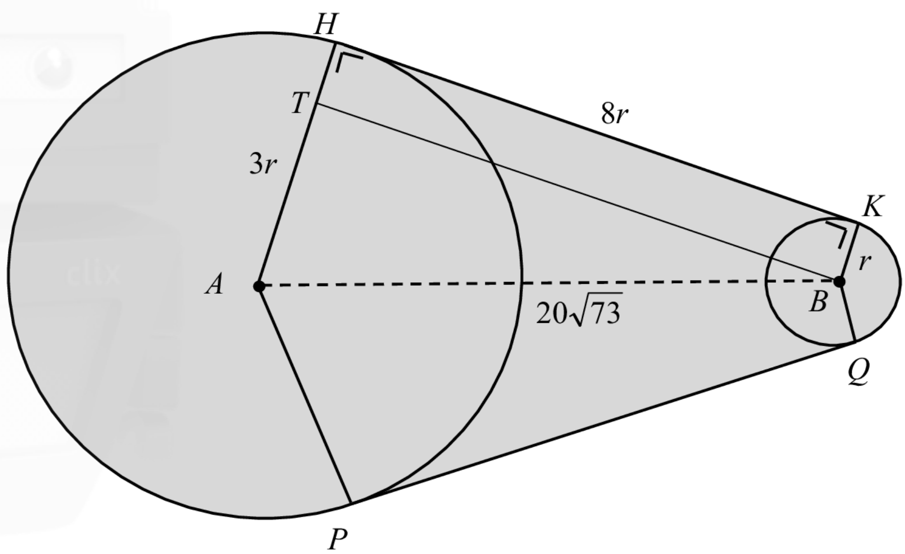
(d)	$\lvert TE\rvert = \sqrt{3}\lvert CT\rvert = 9{\cdot}17986 \text{ m} = 9{\cdot}2 \text{ m}$	**Scale 10B (0, 5, 10)** *Low Partial Credit* • Substitution into formula for $\lvert TE\rvert$
(e)	$\cos\theta = \frac{\lvert CT\rvert}{\lvert FT\rvert} = \frac{\lvert CT\rvert}{\lvert TE\rvert} = \frac{\lvert CT\rvert}{\sqrt{3}\lvert CT\rvert} = \frac{1}{\sqrt{3}}$ $\theta = 54{\cdot}7$	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* • Some relevant substitution for $\cos\theta$ *High Partial Credit:* • Formula for $\cos\theta$ substituted in terms of $\lvert CT\rvert$
(f)	$P = \frac{(54{\cdot}7)(2)}{360}$ $= 0{\cdot}3038$ $= 30{\cdot}4$	**Scale 10C (0, 4, 5, 10)** *Low Partial Credit:* • (Answer to **part (e)**)×2 • 360° *High Partial Credit:* • P fully formulated

Q7	Model Solution – 55 Marks	Marking Notes
(a) (i)	$\lvert EC\rvert^2 = 3^2 + 2{\cdot}5^2 = 15{\cdot}25$ $\lvert EC\rvert = \sqrt{15{\cdot}25}$ $\lvert EC\rvert = 3{\cdot}905$ $\Rightarrow \lvert AC\rvert = 1{\cdot}9525$ $= 1{\cdot}95$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • Pythagoras with relevant substitution *High Partial Credit* • $\lvert EC\rvert$ correct • $\lvert AC\rvert = \frac{1}{2}\sqrt{15{\cdot}25}$
(a) (ii)	$\tan 50° = \frac{\lvert AB\rvert}{1{\cdot}95}$ $\lvert AB\rvert = 1{\cdot}95(1{\cdot}19175) = 2{\cdot}23239$ $\lvert AB\rvert = 2{\cdot}3$	Scale 10B (0, 5, 10) *Partial Credit* • tan formulated correctly
(a) (iii)	$\lvert BC\rvert^2 = 1{\cdot}95^2 + 2{\cdot}3^2$ $\lvert BC\rvert = 3 \cdot 015377$ $\lvert BC\rvert = 3$ Also: $\sin 40° = \frac{1{\cdot}95}{\lvert BC\rvert}$ **or** $\cos 40° = \frac{2{\cdot}3}{\lvert BC\rvert}$ **or** $\cos 50° = \frac{1{\cdot}95}{\lvert BC\rvert}$ **or** $\sin 50° = \frac{2{\cdot}3}{\lvert BC\rvert}$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • Pythagoras with relevant substitution *High Partial Credit* • Pythagoras fully substituted • $\lvert BC\rvert = \frac{1{\cdot}95}{\sin 40°}$ (i.e. $\lvert BC\rvert$isolated)
(a) (iv)	$3^2 = 3^2 + 2{\cdot}5^2 - 2(3)(2{\cdot}5)\cos \propto$ $15 \cos \propto = 6{\cdot}25$ $\propto = 65°$ **or** $\cos \propto = \frac{1{\cdot}25}{3}$ $\propto = 65°$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • cosine rule with some relevant substitution • cosine ratio with some relevant substitutions • identifies three sides of triangle *BCD* *High Partial Credit* • cosine rule with full relevant substitutions • cosine ratio with full relevant substitutions

(a) **(v)**	$A = 2\times$ isosceles triangle $+2\times$ equilateral triangle $= 2\times\left[\frac{1}{2}(2\cdot5)(3)\sin 65°\right] +$ $2\times\left[\frac{1}{2}(3)(3)\sin 60°\right]$ $= 14\cdot59$ A=15	Scale 10D (0,3,5,8,10) *Low Partial Credit* • recognises area of 4 triangles *Mid Partial Credit* • Area of 1 triangle correct *High Partial Credit* • area of isosceles triangle and equilateral triangle **Note**: Area = 4 isosceles or 4 equilateral triangles merit *HPC* at most
(b)	$\tan 60° = \frac{3}{\lvert CA\rvert}$ $\Rightarrow \lvert CA\rvert = \sqrt{3}$ $\lvert CE\rvert = 2\sqrt{3}$ $x^2 + x^2 = \left(2\sqrt{3}\right)^2$ $x = \sqrt{6}$	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • effort at Pythagoras but without $\lvert CA\rvert$ (or $\lvert CE\rvert$) • $\lvert CA\rvert$ found *High Partial Credit* • $\lvert CE\rvert = 2\sqrt{3}$

Question 5 (2015)

(a) Find r, the radius of the smaller circle. (Hint: Draw $BT \parallel KH$, $T \in AH$.)



$$|AT|^2 + |BT|^2 = |AB|^2 \quad \Rightarrow \quad (3r)^2 + (8r)^2 = \left(20\sqrt{73}\right)^2$$
$$\Rightarrow \quad 9r^2 + 64r^2 = 29200$$
$$\Rightarrow \quad r^2 = 400 \quad \Rightarrow \quad r = 20 \text{ cm}$$

(b) Find the area of the quadrilateral $ABKH$.

$$
\begin{aligned}
|ABKH| &= |BKHT| + |\Delta ABT| \\
&= 20\times 160 + \tfrac{1}{2}(60)(160) \\
&= 8000 \text{ cm}^2
\end{aligned}
$$

(c) **(i)** Find $|\angle HAP|$, in degrees, correct to one decimal place.

$$
\begin{aligned}
\tan|\angle HAB| = \frac{160}{60} \quad &\Rightarrow \quad |\angle HAB| = 69\cdot 44^\circ \\
&\Rightarrow \quad |\angle HAP| = 138\cdot 9^\circ
\end{aligned}
$$

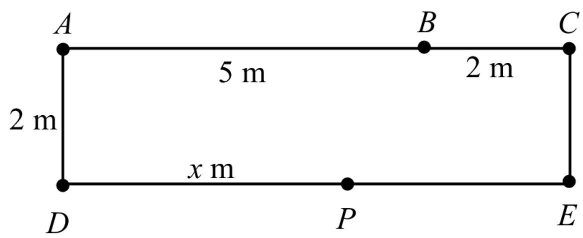
(ii) Find the area of the machine part, correct to the nearest cm^2.

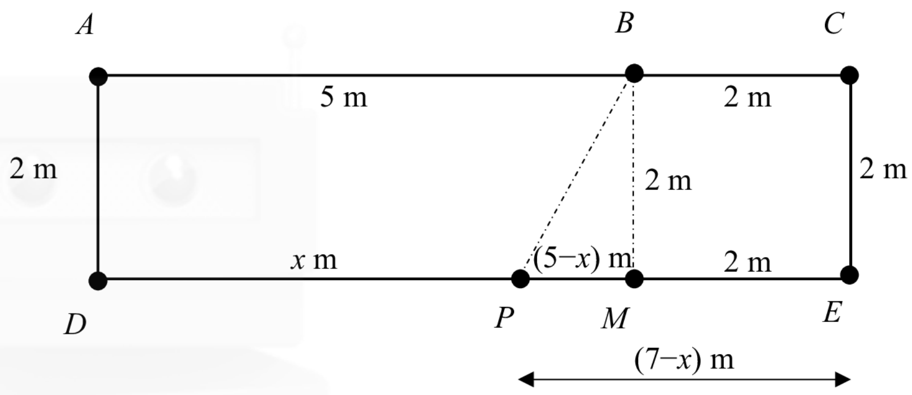
Area large sector HAP + 2 area $HABK$ + area sector KBQ

$$
\begin{aligned}
&= \pi(80)^2\left(\frac{221\cdot 1}{360}\right) + 2\times 8000 + \pi(20)^2\left(\frac{138\cdot 9}{360}\right) \\
&= 12348\cdot 55 + 16000 + 484\cdot 85 \\
&= 28833\cdot 4 \\
&= 28833
\end{aligned}
$$

Question 6 (2014)

(b) $ADEC$ is a rectangle with $|AC| = 7$ m and $|AD| = 2$ m, as shown.
B is a point on $[AC]$ such that $|AB| = 5$ m.
P is a point on $[DE]$ such that $|DP| = x$ m.





(i) Let $f(x) = |PA|^2 + |PB|^2 + |PC|^2$.
Show that $f(x) = 3x^2 - 24x + 86$, for $0 \le x \le 7$, $x \in \mathbb{R}$.

$$|PM| = |PE| - |ME|$$
$$= (7 - x) - 2$$
$$= (5 - x)$$

$$f(x) = |PA|^2 + |PB|^2 + |PC|^2$$
$$= \left[|PD|^2 + |DA|^2\right] + \left[|PM|^2 + |MB|^2\right] + \left[|PE|^2 + |EC|^2\right]$$
$$= x^2 + 2^2 + \left((5-x)^2 + 2^2\right) + \left((7-x)^2 + 2^2\right)$$
$$= x^2 + 4 + 25 - 10x + x^2 + 4 + 49 - 14x + x^2 + 4$$
$$= 3x^2 - 24x + 86$$

(ii) The function $f(x)$ has a minimum value at $x = k$.
Find the value of k and the minimum value of $f(x)$.

$$f(x) = 3x^2 - 24x + 86$$
$$f'(x) = 6x - 24$$
$$f''(x) = 6 > 0 \quad \Rightarrow \quad \text{minimum}$$
$$f'(x) = 0 \Rightarrow 6x - 24 = 0 \Rightarrow x = 4 = k$$
$$f(4) = 3(4)^2 - 24(4) + 86 = 38$$

OR

$$\begin{aligned} f(x) &= 3x^2 - 24x + 86 \\ &= 3\left(x^2 - 8x + \frac{86}{3}\right) \\ &= 3\left[\left(x^2 - 8x + 16\right) + \frac{38}{3}\right] \\ &= 3\left[\left(x - 4\right)^2 + \frac{38}{3}\right] \end{aligned}$$

At $x = 4 \Rightarrow$ minimum value for $f(x)$

$$\begin{aligned} f(4) &= 3x^2 - 24x + 86 \\ &= 3(4)^2 - 24(4) + 86 \\ &= 48 - 96 + 86 \\ &= 38 \end{aligned}$$

Let l be the length of the box and let w be the width of the box, both in centimetres. Then by adding up dimensions as we move left to right across the diagram above, we see that $1+l+h+l+h=31$. Therefore, by isolating l in this equation we obtain

$$l = 15 - h.$$

Going top to bottom, we see that $1+h+w+h+1=22$ and by isolating w, we see that

$$w = 20 - 2h.$$

Therefore

height =	h cm
length =	$15-h$ cm
width =	$20-2h$ cm

(b) Write an expression for the capacity of the box in cubic centimetres, in terms of h.

Capacity = length $\times$ width $\times$ height $= (15-h)(20-2h)h = 2h^3 - 50h^2 + 300h\text{cm}^3$.

(c) Show that the value of h that gives a box with a square bottom will give the correct capacity.

The bottom of the box is square if and only if length = width. In other words, if and only if $15-h = 20-2h$. This is equivalent to $h = 5$. From the solution to part (b), we calculate that, when $h = 5$, the capacity of the box will be $(15-5)(20-2(5))5 = 10(10)(5) = 500\text{cm}^3$, as required.

(d) Find, correct to one decimal place, the other value of h that gives a box of the correct capacity.

We must solve $2h^3 - 50h^2 + 300h = 500$, or

$$2h^3 - 50h^2 + 300h - 500 = 0.$$

From part (c), we know that $h = 5$ is one solution. Therefore, by the Factor Theorem, $(h-5)$ is a factor of $2h^3 - 50h^2 + 300h - 500$. Factorising yields

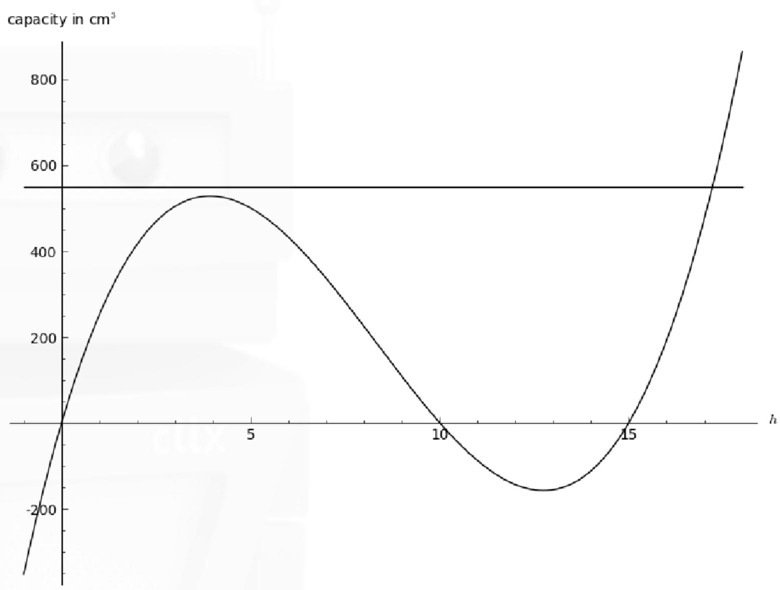
$$2h^3 - 50h^2 + 300h - 500 = (h-5)(2h^2 - 40h + 100).$$

Now, we solve $2h^2 - 40h + 100 = 0$ using the quadratic formula. So

$$h = \frac{40 \pm \sqrt{40^2 - 4(2)(100)}}{2(2)} = \frac{40 \pm \sqrt{800}}{4} = 10 \pm \sqrt{50}$$

So, correct to one decimal place, $h = 17.1$ or $h = 2.9$.
Now, however, we observe that since the length of the box is $15 - h$, we must have $15 - h > 0$ or $h < 15$. Therefore $h \neq 17.1$. So the other value of h that gives the correct capacity is 2.9cm.

MODEL ANSWER BY thamathstutor.ie ONLINE SUPPORT SYSTEM FOR PROJECT MATHS



The capacity of the new box will be $1.1 \times 500 = 550\text{cm}^3$. On the diagram above we have drawn a horizontal line representing the equation

$$\text{Capacity} = 550.$$

We can see from the diagram that this horizontal line only meets the cubic curve at one point and that the h-co-ordinate of that point is greater than 15.
However, as we observed above, for any box constructed as described in the question, we must have $h < 15$. Therefore it is not possible to make the bigger box from the same piece of cardboard as before.

MODEL ANSWER BY thamathstutor.ie ONLINE SUPPORT SYSTEM FOR PROJECT MATHS

(i) Find the value of $f(0.2)$

Substituting 0.2 for x gives

$$f(0.2) = -0.5(0.2)^2 + 5(0.2) - 0.98 = -0.5(0.04) + 1 - 0.98 = 0$$

(ii) Show that f has a local maximum point at $(5, 11.52)$.

First we calculate the derivative of f:

$$f'(x) = -0.5(2x) + 5(1) - 0 = -x + 5.$$

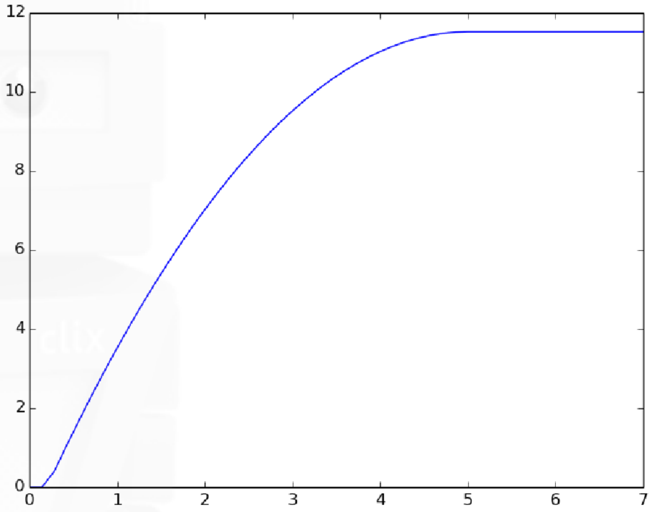
Now $f'(5) = -5 + 5 = 0$. Therefore $x = 5$ is a stationary point.
Now

$$f''(x) = -1.$$

So $f''(5) = -1 < 0$. That means that $x = -5$ is a local maximum. Finally,

$$f(5) = -0.5(5^2) + 5(5) - 0.98 = 11.52.$$

Therefore the graph of f has a local maximum point at $(5, 11.52)$.



Note that between $t = 0$ and $t = 0.2$ the graph is just a horizontal line along the t-axis. Likewise, for $t \geq 5$ the graph is a horizontal line at height $v = 11.52$. In between $t = 0.2$ and $t = 5$ the function is a quadratic so the graph must be a parabola. We can sketch this by evaluating the function at three or four points. For example $v(1) = 3.52$, $v(2) = 7.02$, $v(3) = 9.52$ and $v(4) = 11.02$. So we plot the points $(1, 3.52)$, $(2, 7.02)$, $(3, 9.52)$ and $(4, 11.02)$ and then join them by a smooth curve. Make sure that this parabolic arc starts at $(0.2, 0)$ and ends at $(5, 11.52)$.

MODEL ANSWER BY thematHstutor.ie
ONLINE SUPPORT SYSTEM FOR PROJECT MATHS

(ii) Find the distance travelled by the sprinter in the first 5 seconds of the race.

The distance travelled in the first 5 seconds of the race is given by

$$\int_0^5 v(t) \quad dt.$$

Now

$$\begin{aligned}
\int_0^5 v(t) \quad dt &= \int_0^{0.2} v(t) \quad dt + \int_{0.2}^5 v(t) \quad dt \\
&= \int_0^{0.2} 0 \quad dt + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= 0 + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \Big|_{0.2}^5 \\
&= \frac{0.5(5^3)}{3} + \frac{5(5^2)}{2} - 0.98(5) - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2)\right) \\
&= 36.864
\end{aligned}$$

So the sprinter travels 36.864 metres in the first 5 seconds of the race.

MODEL ANSWER BY thematicstutor.ie ONLINE SUPPORT SYSTEM FOR PROJECT MATHS

(iii) Find the sprinter's finishing time for the race. Give your answer correct to two decimal places.

We have just seen that the sprinter travels 36.864 metres in the first 5 seconds of the race. So he has 63.136metres left to travel to complete the race at that point. Also after 5 seconds, his velocity is a constant 11.52 metres per second. Therefore it will take him a further $\frac{63.136}{11.52}$ seconds to complete the race. Now $\frac{63.136}{11.52} = 5.48$ correct to two decimal places. So his total time is $5 + 5.48 = 10.48$ seconds, correct to two decimal places.

MODEL ANSWER BY thematicstutor.ie ONLINE SUPPORT SYSTEM FOR PROJECT MATHS

After 7 seconds the sprinter's velocity doesn't change. So the first step is to calculate how far he has travelled in the first 7 seconds of the race. As in part (ii) this is equal to the value of the defnite integral

$$\int_{0.2}^{7} (-0.5t^2 + 5t - 0.98) \quad dt.$$

Now

$$\begin{aligned}
\int_{0}^{7} (-0.5t^2 + 5t - 0.98) \quad dt &= \left. \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \right|_{0.2}^{7} \\
&= \frac{0.5(7^3)}{3} + \frac{5(7^2)}{2} - 0.98(7) \\
&\quad - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2) \right) \\
&= 58.571
\end{aligned}$$

So he travels 58.571 metres in 7 seconds. Therefore, he has $100 - 58.571 - 41.429$ metres left to travel at that point. His velocity for rest of the race is 11.52 metres per second. Therefore it will take him another $\frac{41.429}{11.52} = 3.596$ seconds to complete the race. So his total time for the race is $7 + 3.596 = 10.596$. So it takes him 10.60 seconds to finish the race, correct to two decimal places.

(c) A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

(i) Prove that the radius of the snowball is decreasing at a constant rate.

Let t be time. Let r be the radius, A the surface area and V the volume of the snowball. From the Formula and Tables booklet we know that $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$. In particular,

$$\frac{dV}{dr} = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2 = A.$$

Now we are told that the rate of change of volume with respect to time is proportional to the surface area. In other words,

$$\frac{dV}{dt} = kA \tag{1}$$

for some constant k. Clearly $k < 0$ since the volume of the snowball is decreasing as it melts. On the other hand, using the chain rule, we see that

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt} \\ &= A\frac{dr}{dt} \end{aligned} \tag{2}$$

Therefore by combining (1) and (2) , we see that

$$A\frac{dr}{dt} = kA.$$

Now dividing across by A yields

$$\frac{dr}{dt} = k$$

where k is a constant, as required.

(ii) If the snowball loses half of its volume in an hour, how long more will it take for it to melt completely?

Give your answer to the nearest minute.

Let r_0 be the initial radius and let r_2 be the radius after 1 hour.
So the initial volume is $\frac{4}{3}\pi r_0^3$. Therefore after one hour, the volume is $\frac{2}{3}\pi r_0^3$. Therefore

$$\frac{4}{3}\pi r_1^3 = \frac{2}{3}\pi r_0^2.$$

Therefore

$$\left(\frac{r_1}{r_0}\right)^3 = \frac{1}{2}$$

or

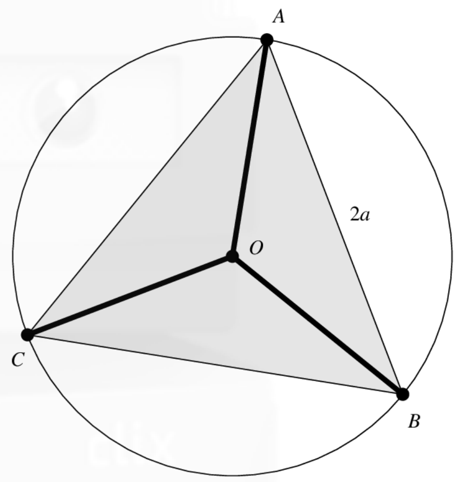
$$r_1 = \frac{1}{\sqrt[3]{2}} r_0.$$

Now the radius is decreasing at a constant rate and we have found that it takes 1 hour for it to decrease from r_0 to $\frac{1}{\sqrt[3]{2}} r_0$. Therefore the rate of change of the radius is $r_0 - \frac{1}{\sqrt[3]{2}} r_0$ units per hour.
Now the snowball will have melted completely when the radius reaches 0. So we calculate the time required to to change from r_0 to 0. This will be

$$\frac{\text{total change}}{\text{rate of change}} = \frac{r_0 - 0}{r_0 - \frac{1}{\sqrt[3]{2}} r_0} = \frac{1}{1 - \frac{1}{\sqrt[3]{2}}} \text{ hours.}$$

This is equal to 4.8473 hours (correct to four decimal places). So it will take 3.8473 more hours (after the 1 already elapsed) for it to melt completely.
Now 3.8473 hours is equal $3.8473 \times 60 = 230.84$.
So, to the nearest minute, it will take a further 231 minutes for the snowball to melt completely.

Consider the base of the cylindrical container together with the base of the tetrahedron drawn in the diagram below:



Now O is the circumcentre of an equilateral triangle. So

$$|\angle BOA| = |\angle AOC| = |\angle COB|$$

and since these angles sum to 360°, we must have $|\angle BOA| = 120^\circ$.
Now consider the triangle $\triangle AOB$. It is an isosceles triangle, so $|\angle ABO| = |\angle BAO|$. Since

$$|\angle ABO| + |\angle BAO| + |\angle BOA| = 180^\circ$$

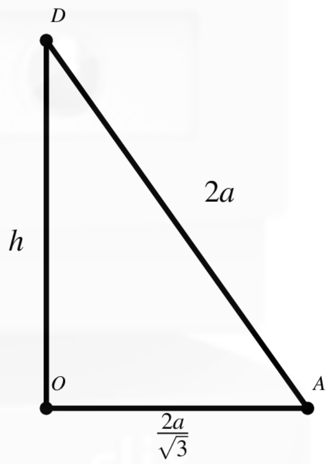
we get $|\angle ABO| = 30^\circ$. Now we apply the Sine Rule to the triangle $\triangle AOB$ to get $\frac{|OA|}{\sin 30^\circ} = \frac{2a}{\sin 120^\circ}$ or

$$|OA| = 2a\frac{\sin 30^\circ}{\sin 120^\circ} = 2a\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2a}{\sqrt{3}}.$$

So the radius of the cylinder is $\frac{2a}{\sqrt{3}}$.

Solution continued on next page ...

Now let D be the top point of the tetrahedron and drop a vertical line from D to O to create a right angled triangle as shown below. Let h be the height of the cylinder.



Therefore, by Pythagoras' Theorem,

$$h^2 + \left(\frac{2a}{\sqrt{3}}\right)^2 = (2a)^2$$

which implies that $h^2 = 4a^2 - \frac{4a^2}{3} = \frac{8a^2}{3}$. Therefore

$$h = a\sqrt{\frac{8}{3}} = 2a\sqrt{\frac{2}{3}}.$$

Now the volume of a cylinder with height $2a\sqrt{\frac{2}{3}}$ and radius $\frac{2a}{\sqrt{3}}$ is

$$\pi\left(\frac{2a}{\sqrt{3}}\right)^2 2a\sqrt{\frac{2}{3}} = \pi\frac{8a^3\sqrt{2}}{3\sqrt{3}}.$$

Now multiply this last expression above and below by $\sqrt{3}$ to obtain a volume of

$$\pi\frac{8a^3\sqrt{6}}{9}$$

as required.

(a) What is the height of the surface at time $t = 0$?

$h(0) = 10^2 = 100$ cm.

(b) After how many seconds will the height of the surface be 64 cm?

$$\left(10 - \frac{t}{200}\right)^2 = 64$$

$$10 - \frac{t}{200} = 8 \quad \text{(since } t > 0\text{)}$$

$$t = 400$$

Answer: 400 seconds.

(c) Find the rate at which the **volume** of water in the tank is decreasing at the instant when the height is 64 cm.
Give your answer correct to the nearest cm^3 per second.

$V = \pi r^2 h = \pi(52)^2 h = 2704\pi h.$

$$\frac{dV}{dt} = 2704\pi \frac{dh}{dt}$$

$$= 2704\pi\left(\tfrac{-2}{25}\right) = -216{\cdot}32\pi$$

$\therefore$ Volume is decreasing at $216{\cdot}32\pi \text{ cm}^3\,\text{s}^{-1} \approx 680\,\text{cm}^3\,\text{s}^{-1}$.

$$\frac{dh}{dt} = 2\left(10 - \frac{t}{200}\right)\frac{-1}{200}$$

$$\left.\frac{dh}{dt}\right|_{t=400} = \frac{-2}{25}$$

(d) The rate at which the volume of water in the tank is decreasing is equal to the speed of the water coming out of the hole, multiplied by the area of the hole. Find the speed at which the water is coming out of the hole at the instant when the height is 64 cm.

$$\frac{dV}{dt} = Av$$

$$216{\cdot}32\pi = \pi 1^2 v$$

$$v = 216{\cdot}32 \text{ cm s}^{-1}$$

(e) Show that, as t varies, the speed of the water coming out of the hole is a constant multiple of $\sqrt{h}$.

$$v = -\frac{1}{\pi}\frac{dV}{dt}$$

$$= -\frac{1}{\pi}2704\pi\frac{dh}{dt}$$

$$= 27{\cdot}04\left(10 - \frac{t}{200}\right)$$

$$= 27{\cdot}04\sqrt{h}$$

which is a constant multiple of $\sqrt{h}$

(f) The speed, in centimetres per second, of water coming out of a hole like this is known to be given by the formula

$$v = c\sqrt{1962h}$$

where c is a constant that depends on certain features of the hole.
Find, correct to one decimal place, the value of c for this hole.

$$c\sqrt{1962} = 27{\cdot}04$$

$$c \approx 0{\cdot}6$$

(a) By expressing r^2 in terms of h, show that the capacity of the cup, in cm^3, is given by the formula

$$V = \frac{\pi}{3} h\left(81 - h^2\right).$$

$$r^2 + h^2 = 9^2$$
$$r^2 = 81 - h^2$$
$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{\pi h}{3}\left(81 - h^2\right)$$

$$V = \frac{\pi}{3} h\left(81 - h^2\right).$$

$$r^2 + h^2 = 9^2$$
$$r^2 = 81 - h^2$$
$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{\pi h}{3}\left(81 - h^2\right)$$

$$\frac{\pi h}{3}\left(81-h^2\right)=\frac{154\pi}{3}$$

$$h\left(81-h^2\right)=154$$

$$h^3-81h+154=0$$

Integer root is a factor of 154 $\Rightarrow \in \{1, 2, 7, 14, 11, 22, 77, 154\}$

$h=1$ is not a solution; $h=2$ is a solution.

$$\begin{array}{r l}
 & h^2+2h-77 \\
h-2 \big) & h^3+0h^2-81h+154 \\
 & \underline{h^3-2h^2} \\
 & \quad 2h^2-81h \\
 & \quad \underline{2h^2-4h} \\
 & \qquad -77h+154 \\
 & \qquad \underline{-77h+154} \\
 & \qquad\qquad 0
\end{array}$$

$$h^2+2h-77=0$$

$$(h+1)^2-78=0$$

$$h=-1\pm\sqrt{78}$$

Positive solutions are $h=2, \quad h\approx 7.83$

$$\frac{\pi h}{3}\left(81-h^2\right)=\frac{154\pi}{3}$$
$$h\left(81-h^2\right)=154$$
$$h^3-81h+154=0$$

Integer root is a factor of 154 $\Rightarrow \in \{1, 2, 7, 14, 11, 22, 77, 154\}$

$h=1$ is not a solution; $h=2$ is a solution.

$$\begin{array}{r|l} & h^2+2h-77 \\ \hline h-2 & h^3+0h^2-81h+154 \\ & \underline{h^3-2h^2} \\ & \quad 2h^2-81h \\ & \quad \underline{2h^2-4h} \\ & \qquad -77h+154 \\ & \qquad \underline{-77h+154} \\ & \qquad\qquad 0 \end{array}$$

$$h^2+2h-77=0$$
$$(h+1)^2-78=0$$
$$h=-1\pm\sqrt{78}$$

Positive solutions are $h=2, \quad h\approx 7.83$

(c) Find the maximum possible volume of the cup, correct to the nearest cm^3.

$$V=\frac{\pi h}{3}\left(81-h^2\right), \qquad h\in[0,9]$$
$$=\frac{\pi}{3}\left(81h-h^3\right)$$
$$\frac{dV}{dh}=\pi\left(27-h^2\right)$$

Local max/min when $\frac{dV}{dh}=0 \;\Rightarrow\; h=\sqrt{27}$. (Clearly a max., since $V(0) = V(9) = 0$.)

$$V_{\max}=\pi\left(27\sqrt{27}-9\sqrt{27}\right)=18\sqrt{27}\pi\approx 294\text{ cm}^3$$

$$V = \frac{\pi h}{3}\left(81 - h^2\right), \qquad h \in [0,9]$$
$$= \frac{\pi}{3}\left(81h - h^3\right)$$
$$\frac{dV}{dh} = \pi\left(27 - h^2\right)$$

Local max/min when $\frac{dV}{dh} = 0 \Rightarrow h = \sqrt{27}$. (Clearly a max., since $V(0) = V(9) = 0$.)

$$V_{\text{max}} = \pi\left(27\sqrt{27} - 9\sqrt{27}\right) = 18\sqrt{27}\pi \approx 294\text{ cm}^3$$

(d) Complete the table below to show the radius, height, and capacity of each of the cups involved in parts **(b)** and **(c)** above.
In each case, give the radius and height correct to two decimal places.

	cups in part **(b)**		cup in part **(c)**
radius (r)	8·77 cm	4·43 cm	7·35 cm
height (h)	2 cm	7·83 cm	5·20 cm
capacity (V)	$\frac{154\pi}{3} \approx 161\text{ cm}^3$	$\frac{154\pi}{3} \approx 161\text{ cm}^3$	294 cm^3

	cups in part **(b)**		cup in part **(c)**
radius (r)	8·77 cm	4·43 cm	7·35 cm
height (h)	2 cm	7·83 cm	5·20 cm
capacity (V)	$\frac{154\pi}{3} \approx 161\text{ cm}^3$	$\frac{154\pi}{3} \approx 161\text{ cm}^3$	294 cm^3

(e) In practice, which one of the three cups above is the most reasonable shape for a conical cup? Give a reason for your answer.

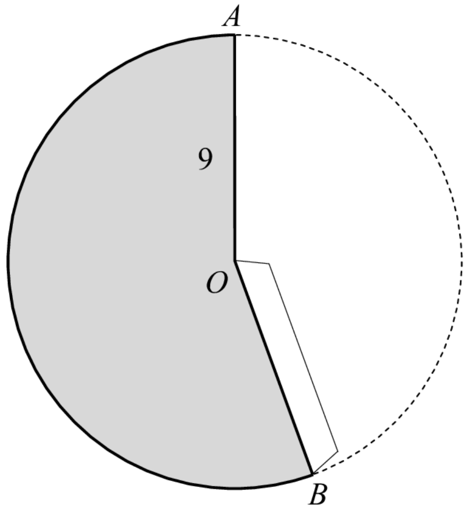
The middle one (radius 4·43 cm, height 7·83 cm).
The others are much too wide and shallow to hold.

The middle one (radius 4·43 cm, height 7·83 cm).
The others are much too wide and shallow to hold.

(f) For the cup you have chosen in part **(e)**, find the measure of the angle AOB that must be cut from the circular disc in order to make the cup.
Give your answer in degrees, correct to the nearest degree.

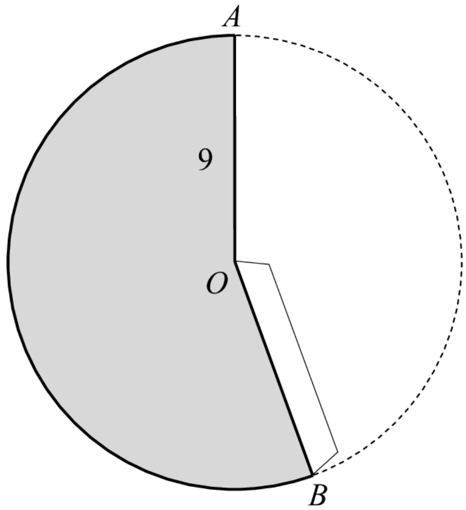
Circumference of rim $= 2\pi r \approx 8{\cdot}86\pi \approx 27{\cdot}86$ cm.

$$\theta = \frac{l}{r} = \frac{27{\cdot}86}{9} = 3{\cdot}096 \text{ rad} \approx 177°$$



Circumference of rim $= 2\pi r \approx 8{\cdot}86\pi \approx 27{\cdot}86$ cm.

$$\theta = \frac{l}{r} = \frac{27{\cdot}86}{9} = 3{\cdot}096 \text{ rad} \approx 177°$$



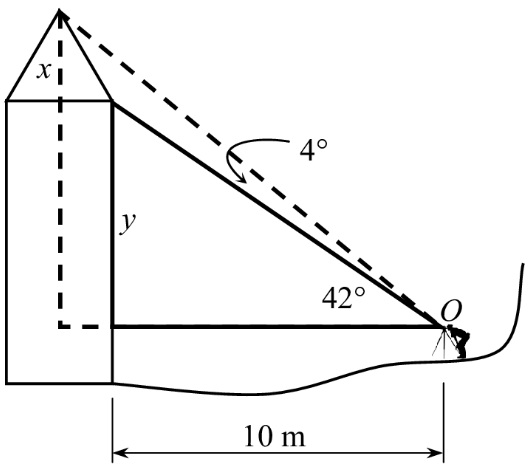
Question 12 (2011)

In the solid triangle:

$$\tan 42° = \frac{y}{10}$$
$$y = 10 \tan 42$$
$$y = 9{\cdot}004$$

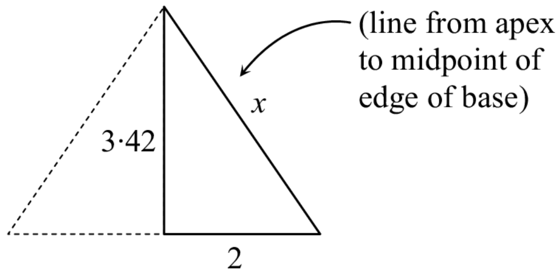
In the dashed triangle:

$$\tan 46° = \frac{x+y}{12}$$
$$x + y = 12 \tan 46$$
$$x = 12{\cdot}426 - 9{\cdot}004 = 3{\cdot}42 \text{ m}$$



$$x = \sqrt{3{\cdot}42^2 + 2^2}$$

$$x = 3{\cdot}964$$



$$\text{Area} = 4\left(\frac{1}{2}(4)(3{\cdot}964)\right) = 31{\cdot}71\text{ m}^2$$

Maximum possible area of roof given by:

Angle of elevation of bottom = 41°

Angle of elevation of top = 47°

$\therefore$ Height $= 12\tan 47 - 10\tan 41 = 4{\cdot}18$ m

$x = \sqrt{4{\cdot}18^2 + 2^2} = 4{\cdot}634$ m

Area = $37{\cdot}07\text{ m}^2$

Minimum possible area of roof given by:

Angle of elevation of bottom = 43°

Angle of elevation of top = 45°

$\therefore$ Height $= 12\tan 45 - 10\tan 43 = 2{\cdot}675$ m

$x = \sqrt{2{\cdot}675^2 + 2^2} = 3{\cdot}34$ m

Area = $26{\cdot}72\text{ m}^2$

$$26{\cdot}72\text{ m}^2 \le \text{area of roof} \le 37{\cdot}07\text{ m}^2.$$