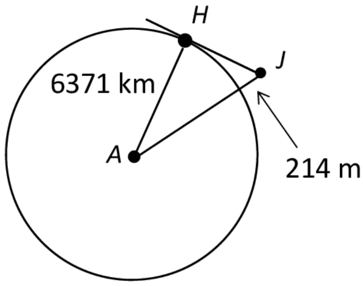
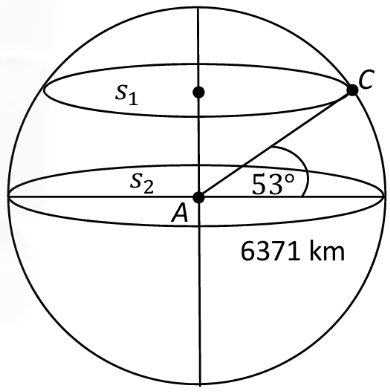
Question 1

Question 6 **(25 marks)**

(a) Take the earth as a sphere with radius 6371 km.
Jack is standing on the Cliffs of Moher at the point J which is 214 metres above sea level.
He is looking out to sea at a point H on the horizon.
Taking A as the centre of the earth, find $|JH|$, the distance from Jack to the horizon.
Give your answer correct to the nearest km.



(b) The Cliffs of Moher, at point C, are at latitude 53° north of the equator.
On the diagram, s_1 represents the circle that is at latitude 53°.
s_2 represents the equator (which is at latitude 0°).
A is the centre of the earth.
s_1 and s_2 are on parallel planes.
Find the length of the circle s_1.
Give your answer correct to the nearest km.



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Question 7 **(40 marks)**

Two solid cones, each of radius R cm and height R cm are welded together at their vertices and placed in the smallest possible hollow cylinder, as shown in **Figure 1** below.

Figure 1

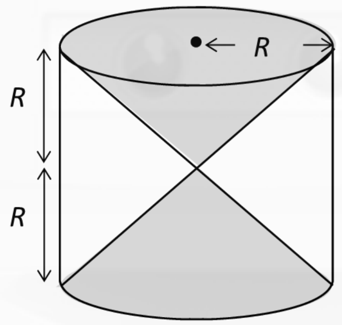
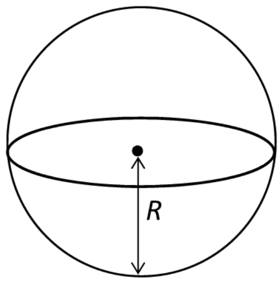


Figure 2



(a) Show that the capacity (volume) of the empty space in the cylinder is equal to the capacity of an empty sphere of radius R cm (**Figure 2**).

(b) In the remainder of this question, $R = 12$ cm. Water is poured into both the cylinder and the sphere to a depth of 6 cm as shown below (**Figure 3** and **Figure 4** respectively).

(i) Find $|AB|$, the radius of the circular surface of the water in the sphere (**Figure 4**). Give your answer in the form $a\sqrt{b}$ cm, where $a, b \in \mathbb{N}$.

Figure 3

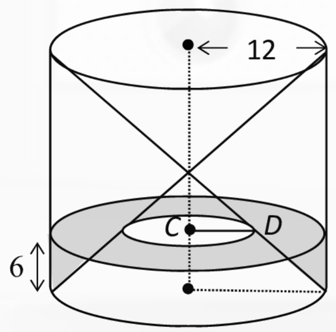
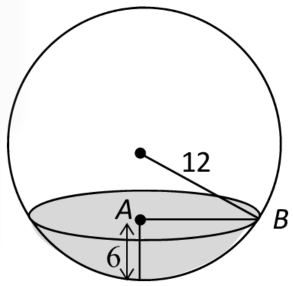


Figure 4



(ii) Find $|CD|$, the radius of the cone at water level, as shown in **Figure 3**.

(iii) Verify that the area of the surface of the water in the sphere is equal to the area of the surface of the water in the cylinder.

(c) The mathematician Cavalieri discovered that, at the same depth, the volume of water in the available space in the cylinder is equal to the volume of water in the sphere.
Use this discovery to find the volume of water in the sphere when the depth is 6 cm.
Give your answer in terms of π.

Question 9 **(50 marks)**

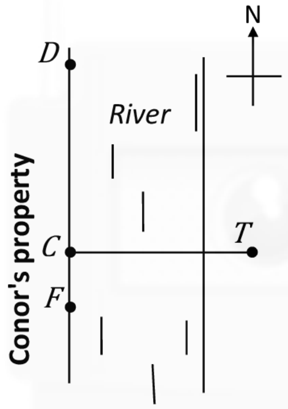


Figure 1

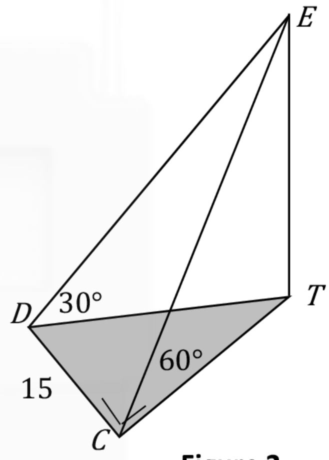


Figure 2

Conor's property is bounded by the straight bank of a river, as shown in **Figure 1** above.
T is the base of a vertical tree that is growing near the opposite bank of the river.
$|TE|$ is the height of the tree, as shown in **Figure 2** above.
From the point C, which is due west of the tree, the angle of elevation of E, the top of the tree, is $60°$.
From the point D, which is 15 m due north of C, the angle of elevation of E is $30°$ (see **Figure 2**).
The land on both sides of the river is flat and at the same level.

(a) Use triangle ECT, to express $|TE|$ in the form $\sqrt{a}|CT|$ metres, where $a \in \mathbb{N}$.

(b) Show that $|TE|$ may also be expressed as $\sqrt{\frac{225+|CT|^2}{3}}$ metres.

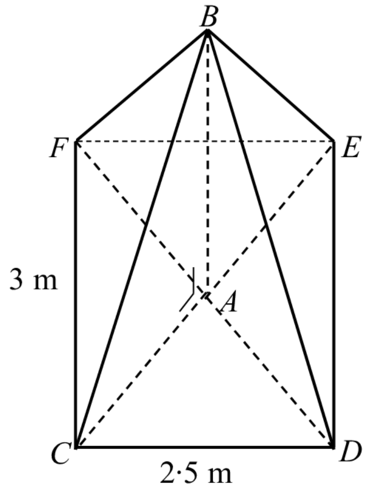
(c) Hence find $|CT|$, the distance from the base of the tree to the bank of the river at Conor's side. Give your answer correct to 1 decimal places.

(d) Find $|TE|$, the height of the tree. Give your answer correct to 1 decimal place.

(e) The tree falls across the river and hits the bank at Conor's side at the point F. Find the maximum size of the angle FTC. Give your answer in degrees, correct to 1 decimal place.

(f) If the tree was equally likely to fall in any direction, find the probability that it would hit the bank at Conor's side, when it falls.
Give your answer as a percentage, correct to 1 decimal place.

A glass Roof Lantern in the shape of a pyramid has a rectangular base *CDEF* and its apex is at *B* as shown. The vertical height of the pyramid is $|AB|$, where *A* is the point of intersection of the diagonals of the base as shown in the diagram.
Also $|CD| = 2{\cdot}5$ m and $|CF| = 3$ m.



(a) **(i)** Show that $|AC| = 1{\cdot}95$ m, correct to two decimal places.

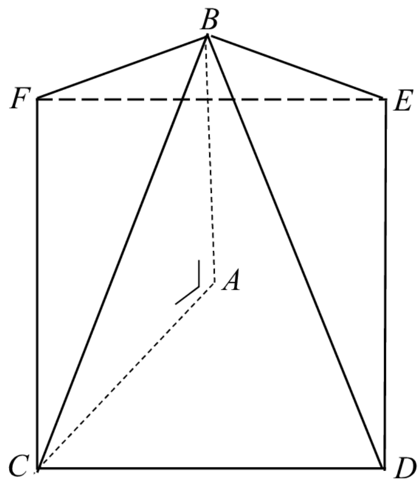
(ii) The angle of elevation of *B* from *C* is 50° (i.e. $|\angle BCA| = 50°$).
Show that $|AB| = 2{\cdot}3$ m, correct to one decimal place.

(iii) Find $|BC|$, correct to the nearest metre.

(iv) Find $|\angle BCD|$, correct to the nearest degree.

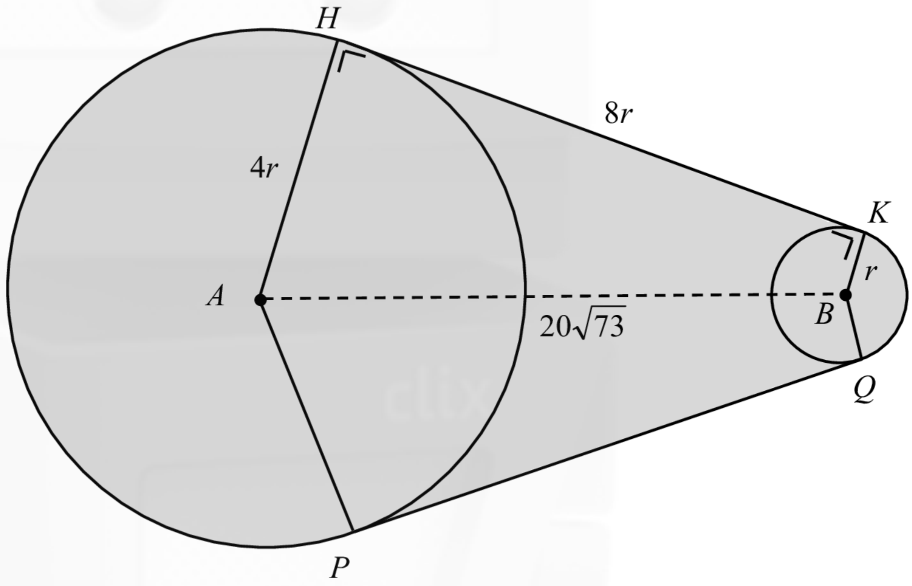
(v) Find the area of glass required to glaze all four triangular sides of the pyramid. Give your answer correct to the nearest m^2.

(b) Another Roof Lantern, in the shape of a pyramid, has a square base *CDEF*. The vertical height $|AB| = 3$ m, where A is the point of intersection of the diagonals of the base as shown.
The angle of elevation of B from C is 60°
(i.e. $|\angle BCA| = 60°$).
Find the length of the side of the square base of the lantern.
Give your answer in the form $\sqrt{a}$ m, where $a \in \mathbb{N}$.



Question 7 **(40 marks)**

A flat machine part consists of two circular ends attached to a plate, as shown (diagram not to scale).
The sides of the plate, HK and PQ, are tangential to each circle.
The larger circle has centre A and radius $4r$ cm.
The smaller circle has centre B and radius r cm.
The length of $[HK]$ is $8r$ cm and $|AB| = 20\sqrt{73}$ cm.



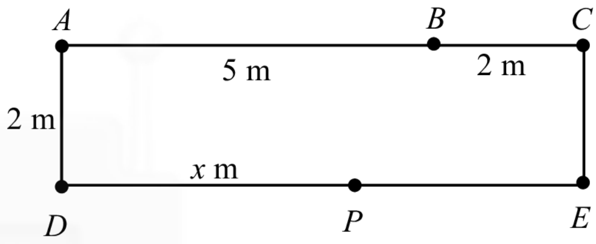
(a) Find r, the radius of the smaller circle. (Hint: Draw $BT \parallel KH$, $T \in AH$.)

(b) Find the area of the quadrilateral $ABKH$.

(c) **(i)** Find $|\angle HAP|$, in degrees, correct to one decimal place.

(ii) Find the area of the machine part, correct to the nearest cm^2.

(b) $ADEC$ is a rectangle with $|AC| = 7$ m and $|AD| = 2$ m, as shown.
B is a point on $[AC]$ such that $|AB| = 5$ m.
P is a point on $[DE]$ such that $|DP| = x$ m.



(i) Let $f(x) = |PA|^2 + |PB|^2 + |PC|^2$.

Show that $f(x) = 3x^2 - 24x + 86$, for $0 \le x \le 7$, $x \in \mathbb{R}$.

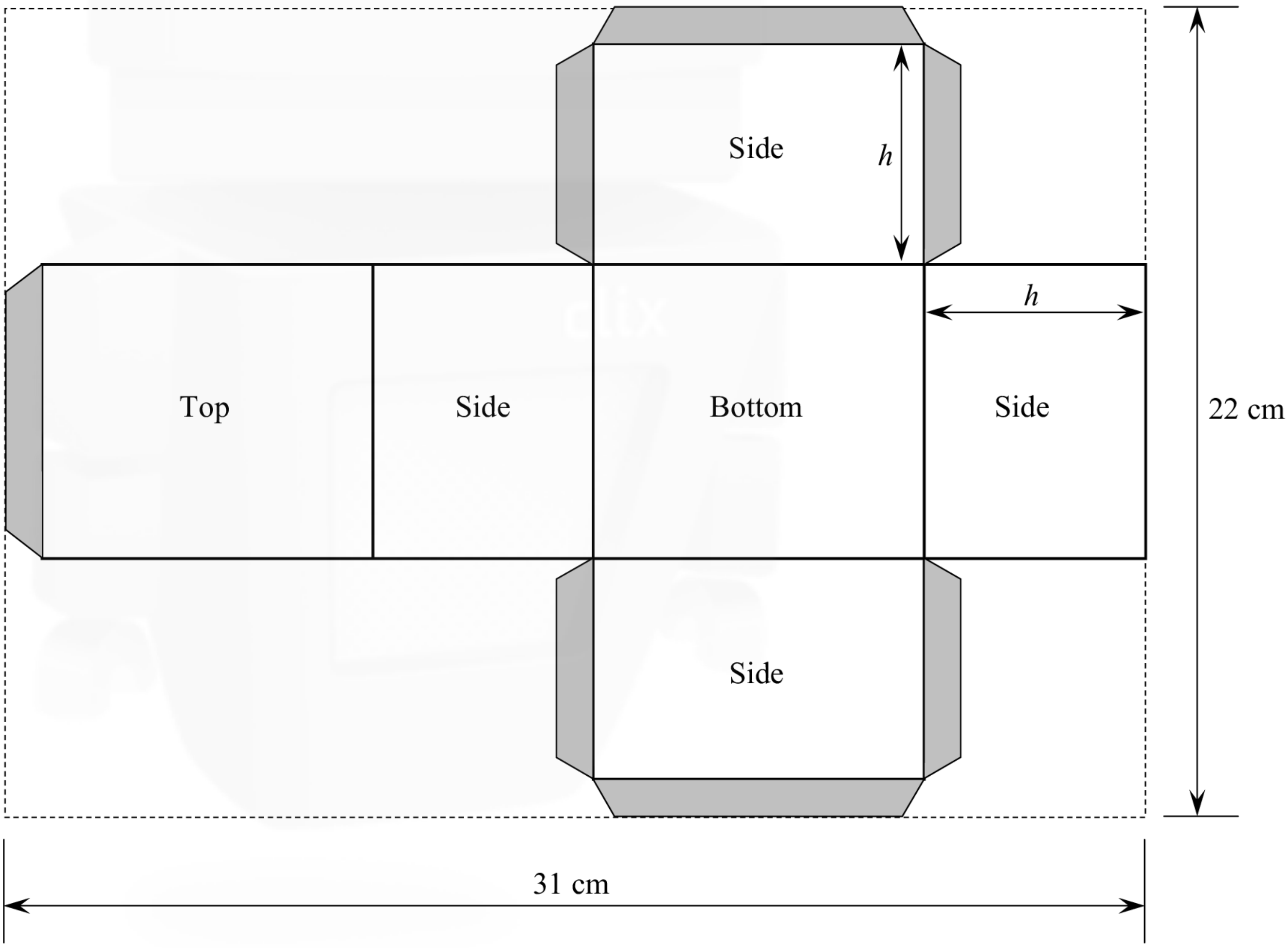
(ii) The function $f(x)$ has a minimum value at $x = k$.

Find the value of k and the minimum value of $f(x)$.

Question 7 **(50 marks)**

A company has to design a rectangular box for a new range of jellybeans. The box is to be assembled from a single piece of cardboard, cut from a rectangular sheet measuring 31 cm by 22 cm. The box is to have a capacity (volume) of 500 cm^3.

The net for the box is shown below. The company is going to use the full length and width of the rectangular piece of cardboard. The shaded areas are flaps of width 1 cm which are needed for assembly. The height of the box is h cm, as shown on the diagram.



(a) Write the dimensions of the box, in centimetres, in terms of h.

height = h cm

length = ______________ cm

width = ______________ cm

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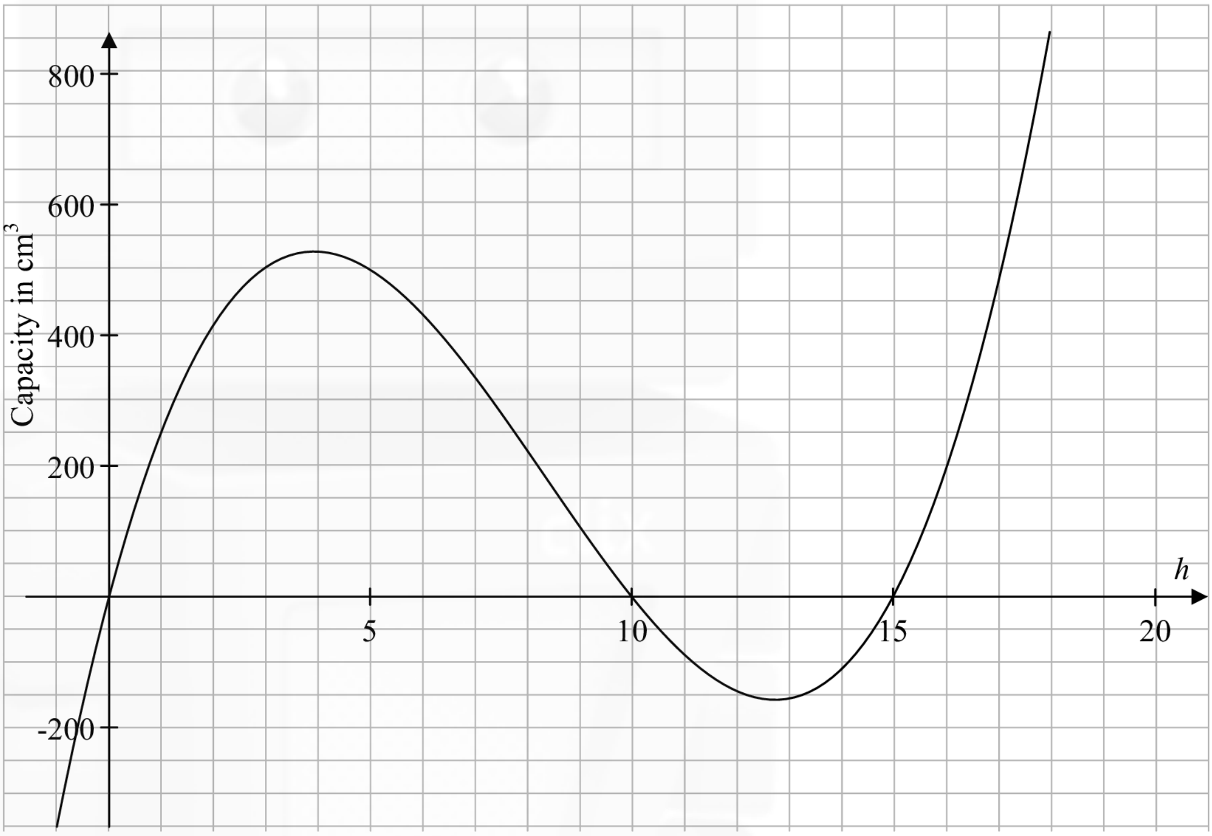
(b) Write an expression for the capacity of the box in cubic centimetres, in terms of h.

(c) Show that the value of h that gives a box with a square bottom will give the correct capacity.

(d) Find, correct to one decimal place, the other value of h that gives a box of the correct capacity.

(e) The client is planning a special "10% extra free" promotion and needs to increase the capacity of the box by 10%. The company is checking whether they can make this new box from a piece of cardboard the same size as the original one (31 cm × 22 cm). They draw the graph below to represent the box's capacity as a function of h. Use the graph to explain why it is *not* possible to make the larger box from such a piece of cardboard.



Explanation:

Question 9 **(50 marks)**

(a) Let $f(x) = -0{\cdot}5x^2 + 5x - 0{\cdot}98$, where $x \in \mathbb{R}$.

(i) Find the value of $f(0{\cdot}2)$.

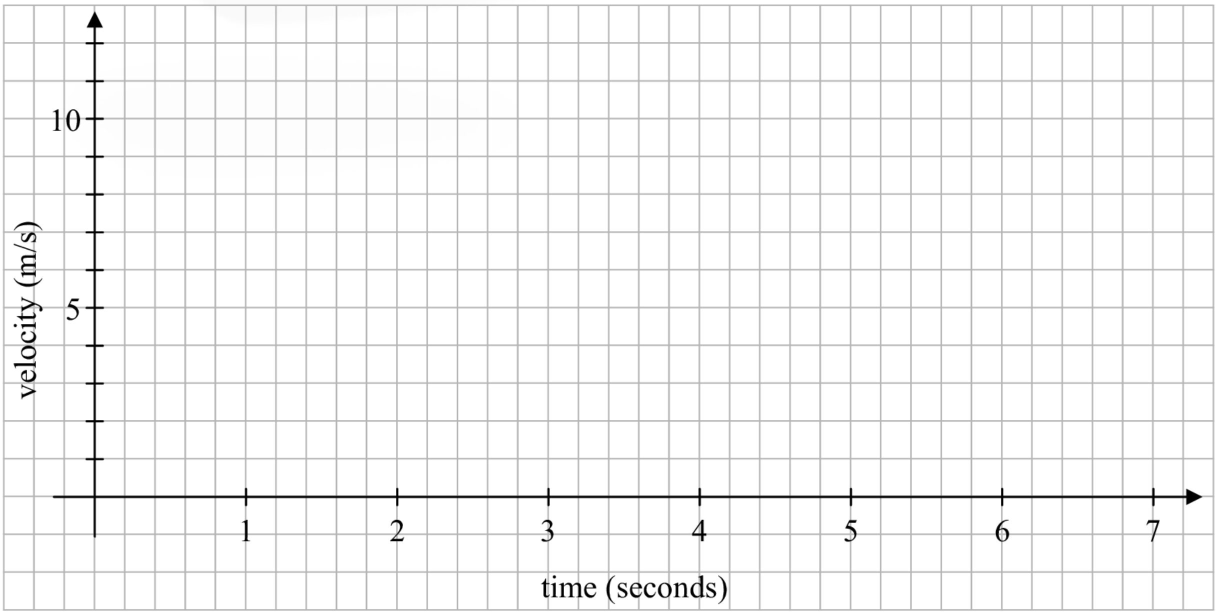
(ii) Show that f has a local maximum point at $(5,\ 11{\cdot}52)$.

(b) A sprinter's velocity over the course of a particular 100 metre race is approximated by the following model, where v is the velocity in metres per second, and t is the time in seconds from the starting signal:

$$v(t) = \begin{cases} 0, & \text{for } 0 \le t < 0{\cdot}2 \\ -0{\cdot}5t^2 + 5t - 0{\cdot}98, & \text{for } 0{\cdot}2 \le t < 5 \\ 11{\cdot}52, & \text{for } t \ge 5 \end{cases}$$

Note that the function in part **(a)** is relevant to $v(t)$ above.

(i) Sketch the graph of v as a function of t for the first 7 seconds of the race.



(ii) Find the distance travelled by the sprinter in the first 5 seconds of the race.

(iii) Find the sprinter’s finishing time for the race. Give your answer correct to two decimal places.

(c) A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

(i) Prove that the radius of the snowball is decreasing at a constant rate.

(ii) If the snowball loses half of its volume in an hour, how long more will it take for it to melt completely?

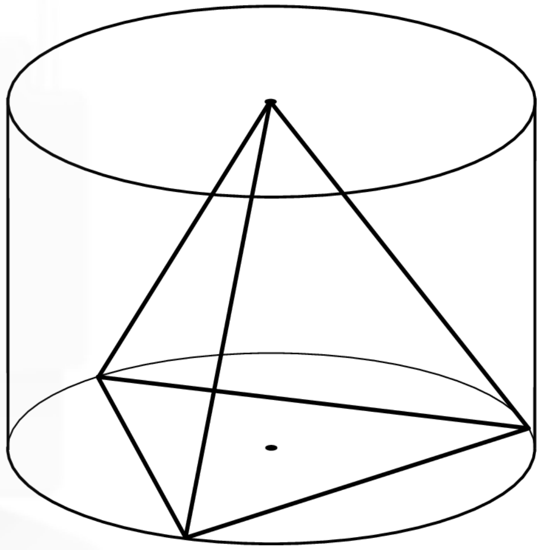
Give your answer correct to the nearest minute.

Question 9 **(25 marks)**

A regular tetrahedron has four faces, each of which is an equilateral triangle.

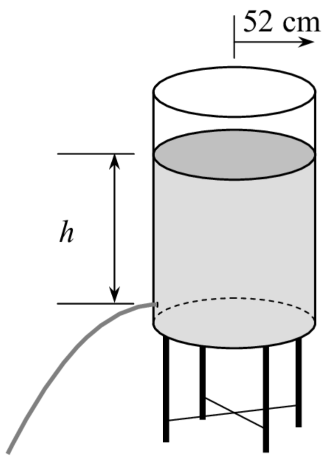
A wooden puzzle consists of several pieces that can be assembled to make a regular tetrahedron. The manufacturer wants to package the assembled tetrahedron in a clear cylindrical container, with one face flat against the bottom.

If the length of one edge of the tetrahedron is $2a$, show that the volume of the smallest possible cylindrical container is $\left(\frac{8\sqrt{6}}{9}\right)\pi a^3$.



Question 7 **(50 marks)**

An open cylindrical tank of water has a hole near the bottom. The radius of the tank is 52 cm. The hole is a circle of radius 1 cm. The water level gradually drops as water escapes through the hole.



Over a certain 20-minute period, the height of the surface of the water is given by the formula

$$h = \left(10 - \frac{t}{200}\right)^2$$

where h is the height of the surface of the water, in cm, as measured from the centre of the hole,
and t is the time in seconds from a particular instant $t = 0$.

(a) What is the height of the surface at time $t = 0$?

(b) After how many seconds will the height of the surface be 64 cm?

(c) Find the rate at which the **volume** of water in the tank is decreasing at the instant when the height is 64 cm.
Give your answer correct to the nearest cm^3 per second.

(d) The rate at which the volume of water in the tank is decreasing is equal to the speed of the water coming out of the hole, multiplied by the area of the hole. Find the speed at which the water is coming out of the hole at the instant when the height is 64 cm.

(e) Show that, as t varies, the speed of the water coming out of the hole is a constant multiple of $\sqrt{h}$.

(f) The speed, in centimetres per second, of water coming out of a hole like this is known to be given by the formula

$$v = c\sqrt{1962h}$$

where c is a constant that depends on certain features of the hole.
Find, correct to one decimal place, the value of c for this hole.

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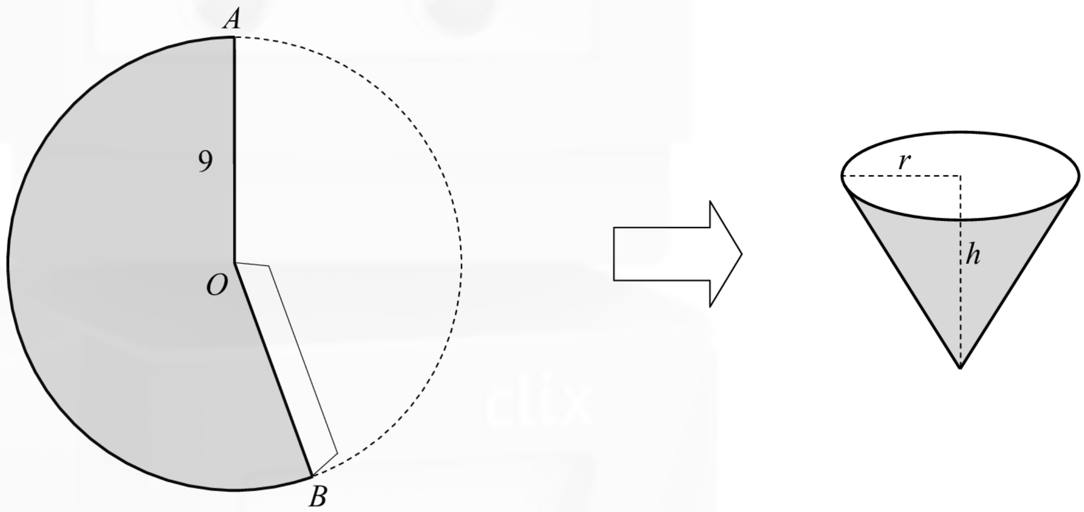
Question 8 **(50 marks)**

A company uses waterproof paper to make disposable conical drinking cups. To make each cup, a sector AOB is cut from a circular piece of paper of radius 9 cm. The edges AO and OB are then joined to form the cup, as shown.

The radius of the rim of the cup is r, and the height of the cup is h.



(a) By expressing r^2 in terms of h, show that the capacity of the cup, in cm^3, is given by the formula

$$V = \frac{\pi}{3} h\left(81 - h^2\right).$$

(b) There are two positive values of h for which the capacity of the cup is $\frac{154\pi}{3}$.

One of these values is an integer.
Find the two values.
Give the non-integer value correct to two decimal places.

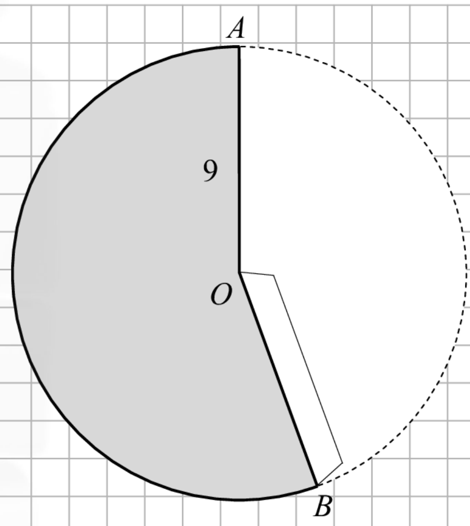
(c) Find the maximum possible volume of the cup, correct to the nearest cm^3.

(d) Complete the table below to show the radius, height, and capacity of each of the cups involved in parts **(b)** and **(c)** above.
In each case, give the radius and height correct to two decimal places.

	cups in part **(b)**		cup in part **(c)**
radius (r)			
height (h)			
capacity (V)	$\frac{154\pi}{3} \approx 161 \text{ cm}^3$	$\frac{154\pi}{3} \approx 161 \text{ cm}^3$	

(e) In practice, which one of the three cups above is the most reasonable shape for a conical cup? Give a reason for your answer.

(f) For the cup you have chosen in part **(e)**, find the measure of the angle AOB that must be cut from the circular disc in order to make the cup.
Give your answer in degrees, correct to the nearest degree.

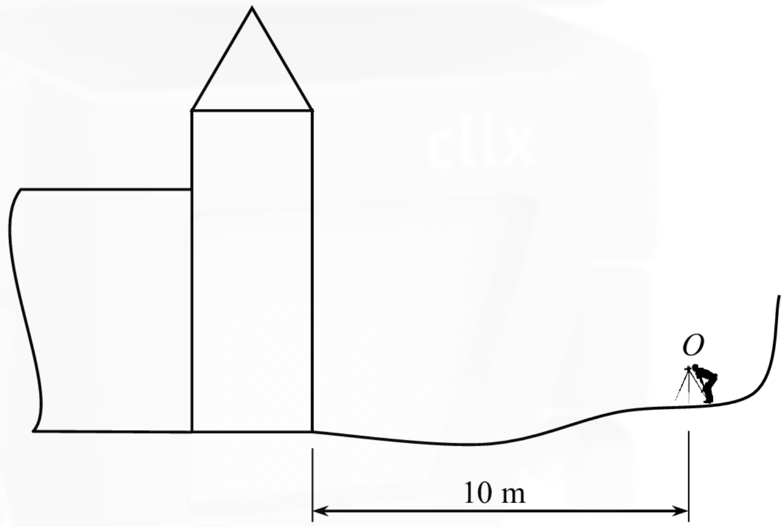


Question 8 **(75 marks)**

(a) A tower that is part of a hotel has a square base of side 4 metres and a roof in the form of a pyramid. The owners plan to cover the roof with copper. To find the amount of copper needed, they need to know the total area of the roof.

A surveyor stands 10 metres from the tower, measured horizontally, and makes observations of angles of elevation from the point O as follows:

The angle of elevation of the top of the roof is 46°.
The angle of elevation of the closest point at the bottom of the roof is 42°.
The angle of depression of the closest point at the bottom of the tower is 9°.



(i) Find the vertical height of the roof.

(ii) Find the total area of the roof.

(iii) If all of the angles observed are subject to a possible error of $\pm 1°$, find the range of possible areas for the roof.