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DEB exams

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Pre-Leaving Certificate Examination, 2014

Applied Mathematics

Marking Scheme

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Pre-Leaving Certificate Examination, 2014

Applied Mathematics

Ordinary Level
Marking Scheme (300 marks)

General Instructions

1. Penalties of three types are applied to students' work as follows:

Slips	- numerical slips	S(–1)
Blunders	- mathematical errors	B(–3)
Misreading	- if not serious	M(–1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

4. Number the grid on each script 1 to 9 in numerical order, not the order of answering.

5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question.
In many cases, there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

1. Four points, A, B, C and D lie on a straight level road. A car, travelling with uniform acceleration, passes A, and then 2 seconds later it passes B with a speed of 16 m s^{-1}. Three seconds after the car passes B, it passes C which is 66 m from B. On passing C, the car immediately decelerates uniformly to rest at D. The total distance from A to D is 160 m.

Find **(i)** the uniform acceleration of the car **(10)**

For $[BC]$, $u = 16,\ s = 66,\ t = 3$

$$s = ut + \frac{1}{2}at^2$$

$$66 = (16)(3) + \frac{1}{2}a(3)^2 \qquad \text{... (5m)}$$

$$66 = 48 + \frac{9}{2}a$$

$$\frac{9}{2}a = 18$$

$$9a = 36$$

$$a = 4\ \text{m s}^{-2} \qquad \text{... (5m)}$$

(ii) the speed of the car at A **(10)**

For $[AB]$, $v = 16,\ a = 4,\ t = 2$

$$v = u + at$$

$$16 = u + (4)(2) \qquad \text{... (5m)}$$

$$16 = u + 8$$

$$u = 8\ \text{m s}^{-1} \qquad \text{... (5m)}$$

(iii) the speed of the car at C **(5)**

For $[BC]$, $u = 16,\ a = 4,\ t = 3$

$$v = u + at$$

$$v = (16) + (4)(3)$$

$$v = 28\ \text{m s}^{-1} \qquad \text{... (5m)}$$

(iv) $|AB|$, the distance from A to B **(10)**

For $[AB]$, $u=8,\ v=16,\ a=4$

$$v^2 = u^2 + 2as$$
$$(16)^2 = (8)^2 + 2(4)s \qquad \text{... (5m)}$$
$$256 = 64 + 8s$$
$$8s = 192$$
$$s = 24 \text{ m} \qquad \text{... (5m)}$$

(v) the time from C to D, i.e. the time of the deceleration. **(15)**

Distance from C to D

$$= 160 - 24 - 66$$
$$= 70 \qquad \text{... (5m)}$$

For $[CD]$, $u=28,\ v=0,\ s=70$

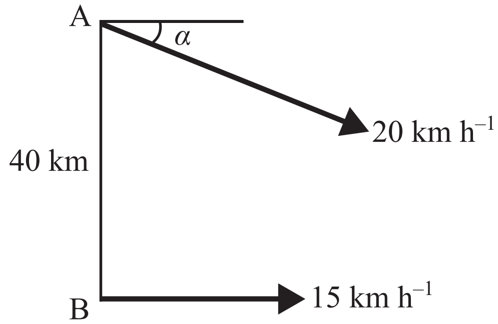
$$v^2 = u^2 + 2as$$
$$(0)^2 = (28)^2 + 2a(70)$$
$$0 = 784 + 140a$$
$$140a = -784$$
$$a = -\frac{28}{5} = -5\cdot 6 \qquad \text{... (5m)}$$

Then

$$v = u + at$$
$$0 = (28) + (-5\cdot 6)t$$
$$5\cdot 6t = 28$$
$$t = 5 \text{ s} \qquad \text{... (5m)}$$

2. At noon, ship A is 40 km due north of ship B. A is moving at a constant speed of 20 km h^{-1} in the direction east α south, where $\tan\alpha = \frac{5}{12}$.

B is moving due east at a constant speed of 15 km h^{-1}.



Find **(i)** the velocity of A in terms of $\vec{i}$ and $\vec{j}$ **(10)**

$\vec{v}_A = 20\cos\alpha\,\vec{i} - 20\sin\alpha\,\vec{j}$ ... (5m)

$\vec{v}_A = 18\cdot 46\vec{i} - 7\cdot 70\,\vec{j}$ ... (5m)

(ii) the velocity of B in terms of $\vec{i}$ and $\vec{j}$ **(5)**

$\vec{v}_B = 15\vec{i}$ ... (5m)

(iii) the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$ **(10)**

$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$ ... (5 m)

$\vec{v}_{AB} = \left(18\cdot 46\vec{i} - 7\cdot 70\,\vec{j}\right) - \left(15\vec{i}\right)$

$\vec{v}_{AB} = 3\cdot 46\vec{i} - 7\cdot 70\,\vec{j}$ ... (5m)

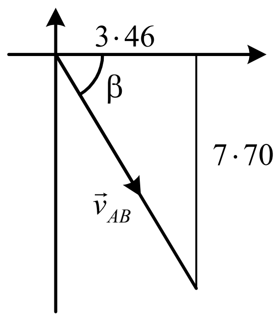
(iv) the shortest distance between A and B in the subsequent motion. **(15)**

$\left|\vec{v}_{AB}\right| = \sqrt{3\cdot 46^2 + 7\cdot 70^2}$

$\left|\vec{v}_{AB}\right| = 8\cdot 44 \text{ km h}^{-1}$

and

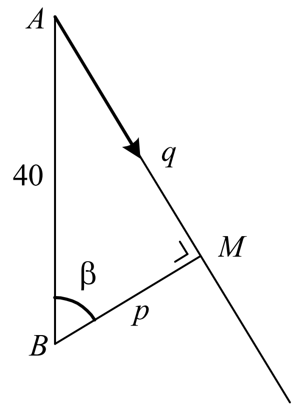
$\beta = \tan^{-1}\frac{7\cdot 70}{3\cdot 46} = 65\cdot 80°$ ... (5m)



Further answers overleaf

Let M be the point of closest approach on the path of A relative to B.
Shortest distance:

$$= p$$
$$= |BM|$$
$$= 40\cos\beta \quad \text{... (5m)}$$
$$= 40\cos 65\cdot 80°$$
$$= 16\cdot 40 \text{ km} \quad \text{... (5m)}$$



(v) the time, to the nearest minute, at which the ships are nearest to each other. **(10)**

Distance travelled by A relative to B

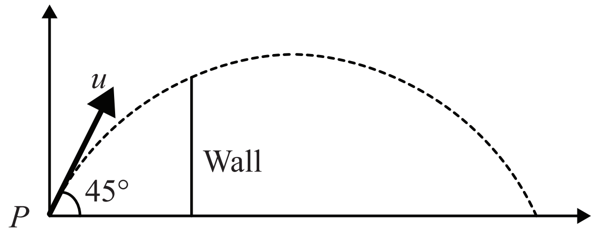
$$= q$$
$$= |AM|$$
$$= \sqrt{40^2 - 16\cdot 40^2}$$
$$= 36\cdot 48 \text{ km} \quad \text{... (5m)}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$
$$= \frac{36\cdot 48}{8\cdot 44}$$
$$= 4\cdot 32 \text{ hours}$$
$$= 4 \text{ hours } 19 \text{ minutes} \quad \text{... (5m)}$$

3. A particle is projected from a point P on horizontal ground with an initial speed of u m s^{-1} at an angle of 45° to the horizontal.

The particle strikes the ground $5\sqrt{2}$ seconds later.

Two seconds after being projected, the particle just clears the top of a vertical wall.



Find **(i)** the value of u **(15)**

For the time of flight, $t = 5\sqrt{2}$ and

$s_y = 0$ ... (5m)

$u\sin 45°\left(5\sqrt{2}\right) - \frac{1}{2}(10)\left(5\sqrt{2}\right)^2 = 0$ ... (5m)

$u\left(\frac{1}{\sqrt{2}}\right)\left(5\sqrt{2}\right) - 250 = 0$

$5u = 250$

$u = 50$ ... (5m)

(ii) the range of the particle on the horizontal ground **(10)**

Range $= (50)\cos 45°\left(5\sqrt{2}\right)$ ... (5m)

$= 50\left(\frac{1}{\sqrt{2}}\right)\left(5\sqrt{2}\right)$

$= 250$ m ... (5m)

(iii) the greatest height reached by the particle **(15)**

Time to greatest height $= \frac{1}{2}\left(5\sqrt{2}\right) = \frac{5}{\sqrt{2}}$ ... (5m)

Greatest height $= (50)\sin 45°\left(\frac{5}{\sqrt{2}}\right) - \frac{1}{2}(10)\left(\frac{5}{\sqrt{2}}\right)^2$ ... (5m)

$= \frac{250}{2} - \frac{125}{2}$

$= 62 \cdot 5$ m ... (5m)

Further answers overleaf

(iv) the height of the vertical wall, correct to two decimal places. **(10)**

When $t = 2$,

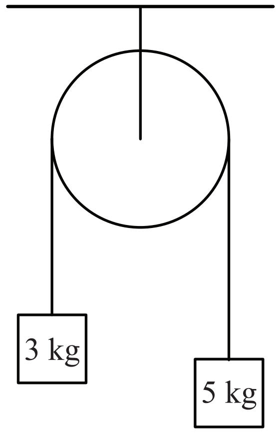
height of wall = $(50)\sin 45^\circ(2) - \frac{1}{2}(10)(2)^2$ ... (5m)

$= 50\sqrt{2} - 20$

$= 50 \cdot 71$ m ... (5m)

4. (a) Two particles of masses 3 kg and 5kg are connected by a taut, light, inextensible string which passes over a smooth, light pulley.

The system is released from rest.



Find **(i)** the common acceleration of the particles **(15)**

$T - 3g = 3a$ ... (5m)

$5g - T = 5a$ ... (5m)

$2g = 8a$

$a = \dfrac{g}{4}$

$a = 2\cdot 5 \text{ m s}^{-2}$ ... (5m)

(ii) the tension in the string. **(5)**

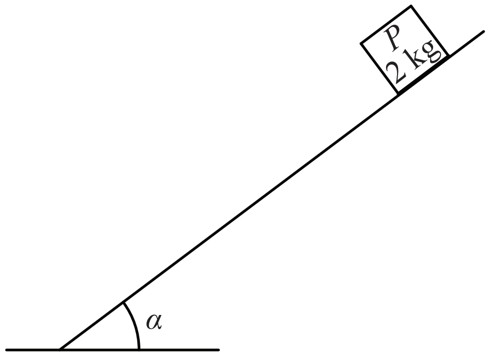
$T = 3a + 3g$

$T = 3(2\cdot 5) + 3(10)$

$T = 37\cdot 5 \text{ N}$ ... (5m)

Further answers overleaf

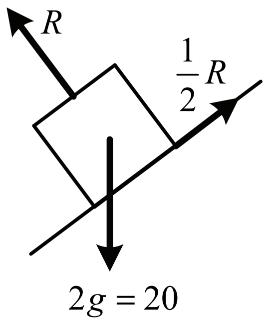
(b) A particle, P, of mass 2 kg sits on a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$.

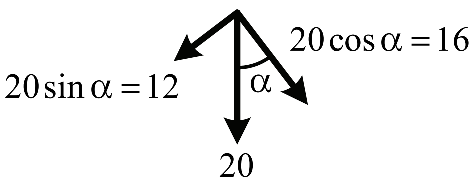


The coefficient of friction between P and the plane is $\frac{1}{2}$.

P is released from rest, and travels 4 m to the bottom of the plane.

Find **(i)** the acceleration of P down the plane. **(25)**





... (10 m)

$R = 16$ ... (5m)

$12 - \frac{1}{2}R = 2a$ ... (5m)

$12 - 8 = 2a$

$4 = 2a$

$a = 2 \text{ m s}^{-2}$ ... (5m)

(ii) the time taken by P to reach the bottom of the plane. **(5)**

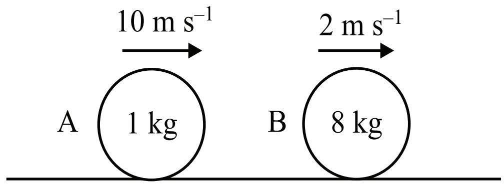
$s = ut + \frac{1}{2}at^2$

$4 = (0)t + \frac{1}{2}(2)t^2$

$4 = t^2$

$t = 2 \text{ s}$ ... (5m)

5. A smooth sphere A, of mass 1 kg, collides directly with another smooth sphere B, of mass 8 kg, on a smooth horizontal table.



A and B are moving in the same direction with speeds of 10 m s^{-1} and 2 m s^{-1}, respectively.

The coefficient of restitution for the collision is $\frac{1}{8}$.

Find **(i)** the speed of A and the speed of B after the collision **(30)**

PCM: $(1)(10)+(8)(2)=(1)v_1+(8)v_2$ ... (10m)

$26 = v_1 + 8v_2$

$v_1 + 8v_2 = 26$

NEL: $v_1 - v_2 = -\frac{1}{8}(10-2)$ ... (10m)

$v_1 - v_2 = -1$

$-v_1 + v_2 = 1$

Thus $9v_2 = 27$

$v_2 = 3\ \text{m s}^{-1}$

and $v_1 = 2\ \text{m s}^{-1}$ ... (10m)

(ii) the loss in kinetic energy of A due to the collision **(15)**

KE of A before collision $= \frac{1}{2}(1)(10)^2 = 50$ ... (5m)

KE of A after collision $= \frac{1}{2}(1)(2)^2 = 2$ ... (5m)

Loss in KE of A $= 50 - 2 = 48$ J ... (5m)

(iii) the magnitude of the impulse imparted to B due to the collision. **(5)**

Impulse $= \left|(8)(3)-(8)(2)\right| = 8$ Ns ... (5m)

6. **(a)** Particles of weight 5 N, 4 N, 2 N and 3 N are placed at the points $(1, q)$, (q, p), $(p, 6)$ and $(q + 1, q)$, respectively.

The co-ordinates of the centre of gravity of the system are (p, q).

Find **(i)** the value of q
(ii) the value of p. **(25)**

$$p = \frac{5(1)+4(q)+2(p)+3(q+1)}{5+4+2+3}$$ ... (10m)

$$14p = 5+4q+2p+3q+3$$
$$12p-7q=8$$

and

$$q = \frac{5(q)+4(p)+2(6)+3(q)}{5+4+2+3}$$ ... (10m)

$$14q = 5q+4p+12+3q$$
$$-4p+6q=12$$

then

$$12p-7q=8$$
$$-12p+18q=36$$
$$11q=44$$
$$q=4$$

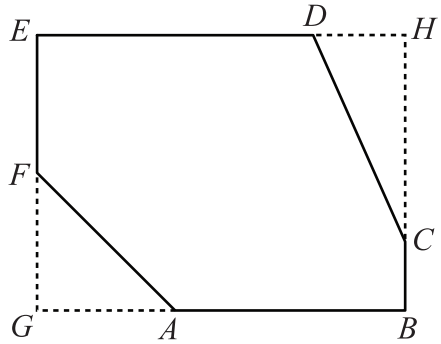
and

$$12p-28=8$$
$$12p=36$$
$$p=3$$ ... (5m)

(b) A uniform lamina *ABCDEF* consists of a rectangle *GBHE* from which the two triangles *GAF* and *CHD* have been removed.

The co-ordinates of the points are *A*(6, 0), *B*(15, 0), *C*(15, 3), *D*(12, 12), *E*(0, 12), *F*(0, 6), *G*(0, 0) and *H*(15, 12).

Find the co-ordinates of the centre of gravity of the lamina *ABCDEF*. **(25)**



	Area	Centre of gravity	
GBHE	$15\times12=180$	$(7\cdot5,6)$	... (5m)
GAF	$\frac{1}{2}(6)(6)=18$	$(2,2)$	... (5m)
CHD	$\frac{1}{2}(3)(9)=13\cdot5$	$(14,9)$	... (5m)
ABCDEF	$180-18-13\cdot5$ $=148\cdot5$	(x,y)	

then

$$7\cdot5=\frac{18(2)+13\cdot5(14)+148\cdot5(x)}{180}$$
$$1350=36+189+148\cdot5x$$
$$148\cdot5x=1125$$
$$x=7\cdot58$$ ... (5m)

and

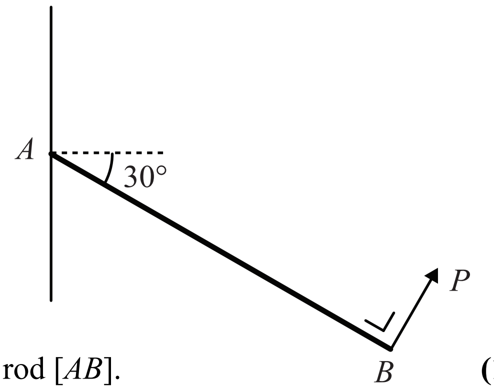
$$6=\frac{18(2)+13\cdot5(9)+148\cdot5(y)}{180}$$
$$1080=36+121\cdot5+148\cdot5y$$
$$148\cdot5y=922\cdot5$$
$$y=6\cdot21$$ ... (5m)

Thus the co-ordinates of the centre of gravity are $(7\cdot58,6\cdot21)$.

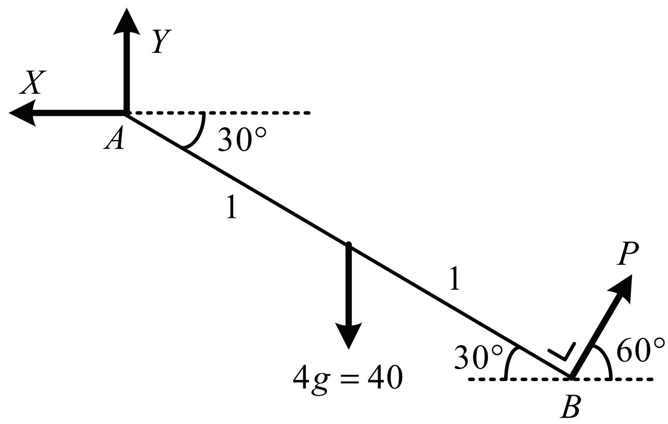
7. A uniform rod, [*AB*], of weight 40 N and length 2 m, is smoothly hinged at end *A* to a vertical wall.

The rod is held inclined at 30° below the horizontal by a force *P* applied at *B*.

The force *P* is at right angles to the rod [*AB*].



(i) Show on a diagram all the forces acting on the rod [*AB*]. **(15)**



... (15m)

(ii) Find the magnitude of *P*. **(15)**

Moments about A:

Anticlockwise Moments = Clockwise Moments

$P(2) = 40(1\cos 30°)$ ... (5m, 5m)

$P = 20\left(\frac{\sqrt{3}}{2}\right)$

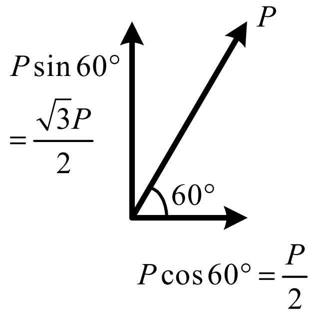
$P = 10\sqrt{3}$ N ... (5m)

(iii) Find the magnitude of the reaction at the hinge, A. **(20)**

The rod is in equilibrium.

$\uparrow = \downarrow$: $\quad Y + \dfrac{\sqrt{3}P}{2} = 40$ ...(5m)

$\leftarrow = \rightarrow$: $\quad X = \dfrac{P}{2}$ ... (5m)



thus

$$Y + \frac{\sqrt{3}}{2}\left(10\sqrt{3}\right) = 40$$
$$Y + 15 = 40$$
$$Y = 25$$

and

$$X = \frac{1}{2}\left(10\sqrt{3}\right) = 5\sqrt{3}$$ ... (5m)

The magnitude of the reaction force at hinge A is

$$\sqrt{X^2 + Y^2} = \sqrt{(5\sqrt{3})^2 + 25^2}$$
$$= \sqrt{75 + 625}$$
$$= \sqrt{700}$$
$$= 10\sqrt{7} = 26 \cdot 46 \text{ N}$$ ... (5m)

8. **(a)** A particle describes a horizontal circle of radius *r* metres with uniform angular velocity ω radians per second.
Its speed is 12 m s^{-1} and its acceleration is 24 m s^{-2}.

Find **(i)** the value of ω **(15)**

$$\text{Speed } = 12$$
$$r\omega = 12 \qquad \text{...(5m)}$$

and

$$\text{Acceleration } = 24$$
$$r\omega^2 = 24 \qquad \text{...(5m)}$$

then

$$(r\omega)\omega = 24$$
$$12\omega = 24$$
$$\omega = 2 \text{ rad s}^{-1} \qquad \text{...(5m)}$$

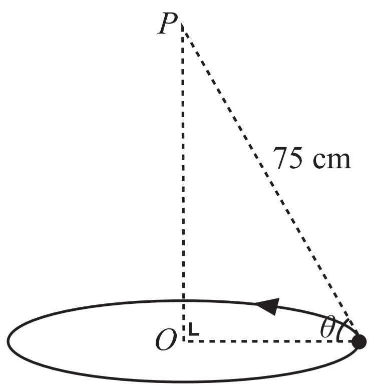
(ii) the value of *r*. **(5)**

Then

$$r(2) = 12$$
$$r = 6 \text{ m} \qquad \text{...(5m)}$$

(b) A particle of mass 3 kg is attached by a light, inextensible string of length 75 cm to a fixed point, *P*.

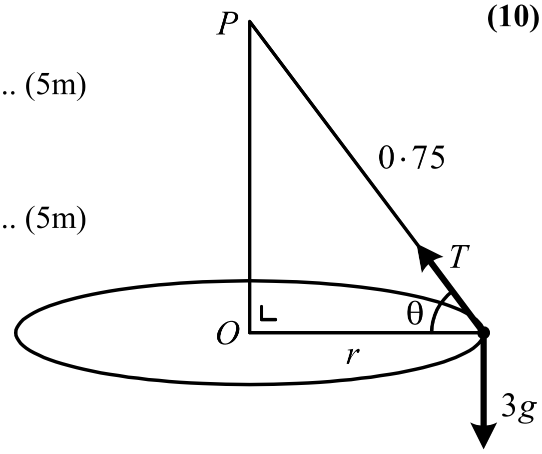
The particle moves in a horizontal circle, centre *O*, with constant speed such that the string makes an angle θ with the horizontal, where $\tan\alpha = \dfrac{4}{3}$.



Find **(i)** the value of *r*, the radius of the circular motion **(10)**

$$\tan\theta = \frac{4}{3}$$
$$\cos\theta = \frac{3}{5} \text{ and } \sin\theta = \frac{4}{5}$$
$$\frac{r}{0\cdot 75} = \cos\theta \qquad \text{... (5m)}$$
$$r = 0\cdot 75\left(\frac{3}{5}\right) = 0\cdot 45 \text{ m} \qquad \text{... (5m)}$$

(ii) the tension in the string **(10)**



$T\sin\theta = 3g$ ... (5m)

$T\left(\frac{4}{5}\right) = 30$

$T = 37\cdot5$ N ... (5m)

(iii) the speed of the particle. **(10)**

Circular motion:

$T\cos\theta = \frac{mv^2}{r}$ ... (5m)

$(37\cdot5)\left(\frac{3}{5}\right) = \frac{(3)v^2}{0\cdot45}$

$(7\cdot5)(0\cdot45) = v^2$

$v^2 = 3\cdot375$

$v = 1\cdot84 \text{ m s}^{-1}$ ... (5m)

9. **(a)** State the Principle of Archimedes. **(10)**

Statement ... (10m)

A solid piece of metal has a weight of 800 N.
When it is completely immersed in a liquid of relative density $1 \cdot 5$, the metal weighs 500 N.

Find **(i)** the volume of the metal **(10)**

$B =$ weight of displaced liquid

$300 = \rho V g$

$300 = (1500)V(10)$ ... (5m)

$V = 0 \cdot 02 \text{ m}^3$ ... (5m)

(ii) the relative density of the metal. **(10)**

Weight of metal $= \rho V g$

$800 = \rho(0 \cdot 02)(10)$ ... (5m)

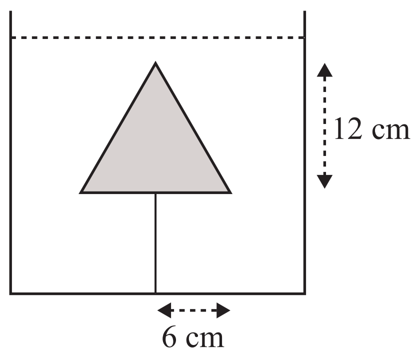
$\rho = 4000$

$s = 4$ ... (5m)

(b) A solid cone has a base of radius 6 cm and a height of 12 cm.

The cone is made from plastic of relative density $0 \cdot 81$ and is completely immersed in a tank of water.

The cone is held at rest with its axis vertical by a vertical string which is attached to the centre of its base and to the base of the water tank.



Find the tension in the string.
[Density of water = 1000 kg m^{-3}.] **(20)**

$$B = 1000\left\{\frac{1}{3}\pi(0 \cdot 06)^2(0 \cdot 12)\right\}(10)$$

$B = 4 \cdot 52$ ... (5m)

$$W = 810\left\{\frac{1}{3}\pi(0 \cdot 06)^2(0 \cdot 12)\right\}(10)$$

$W = 3 \cdot 66$ ... (5m)

and

$T + W = B$ ... (5m)

$T + 3 \cdot 66 = 4 \cdot 52$

$T = 0 \cdot 86 \text{ N}$ ... (5m)

Pre-Leaving Certificate Examination, 2014

Applied Mathematics

Higher Level
Marking Scheme (300 marks)

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Misreading	- if not serious	M(–1)

Serious blunder or omission or misreading which oversimplifies:
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Attempt marks are awarded as follows: 5 (Att. 2)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

4. Number the grid on each script 1 to 10 in numerical order, not the order of answering.

5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question.
In many cases, there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

1. **(a)** An underground train has to travel a distance d from rest at one station to rest at the next station. The train has a maximum acceleration of a and a maximum deceleration of b.

If the train makes the journey in the shortest possible time without having to travel at constant speed, show that the speed limit on the track, v, satisfies

$$v \geq \sqrt{\frac{2abd}{a+b}}\ .$$

(25)

If the train makes the journey without having to travel at constant speed, then it decelerates immediately after accelerating. Let s_1 and s_2 be the distances travelled while accelerating and decelerating, respectively. Let v_1 be the maximum speed attained.

Then

$$s_1 + s_2 = d \qquad \textbf{... 1}$$

Period of acceleration: Let t_1 be the time for acceleration. Then

$$v_1 = (0) + (a)(t_1)$$

$$t_1 = \frac{v_1}{a} \qquad \text{... (5m)}$$

and

$$s_1 = (0)(t_1) + \frac{1}{2}(a)(t_1)^2$$

$$s_1 = \frac{1}{2}a\left(\frac{v_1^2}{a^2}\right) = \frac{v_1^2}{2a} \qquad \text{... (5m)}$$

Likewise,

$$s_2 = \frac{v_1^2}{2b}$$

From **1**,

$$\frac{v_1^2}{2a} + \frac{v_1^2}{2b} = d \qquad \text{... (5m)}$$

$$v_1^2\left(\frac{b+a}{2ab}\right) = d$$

$$v_1^2 = \frac{2abd}{a+b}$$

$$v_1 = \sqrt{\frac{2abd}{a+b}} \qquad \text{... (5m)}$$

To make the journey without a period of constant speed, we need

$$v \geq v_1$$

$$v \geq \sqrt{\frac{2abd}{a+b}} \qquad \text{... (5m)}$$

(b) A stone is dropped from the top of a tower which is located on horizontal ground through the base of the tower. One second later, another stone is thrown vertically downwards from the same point with a speed of 15 m s^{-1}.

(i) If the two stones reach the ground simultaneously, find the height of the tower. **(15)**

Let h m be the height of the tower. If the first stone takes t s to reach the ground, then the second stone will take $(t-1)$ s.

1st stone: $h = (0)(t) + \frac{1}{2}(g)(t)^2$

$h = \frac{1}{2}gt^2$ ... (5m)

2nd stone: $h = (15)(t-1) + \frac{1}{2}g(t-1)^2$ ... (5m)

$h = 15t - 15 + \frac{1}{2}g\left(t^2 - 2t + 1\right)$

$h = 15t - 15 + \frac{1}{2}gt^2 - gt + \frac{1}{2}g$

then $15t - 15 + \frac{1}{2}gt^2 - gt + \frac{1}{2}g = \frac{1}{2}gt^2$

$15t - 15 - gt + \frac{1}{2}g = 0$

$(15 - g)t = 15 - \frac{1}{2}g$

$5 \cdot 2t = 10 \cdot 1$

$t = 1 \cdot 94$ s

The height of the tower is

$h = (4 \cdot 9)(1 \cdot 94)^2 = 18 \cdot 44$ m ... (5m)

(ii) If a third stone is thrown downwards from the same point half a second after the second stone, what initial speed must it have to reach the ground at the same time as the first two stones? **(10)**

Let v m s^{-1} be the initial speed of the third stone. It travels for $0 \cdot 94 - 0 \cdot 5 = 0 \cdot 44$ s while covering a distance of $18 \cdot 44$ m.
Thus

$18 \cdot 44 = (v)(0 \cdot 44) + \frac{1}{2}(9 \cdot 8)(0 \cdot 44)^2$ ... (5m)

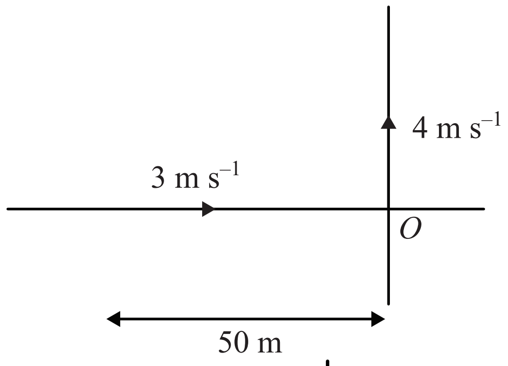
$18 \cdot 44 = 0 \cdot 44v + 0 \cdot 94864$

$0 \cdot 44v = 17 \cdot 49136$

$v = 39 \cdot 75$

The initial speed is $39 \cdot 75$ m s^{-1}. ... (5m)

2. **(a)** Two straight roads cross at right angles at O. On one road, a car is travelling due north at 4 m s^{-1}. As the car passes through O, a bus is 50 m from O and travelling east towards O at a speed of 3 m s^{-1}.



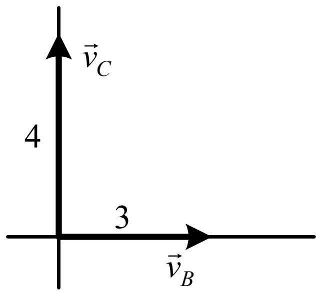
(i) Find the velocity of the bus relative to the car. **(10)**

$\vec{v}_C = 4\vec{j}$

$\vec{v}_B = 3\vec{i}$

then

$$\vec{v}_{BC} = \vec{v}_B - \vec{v}_C$$
$$= 3\vec{i} - 4\vec{j} \qquad \text{... (5m)}$$

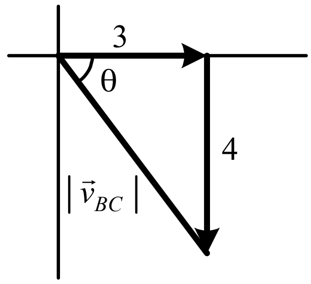


thus

$$|\vec{v}_{BC}| = \sqrt{3^2 + 4^2} = 5 \text{ m s}^{-1}$$

and

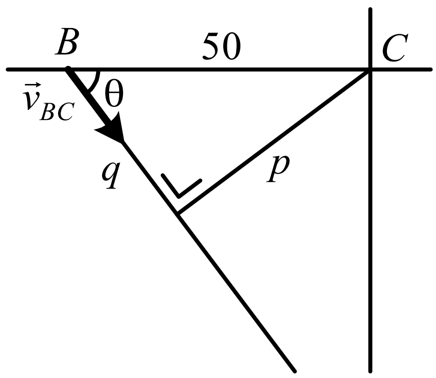
$$\tan\theta = \frac{4}{3}$$
$$\theta = \tan^{-1}\frac{4}{3} = 53{\cdot}13^\circ \qquad \text{... (5m)}$$



(ii) Calculate the shortest distance between the car and the bus and the time at which this occurs. **(10)**

If p is the shortest distance,

$$\frac{p}{50} = \sin\theta$$
$$p = 50\left(\frac{4}{5}\right)$$
$$p = 40 \text{ m} \qquad \text{... (5m)}$$



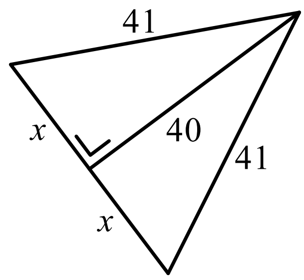
and

$$\frac{q}{50} = \cos\theta$$
$$q = 50\left(\frac{3}{5}\right)$$
$$q = 30 \text{ m}$$

then

$$\text{time} = \frac{\text{relative distance}}{\text{relative speed}}$$
$$= \frac{30}{5}$$
$$= 6 \text{ s} \qquad \text{... (5m)}$$

(iii) Find the length of time during which the car and the bus are within 41 m of one another. **(5)**



$x^2 = 41^2 - 40^2$

$x^2 = 81$

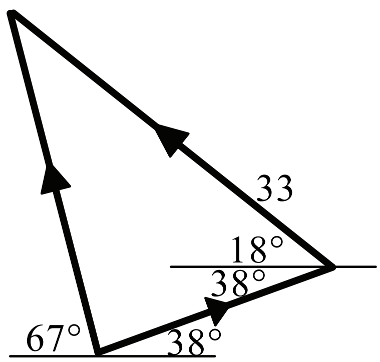
$x = 9$

Distance travelled $= 2 \times 9 = 18$ m

Time $= \dfrac{18}{5} = 3 \cdot 6$ s. ... (5m)

(b) A ship P is travelling in the direction 38° north of east. To an observer on another ship Q, which is travelling 18° south of east at 33 km h^{-1}, P appears to be travelling in the direction 67° north of west.

(i) Find the actual speed of P, correct to one decimal place. **(15)**



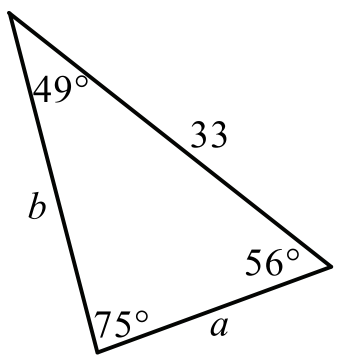


Diagram with angles: ... (10m)

Let $a = |\vec{v}_P|$. Then

$$\frac{a}{\sin 49^\circ} = \frac{33}{\sin 75^\circ}$$

$a = 25 \cdot 8$ km h^{-1} $= |\vec{v}_P|$ ... (5m)

(ii) Find the magnitude of the velocity of P relative to Q, correct to one decimal place. **(5)**

Let $b = |\vec{v}_{PQ}|$. Then

$$\frac{b}{\sin 56^\circ} = \frac{33}{\sin 75^\circ}$$

$b = 28 \cdot 3$ km h^{-1} $= |\vec{v}_{PQ}|$ ... (5)

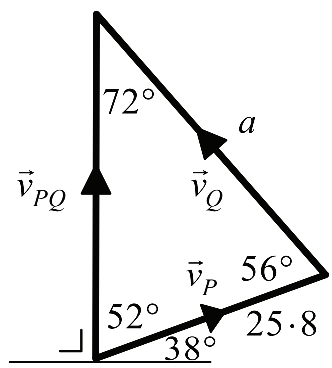
Further answers overleaf

(iii) Q reduces its speed, without changing direction, so that P appears to be travelling due north. Find the reduced speed of Q, correct to one decimal place. **(5)**

Let a be the new speed of Q.

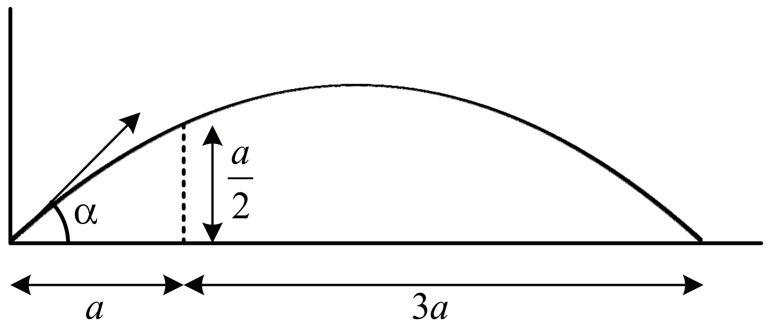
$$\frac{a}{\sin 52^\circ} = \frac{25 \cdot 8}{\sin 72^\circ}$$

$$a = 21 \cdot 4 \text{ km h}^{-1} = \left| \vec{v}_Q \right| \qquad \text{... (5)}$$



3. **(a)** A particle is projected from a point on a horizontal plane so that it just clears a vertical wall of height $\frac{a}{2}$ at a horizontal distance of a from the point of projection and strikes the plane at a horizontal distance $3a$ **beyond the wall**.

(i) Express the range of the particle on the horizontal plane in terms of a. **(5)**



From the diagram, the range is

$a + 3a = 4a$ ... (5m)

(ii) If the angle of projection measured to the horizontal is $\tan^{-1}\frac{2}{k}$, find the value of k. **(20)**

Let α be the angle of projection.

TOF: $s_y = 0$

$$u\sin\alpha t - \frac{1}{2}gt^2 = 0$$

$$t = \frac{2u\sin\alpha}{g} \qquad \text{... (5m)}$$

$$R = 4a: \qquad 4a = u\cos\alpha\left(\frac{2u\sin\alpha}{g}\right)$$

$$4a = \frac{2u^2\sin\alpha\cos\alpha}{g}$$

$$u^2 = \frac{2ga}{\sin\alpha\cos\alpha} \qquad \textbf{... 1} \qquad \text{... (5m)}$$

Path contains the point $\left(a, \frac{a}{2}\right)$:

$$u\cos\alpha t = a, \qquad u\sin\alpha t - \frac{1}{2}gt^2 = \frac{a}{2}$$

$$t = \frac{a}{u\cos\alpha}$$

Further answers overleaf

thus

$$u\sin\alpha\left(\frac{a}{u\cos\alpha}\right)-\frac{g}{2}\left(\frac{a^2}{u^2\cos^2\alpha}\right)=\frac{a}{2} \qquad \text{... (5m)}$$

$$a\tan\alpha-\frac{ga^2}{2\cos^2\alpha}\left(\frac{\sin\alpha\cos\alpha}{2ga}\right)=\frac{a}{2} \qquad \text{... by } \mathbf{1}$$

$$a\tan\alpha-\frac{1}{4}a\tan\alpha=\frac{a}{2}$$

$$\frac{3}{4}\tan\alpha=\frac{1}{2}$$

$$\tan\alpha=\frac{2}{3}$$

$$\alpha=\tan^{-1}\frac{2}{3}.$$

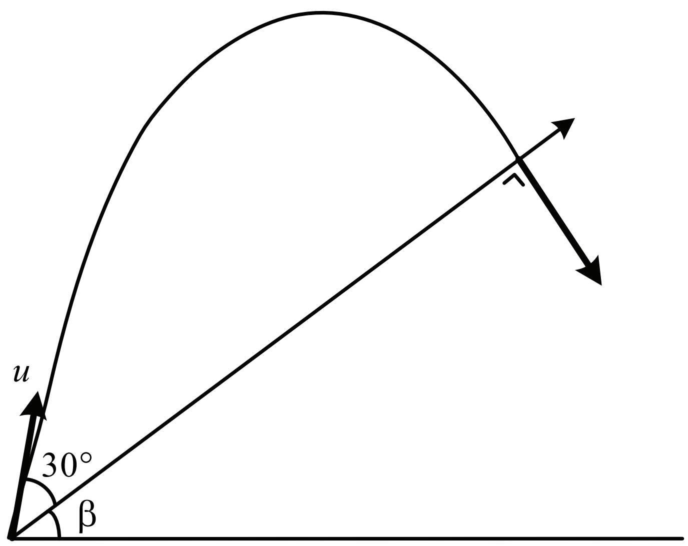
hence $k=3$ ... (5m)

(b) A plane is inclined at an angle β to the horizontal.

A body is projected up the plane with velocity u m s^{-1} at an angle of 30° to the inclined plane.

The plane of projection is vertical and contains the line of greatest slope. The particle strikes the plane at right angles.

(i) Find the value of c and the value of d if $\tan\beta=\frac{\sqrt{c}}{d}$. **(15)**



At the same time, $s_y = 0$ and $v_x = 0$.

$$s_y = 0$$

$$\frac{ut}{2} - \frac{1}{2} g \cos\beta t^2 = 0$$

$$t = \frac{u}{g\cos\beta} \qquad \text{... (5m)}$$

At $t = \dfrac{u}{g\cos\beta}$:

$$v_x = 0$$

$$\frac{\sqrt{3}u}{2} - g\sin\beta\left(\frac{u}{g\cos\beta}\right) = 0 \qquad \text{... (5m)}$$

$$\frac{\sqrt{3}}{2} = \tan\beta$$

thus $c = 3$ and $d = 2$ ... (5m)

(ii) If the range of the body on the inclined plane is $10\sqrt{21}$ m, find the value of u. **(10)**

Then

$$\text{TOF} = \frac{u}{g}.\frac{\sqrt{7}}{2} = \frac{\sqrt{7}u}{2g}$$

and

$$\text{Range } = 10\sqrt{21}$$

$$\frac{\sqrt{3}u}{2}\left(\frac{\sqrt{7}u}{2g}\right) - \frac{1}{2}g\left(\frac{\sqrt{3}}{\sqrt{7}}\right)\left(\frac{7u^2}{4g^2}\right) = 10\sqrt{21} \qquad \text{... (5m)}$$

$$\frac{\sqrt{21}u^2}{4g} - \frac{\sqrt{21}u^2}{8g} = 10\sqrt{21}$$

$$2u^2 - u^2 = 80g$$

$$u^2 = 80g$$

$$u = 28 \qquad \text{... (5m)}$$

4. **(a)** A light inextensible string is attached at one end to a particle A, of mass 5 kg, hanging freely.

The string passes over a smooth fixed pulley and its other end is attached to another particle B, of mass 3 kg, which is held at a height of 5 m above the horizontal plane.

Initially the string is slack. The system is released from rest.

After B has fallen through 1 m, the string becomes taut. Find

(i) the speed of B directly after the string becomes taut **(10)**

First metre:

$u = 0$	$v^2 = u^2 + 2as$
$v = ?$	$v^2 = 0^2 + 2g(1)$
$a = g$	$v^2 = 2g$
$s = 1$	$v = \sqrt{2g}$ ... (5m)

Let v be the velocity of each particle after the jerk.
Momentum before = Momentum of B

$= (3)(\sqrt{2g})$

$= 3\sqrt{2g}$

Momentum after = Momentum of A + Momentum of B

$= (5)(v) + (3)(v)$

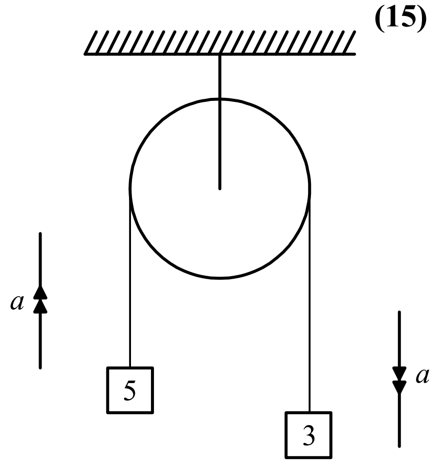
$= 8v$

PCM: Momentum after = Momentum before

$8v = 3\sqrt{2g}$

$v = \dfrac{3\sqrt{2g}}{8}$ m s^{-1} ... (5m)

(ii) the nearest that B gets to the horizontal plane. **(15)**

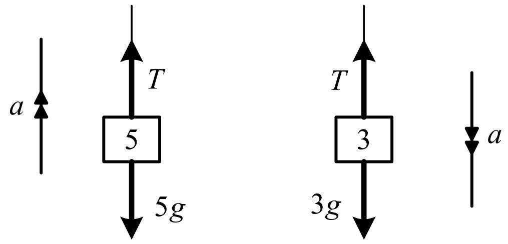


5 kg mass

$\updownarrow$: Newton's 2nd Law

$5a = T - 5g$ **... 1**

3 kg mass



$\updownarrow$: Newton's 2nd Law

$3a = 3g - T$ **… 2**

Adding 1 and 2,

$8a = -2g$

$a = -\dfrac{g}{4}$ ... (5m)

$u = \dfrac{3\sqrt{2g}}{8}$ $\quad v^2 = u^2 + 2as$

$v = 0$ $\quad 0^2 = \left(\dfrac{3\sqrt{2g}}{8}\right)^2 + 2\left(-\dfrac{g}{4}\right)s$

$a = -\dfrac{g}{4}$ $\quad \dfrac{gs}{2} = \dfrac{18g}{64}$

$s = ?$ $\quad s = \dfrac{36}{64}$

$s = \dfrac{9}{16} = 0 \cdot 5625$ m ... (5m)

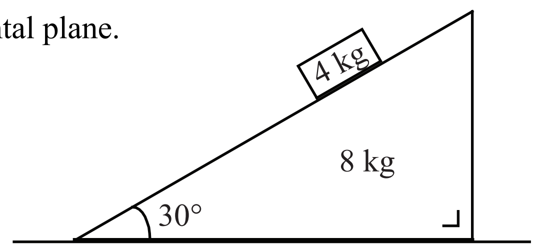
Nearest B gets to the plane $= 5 - (1 + 0 \cdot 5625)$

$= 3 \cdot 4375$

$= 3 \cdot 44$ m ... (5m)

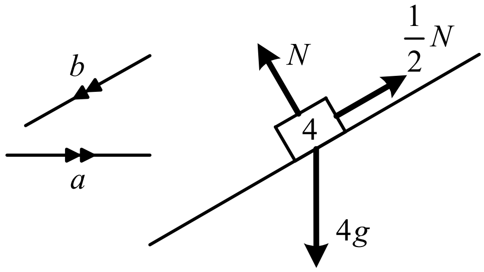
(b) A wedge of mass 8 kg sits on a smooth horizontal plane. A particle of mass 4 kg sits on a face of the wedge which is inclined at 30° to the horizontal. The coefficient of friction between the particle and the wedge is $\frac{1}{2}$.

The system is released from rest.

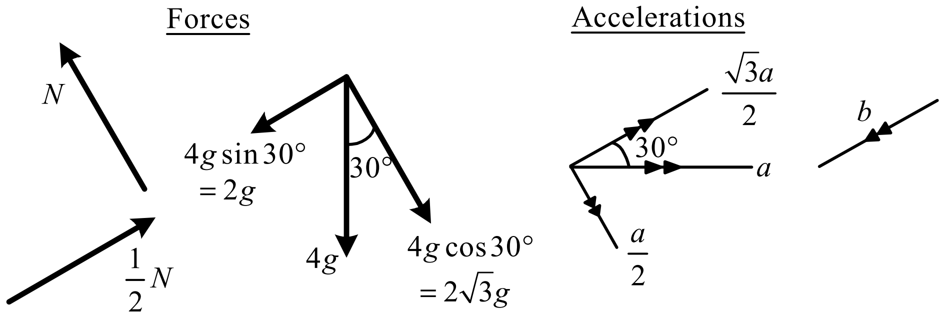


Find the acceleration of the wedge. **(25)**

4 kg particle



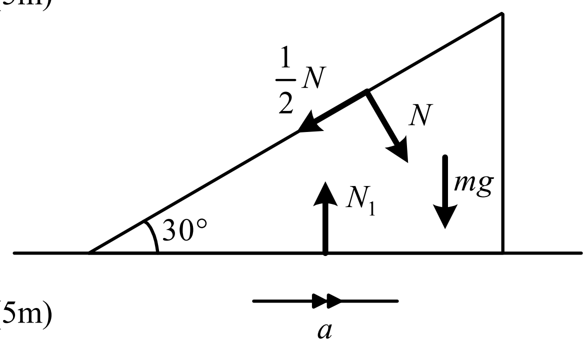
Further answers overleaf



↘ : Newton's 2nd Law

$$4\left(\frac{a}{2}\right) = 2\sqrt{3}g - N \quad \text{... (5m)}$$

$$2a = 2\sqrt{3}g - N \quad \textbf{... 1}$$



Wedge

↔ : Newton's 2nd Law

$$8a = \frac{N}{2} - \frac{\sqrt{3}}{4}N \quad \text{... (5m)}$$

$$32a = 2N - \sqrt{3}N \quad \textbf{... 2}$$

2 : $(2-\sqrt{3})N = 32a$

$$N = \frac{32a}{2-\sqrt{3}} \quad \text{... (5m)}$$

1 : $2a = 2\sqrt{3}g - \frac{32a}{2-\sqrt{3}}$ ... (5m)

$$2(2-\sqrt{3})a = 2\sqrt{3}(2-\sqrt{3})g - 32a$$

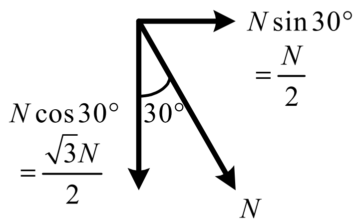
$$(4-2\sqrt{3}+32)a = (4\sqrt{3}-6)g$$

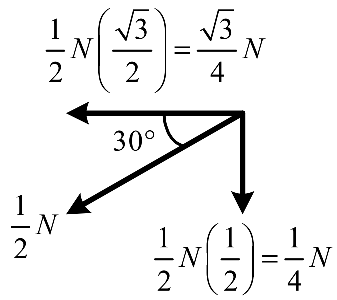
$$(36-2\sqrt{3})a = (4\sqrt{3}-6)g$$

$$(18-\sqrt{3})a = (2\sqrt{3}-3)g$$

$$a = \frac{2\sqrt{3}-3}{18-\sqrt{3}}g \text{ m s}^{-2}$$

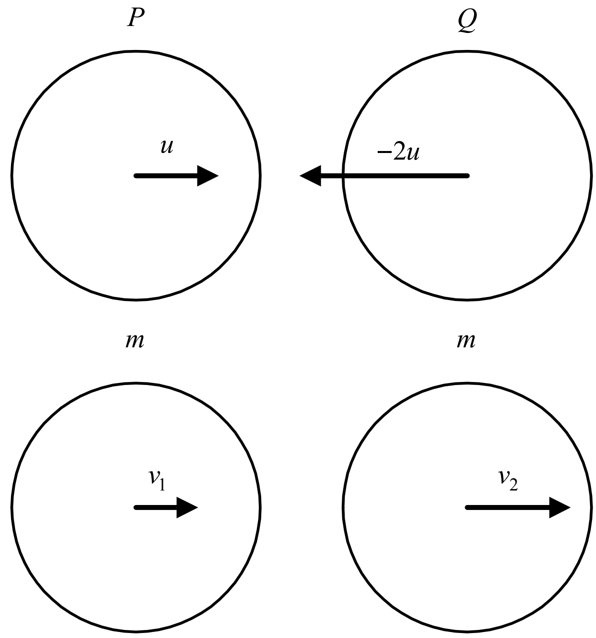
or $a = 0\cdot 28 \text{ m s}^{-2}$ ... (5m)





5. **(a)** Two smooth spheres, P and Q, of equal mass, are travelling directly towards each other with speeds u and $2u$, respectively.

(i) Determine if it is possible for the sphere Q to be at rest after the collision. **(15)**



PCM: $mv_1 + mv_2 = mu - 2mu$

$v_1 + v_2 = -u$ **... 1**

NEL: $v_1 - v_2 = -3eu$ **... 2** ... (5m)

1 : $v_1 + v_2 = -u$

2$\times -1$: $-v_1 + v_2 = 3eu$

$2v_2 = (3e-1)u$

$v_2 = \dfrac{(3e-1)u}{2}$ ... (5m)

For Q to be at rest after the collision, $v_2 = 0$. Thus

$$\frac{(3e-1)u}{2} = 0$$

$$3e - 1 = 0$$

$$e = \frac{1}{3}$$

As $0 < e < 1$, it is possible for Q to be at rest after the collision. ... (5m)

(ii) If the fraction of the kinetic energy lost during the collision is $\frac{27}{40}$, find the value of *e*, the coefficient of restitution. **(15)**

Adding **1** and **2**,

$$2v_1 = -(1+3e)u$$

$$v_1 = \frac{-(1+3e)u}{2}$$

Then:

$$\text{KE before impact} = \frac{1}{2}mu^2 + \frac{1}{2}m(-2u)^2 = \frac{5}{2}mu^2$$

$$\text{KE after impact} = \frac{1}{2}mu^2\left(\frac{1+6e+9e^2}{4}\right) + \frac{1}{2}mu^2\left(\frac{9e^2-6e+1}{4}\right) \qquad \text{... (5m)}$$

$$= \frac{1}{8}mu^2(18e^2+2) = \frac{1}{4}mu^2(1+9e^2)$$

$$\text{Loss in KE} = \frac{10}{4}mu^2 - \frac{1}{4}mu^2(1+9e^2) \qquad \text{... (5m)}$$

$$= \frac{1}{4}mu^2(10-1-9e^2)$$

$$= \frac{9}{4}mu^2(1-e^2)$$

$$\text{Fraction lost} = \frac{\frac{9}{4}mu^2(1-e^2)}{\frac{5}{2}mu^2}$$

$$\Rightarrow \quad \frac{2}{5}.\frac{9}{4}(1-e^2) = \frac{27}{40}$$

$$\Rightarrow \quad 4(1-e^2) = 3$$

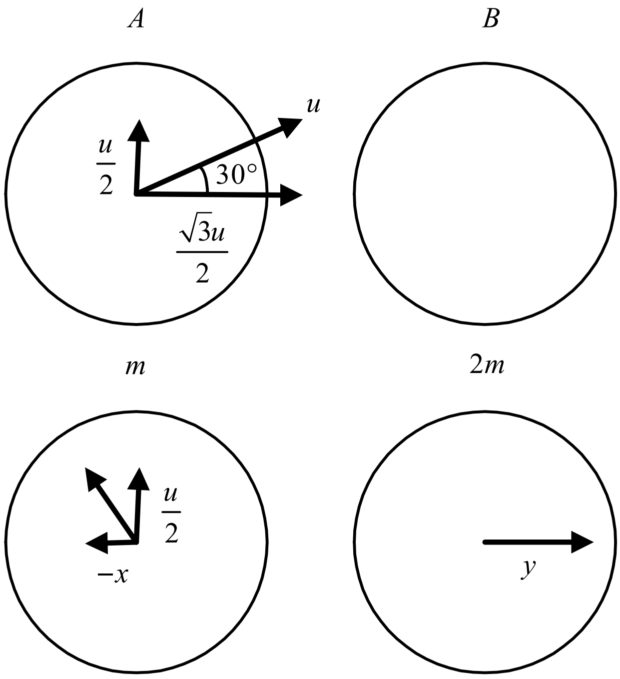
$$\Rightarrow \quad 1 = 4e^2$$

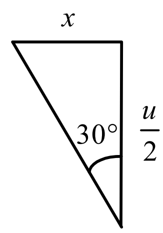
$$\Rightarrow \quad e^2 = \frac{1}{4} \quad \Rightarrow \quad e = \frac{1}{2} \qquad \text{... (5m)}$$

(b) A smooth sphere A, of mass m, collides obliquely with a smooth sphere B, of mass $2m$, which is at rest.

Before the collision, A has a velocity of u in a direction which makes an angle of 30° with the line of centres.

If A is deflected through an angle of 90° by the collision, find the value of e, the coefficient of restitution. **(20)**





$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{x}{\frac{u}{2}} \quad \Rightarrow \quad x = \frac{u}{2\sqrt{3}} \qquad \text{... (5m)}$$

PCM $\vec{i}$:

$$m\left(\frac{-u}{2\sqrt{3}}\right) + 2my = m\left(\frac{\sqrt{3}u}{2}\right) \qquad \text{... (5m)}$$

$$2y = \frac{\sqrt{3}u}{2} + \frac{u}{2\sqrt{3}}$$

$$2y = \frac{3u+u}{2\sqrt{3}} = \frac{2u}{\sqrt{3}}$$

$$y = \frac{u}{\sqrt{3}}$$

NEL $\vec{i}$:

$$\left(\frac{-u}{2\sqrt{3}} - \frac{u}{\sqrt{3}}\right) = -e\left(\frac{\sqrt{3}u}{2}\right) \qquad \text{... (5m)}$$

$$-\frac{3u}{2\sqrt{3}} = -e\frac{\sqrt{3}u}{2}$$

$$\frac{\sqrt{3}u}{2} = e\frac{\sqrt{3}u}{2}$$

$$e = 1 \qquad \text{... (5m)}$$

6. **(a)** A particle is performing simple harmonic motion of amplitude $0 \cdot 8$ m about a fixed point *O*.

A and *B* are two points on the path of the particle such that $|OA| = 0 \cdot 6$ m and $|OB| = 0 \cdot 4$ m. The particle takes 2 seconds to travel from *A* to *B*.

Find, correct to two decimal places, the periodic time of the motion if

(i) *A* and *B* are on the same side of *O* **(15)**

$x = a\cos(\omega t + \varepsilon)$

$x = 0 \cdot 6,\ t = 0:$ $\quad 0 \cdot 6 = 0 \cdot 8\cos\varepsilon$

$\cos\varepsilon = 0 \cdot 75$

$\varepsilon = \cos^{-1} 0 \cdot 75$

$\varepsilon = 0 \cdot 7227$ ... (5m)

$x = 0 \cdot 4,\ t = 2:$ $\quad 0 \cdot 4 = 0 \cdot 8\cos(2\omega + \varepsilon)$

$\cos(2\omega + \varepsilon) = 0 \cdot 5$

$2\omega + \varepsilon = \cos^{-1} 0 \cdot 5$

$2\omega + 0 \cdot 7227 = 1 \cdot 0472$

$2\omega = 0 \cdot 3245$

$\omega = 0 \cdot 16225$ ... (5m)

Period: $T = \dfrac{2\pi}{\omega}$

$T = \dfrac{2\pi}{0 \cdot 16225} = 38 \cdot 73$ s ... (5m)

(ii) *A* and *B* are on opposite sides of *O*. **(10)**

$x = a\cos(\omega t + \varepsilon)$

$x = -0 \cdot 4,\ t = 2:$ $\quad -0 \cdot 4 = 0 \cdot 8\cos(2\omega + \varepsilon)$

$\cos(2\omega + \varepsilon) = -0 \cdot 5$

$2\omega + \varepsilon = \cos^{-1}(-0 \cdot 5)$

$2\omega + 0 \cdot 7227 = 2 \cdot 0944$

$2\omega = 1 \cdot 3717$

$\omega = 0 \cdot 68585$ ... (5m)

Period: $T = \dfrac{2\pi}{\omega}$

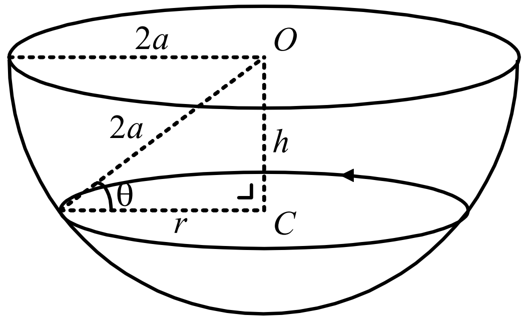
$T = \dfrac{2\pi}{0 \cdot 68585} = 9 \cdot 16$ s ... (5m)

(b) A particle P is moving on the inner surface of a smooth hemispherical bowl with centre O and radius $2a$.

The particle is describing a horizontal circle, centre C, with angular speed $\sqrt{\dfrac{g}{a}}$.

Find **(i)** the magnitude of the force exerted on P by the surface of the bowl **(20)**

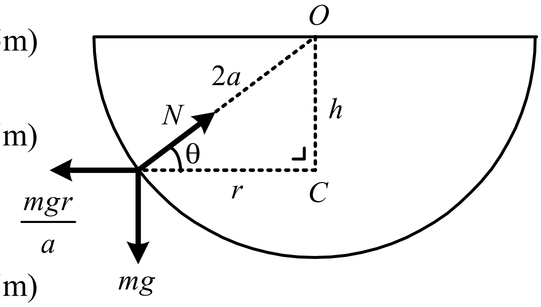
$$\omega = \sqrt{\frac{g}{a}}$$



The circular motion force is

$$m\omega^2 r = m\left(\frac{g}{a}\right)r$$

$$= \frac{mgr}{a}$$



$\uparrow = \downarrow :$ $\quad N\sin\theta = mg$ **… 1** ... (5m)

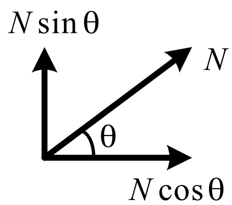
Circular : $\quad N\cos\theta = \dfrac{mgr}{a}$ **… 2** ... (5m)

Also, from the triangle,

$$\cos\theta = \frac{r}{2a} \quad \textbf{… 3} \quad \text{... (5m)}$$

2, 3 : $N\left(\dfrac{r}{2a}\right) = \dfrac{mgr}{a}$

$N = 2mg$ ... (5m)



(ii) the depth of C below O. **(5)**

1 : $\quad 2mg\sin\theta = mg$

$$\sin\theta = \frac{1}{2}$$

$$\theta = 30°$$

then

$$\frac{h}{2a} = \sin\theta$$

$$h = 2a\left(\frac{1}{2}\right)$$

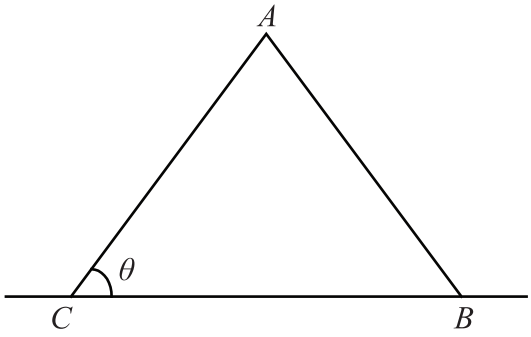
$h = a$ ... (5m)

7. Two uniform rods, AB and AC, each of length $2a$ and weight W, are smoothly jointed at A. The end C is freely hinged to a point on a rough horizontal plane.

The end B rests on the plane and is on the point of slipping.

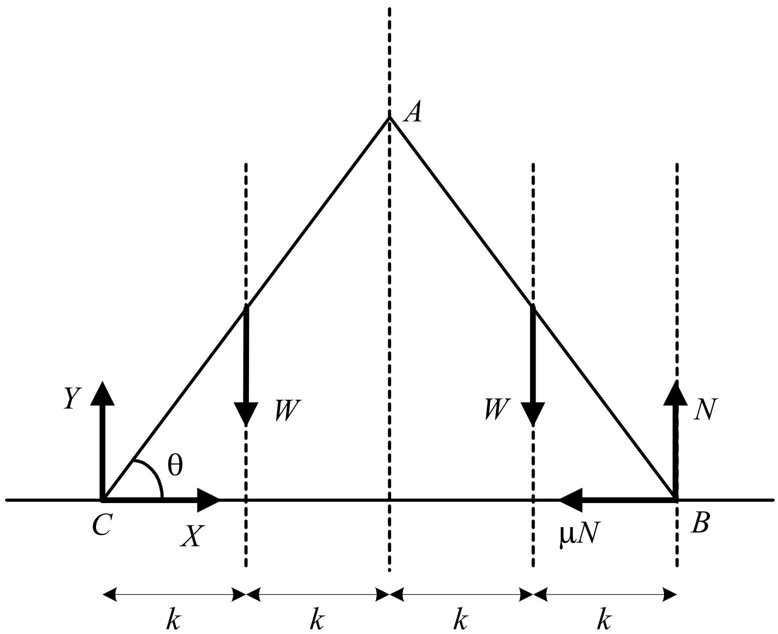
Both rods are in a vertical plane.

The coefficient of friction between B and the plane is μ and the angle between AC and the horizontal is θ.



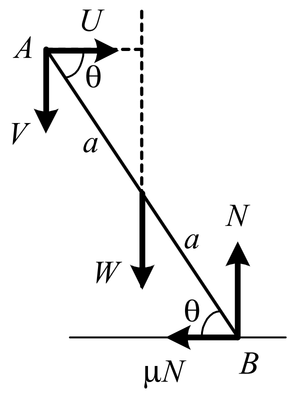
(i) Show, on separate diagrams, all the forces acting on the structure ABC and on the rod AB. **(10)**

Structure *ABC*:



... (5m)

Rod *AB*:



... (5m)

(ii) Prove that $\mu = \dfrac{1}{2\tan\theta}$. **(15)**

For structure ABC, moments about C:

$W.k + W.3k = N.4k$ ... (5m)

$N = W$ … **1**

For rod AB, moments about A:

$W.a\cos\theta + \mu N.2a\sin\theta = N.2a\cos\theta$ ... (5m)

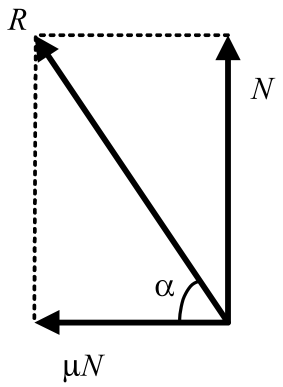
$W + 2\mu N\tan\theta = 2N$ **… 2**

1, **2:** $W + 2\mu W\tan\theta = 2W$

$2\mu\tan\theta = 1$

$\mu = \dfrac{1}{2\tan\theta}$ ... (5m)

(iii) Find, in terms of W and μ, the magnitude and direction of the reaction force at the end B resting on the plane. **(15)**



The total reaction, R, at B is given by

$R = \sqrt{N^2 + (\mu N)^2}$ ... (5m)

$= \sqrt{N^2 + \mu^2 N^2}$

$= \sqrt{N^2(1+\mu^2)}$

$= N\sqrt{1+\mu^2}$

$= W\sqrt{1+\mu^2}$ ... (5m)

The direction of this reaction force makes an angle α with the horizontal, where

$\tan\alpha = \dfrac{N}{\mu N} = \dfrac{1}{\mu}$

i.e. $\alpha = \tan^{-1}\dfrac{1}{\mu}$ ... (5m)

(iv) Find, in terms of W and θ, the magnitude of the reaction at the joint A**.** **(10)**

Rod AB

$\updownarrow$: $V + W = N$

$V + W = W$

$V = 0$ ... (5m)

$\leftrightarrow$: $U = \mu N$

$U = \mu W$

or $U = \dfrac{W}{2\tan\theta}$ ... (5m)

which is the magnitude of the reaction at the joint A.

8. (a) Prove that the moment of inertia of a uniform rod of mass m and length $2l$ about an axis through its centre perpendicular to the rod is $\frac{1}{3}ml^2$. **(20)**

<u>Standard Proof</u>

Moment of mass element	... (5m)
Moment of body	... (5m)
Integral	... (5m)
Deduce	... (5m)

(b) A uniform rod AB of mass m and length $2l$ is free to rotate in a vertical plane about a horizontal axis through A. A particle of mass $2m$ is attached to the rod at B.

The system is released from rest with B vertically above A.

(i) Find the angular velocity when the system is next vertical. **(20)**

For the system,

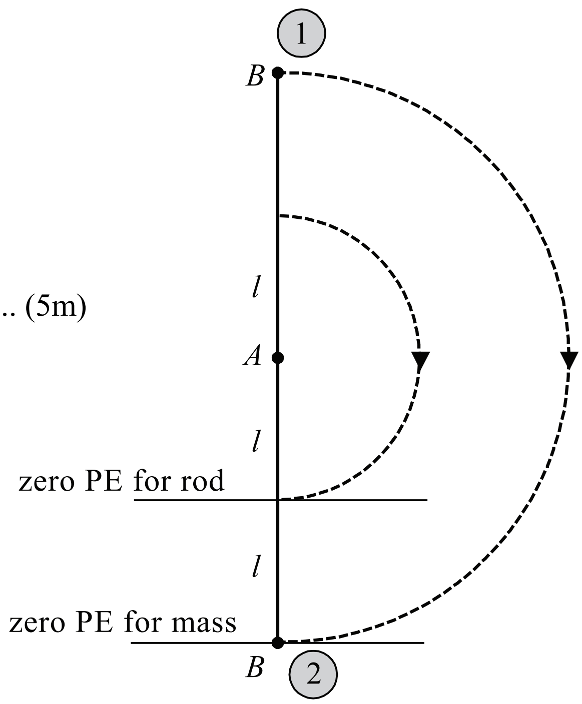
$$I = I_{\text{rod}} + I_B$$

$$I_{\text{rod}} = \frac{4}{3}ml^2$$

$$I_B = mb^2$$
$$= (2m)(2l)^2$$
$$= 8ml^2$$

then

$$I = \frac{4}{3}ml^2 + 8ml^2$$
$$= \frac{28}{3}ml^2 \quad \text{... (5m)}$$



Position 1: B above A

$$KE_1 = 0$$

$$PE_1 = PE_{\text{rod}} + PE_B$$
$$= mg(2l) + (2m)g(4l)$$
$$= 10mgl$$

Position 2: B below A

$$KE_2 = \frac{1}{2}I\omega^2$$
$$= \frac{1}{2}\left(\frac{28}{3}ml^2\right)\omega^2$$

$$= \frac{14}{3} ml^2 \omega^2$$

$$PE_2 = 0$$

PCE: $KE_2 + PE_2 = KE_1 + PE_1$

$$\frac{14}{3} ml^2 \omega^2 + 0 = 0 + 10mgl \qquad \text{... (5m, 5m)}$$

$$7l\omega^2 = 15g$$

$$\omega^2 = \frac{15g}{7l}$$

$$\omega = \sqrt{\frac{15g}{7l}} \text{ rad s}^{-1} \qquad \text{... (5m)}$$

At this point the mass at B falls off.

(ii) Find the height of B when it next comes to rest. **(10)**

For the rod after the mass falls off,

$$I = \frac{4}{3} ml^2$$

(Let the position of zero PE be when B is vertically below A.)

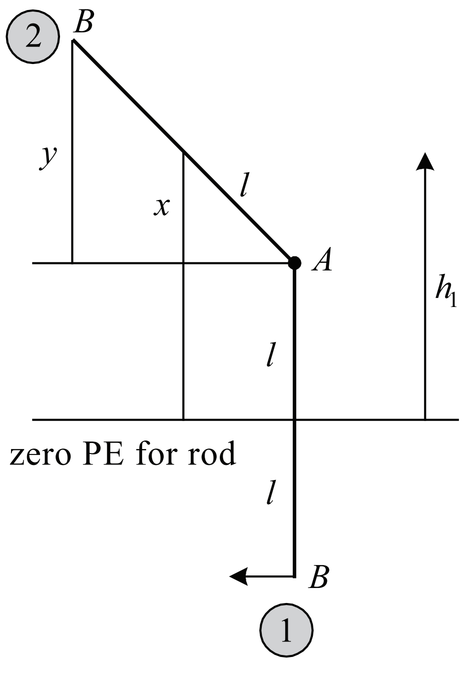
Position 1: B below A.

$$KE_1 = \frac{1}{2} I\omega^2$$

$$= \frac{1}{2}\left(\frac{4}{3}ml^2\right)\left(\frac{15g}{7l}\right)$$

$$= \frac{10}{7} mgl$$

$$PE_1 = 0$$



Position 2: rod comes to rest. Let the height of the centre of mass above the zero PE level be h_1.

$$KE_2 = 0$$

$$PE_2 = mgh_1$$

PCE: $KE_2 + PE_2 = KE_1 + PE_1$

$$0 + mgh_1 = \frac{10}{7} mgl + 0 \qquad \text{... (5m)}$$

$$h_1 = \frac{10}{7} l$$

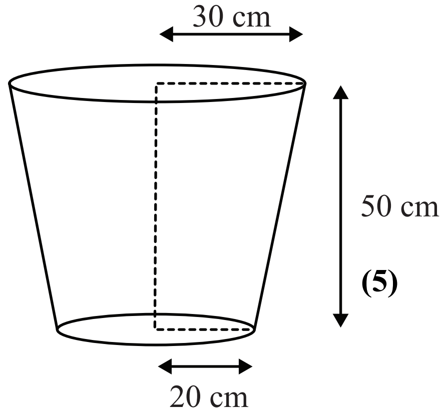
Further answers overleaf

From the diagram, $x = \frac{10}{7}l - l = \frac{3}{7}l$

then $y = 2x = \frac{6}{7}l$

hence, at the highest point B is $\frac{6}{7}l$ above the level of A. ... (5m)

9. **(a)** A bucket, in the shape of a frustum, has a height of 50 cm, a base of radius 20 cm and a top of radius 30 cm.



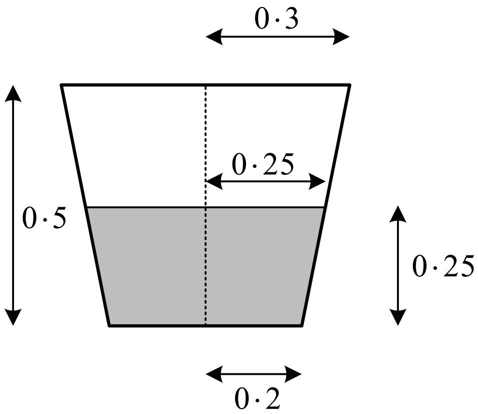
Find, in terms of π and g,

(i) the thrust on the base if the bucket is filled with water **(5)**

Let T be the thrust on the base.

$$T = \rho g h A$$
$$= 1000(1)g(0\cdot5)\left(\pi(0\cdot2)^2\right) \quad \text{... (5m)}$$
$$= 20\pi g \text{ N}$$

(ii) the thrust on the curved side if water is poured into the bucket to a depth of 25 cm. **(25)**



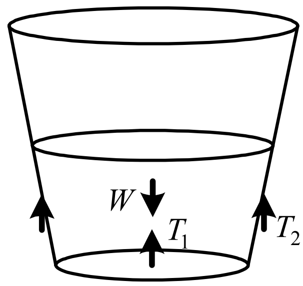
Height of top of water is $25 \text{ cm} = 0\cdot25 \text{ m}$ ... (5m)

Volume of water:

$$V = \frac{1}{3}\pi(0\cdot25)\left((0\cdot25)^2 + (0\cdot25)(0\cdot2) + (0\cdot2)^2\right)$$
$$= \frac{61\pi}{4800} \text{ m}^3 \quad \text{... (5m)}$$

Weight of water:

$$W = \rho V g$$
$$= 1000\left(\frac{61\pi}{4800}\right)g$$
$$= 12\cdot71\pi g \text{ N} \quad \text{... (5m)}$$



Further answers overleaf

Thrust on base:

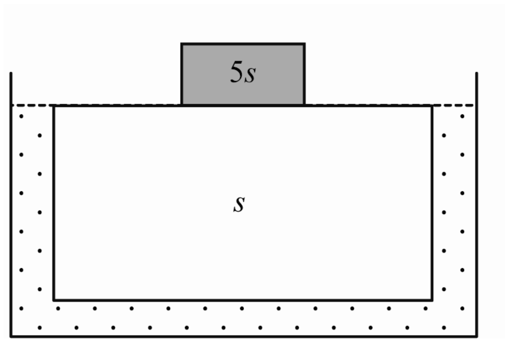
$$T_1 = \rho g h A$$
$$= 1000(1)g(0\cdot25)\left(\pi(0\cdot2)^2\right)$$
$$= 10\pi g \text{ N}$$ ... (5m)

Let T_2 be the vertical thrust on the curved side. Then

$$T_1 + T_2 = W$$
$$T_2 = 12\cdot71\pi g - 10\pi g$$
$$T_1 = 2\cdot71\pi g \text{ N}$$ ... (5m)

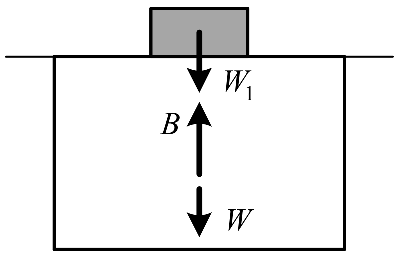
(b) A block of wood, of volume V and relative density s, floats in water.

A smaller block of metal, of volume V_1 and relative density $5s$, is placed on top of the wood, such that the upper surface of the wooden block is in the free surface of the water.



Prove that $\dfrac{V}{V_1} = \dfrac{5s}{1-s}$. **(20)**

Let B be the buoyancy, W the weight of the block of wood and W_1 the weight of the block of metal. Let V be the volume of the wood and V_1 be the volume of the metal.



$$B = 1000(1)Vg$$
$$= 1000gV$$ ... (5m)

and

$$\text{weight} = W + W_1$$
$$= 1000(s)Vg + 1000(5s)V_1 g$$ ... (5m)
$$= 1000sgV + 5000sgV_1$$

In equilibrium:

$$W + W_1 = B$$
$$1000sgV + 5000sgV_1 = 1000gV$$ ... (5m)
$$sV + 5sV_1 = V$$
$$5sV_1 = V - sV$$
$$5sV_1 = (1-s)V$$
$$\frac{5s}{1-s} = \frac{V}{V_1}$$ ... (5m)

10. **(a)** Solve the differential equation

$$x\frac{dy}{dx} = y(1+x)$$

given that $y = 3$ when $x = 1$. **(20)**

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y(1+x)$$

$$\frac{\mathrm{d}y}{y} = \frac{1+x}{x}\mathrm{d}x$$

$$\int\frac{\mathrm{d}y}{y} = \int\left(\frac{1}{x}+1\right)\mathrm{d}x \qquad \text{... (5m)}$$

$$\ln|y| = \ln|x| + x + c \qquad \text{... (5m)}$$

I.C.: $\underline{y = 3,\ x = 1}$

$$\ln 3 = 0 + 1 + c$$

$$c = -1 + \ln 3 \qquad \text{... (5m)}$$

Unique solution:

$$\ln|y| = \ln|x| - x - 1 + \ln 3$$

$$\ln|y| = \ln|x| + \ln e^{-1-x} + \ln 3$$

$$\ln|y| = \ln 3|x|e^{-1-x}$$

$$y = 3xe^{-1-x} = \frac{3x}{e^{1+x}} \qquad \text{... (5m)}$$

(b) A particle of mass m moves with a velocity v m s^{-1} in a straight line through a medium in which the resistance to motion is mkv^3, where k is a constant. No other force acts on the body.

The velocity of the particle falls from 15 m s^{-1} to $7\cdot5$ m s^{-1} in a time of t_1 seconds.

Show that the distance travelled in this time is $10t_1$ m. **(30)**

$$R = mkv^3$$

Newton's 2$^{\text{nd}}$ Law:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -mkv^3$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kv^3$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -kv^3 \qquad \text{... (5m)}$$

Further answers overleaf

$$v^{-3}\mathrm{d}v = -k\,\mathrm{d}t$$

$$\int v^{-3}\mathrm{d}v = -k\int \mathrm{d}t$$

$$\frac{1}{-2}v^{-2} = -kt - c \qquad \text{... (5m)}$$

$$\frac{1}{2v^2} = kt + c \qquad \text{I.C.: } \underline{v = 15,\ t = 0}$$

$$\frac{1}{450} = c$$

Unique solution:

$$\frac{1}{2v^2} = kt + \frac{1}{450} \qquad \text{... (5m)}$$

also, $t = t_1$ when $v = 7\cdot 5$:

$$\frac{1}{112\cdot 5} = kt_1 + \frac{1}{450}$$

$$kt_1 = \frac{4}{450} - \frac{1}{450}$$

$$kt_1 = \frac{3}{450} = \frac{1}{150}$$

$$k = \frac{1}{150t_1} \qquad \text{... (5m)}$$

then

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -kv^3$$

$$v^{-2}\mathrm{d}v = -k\,\mathrm{d}x$$

$$\int v^{-2}\mathrm{d}v = -k\int \mathrm{d}x \qquad \text{... (5m)}$$

$$-\frac{1}{v} = -kx - d$$

$$\frac{1}{v} = kx + d \qquad \text{I.C.: } \underline{v = 15,\ x = 0}$$

$$\frac{1}{15} = d$$

When $v = 7\cdot 5$,

$$\frac{1}{7\cdot 5} = kx + \frac{1}{15}$$

$$kx = \frac{2}{15} - \frac{1}{15}$$

$$\frac{1}{150t_1}x = \frac{1}{15}$$

$$x = \frac{150t_1}{15}$$

$$x = 10t_1 \qquad \text{... (5m)}$$