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Pre-Leaving Certificate Examination, 2013

Applied Mathematics

Marking Scheme

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Pre-Leaving Certificate Examination, 2013

Applied Mathematics

Ordinary Level
Marking Scheme (300 marks)

General Instructions

1. Penalties of three types are applied to students' work as follows:

Slips	- numerical slips	S(–1)
Blunders	- mathematical errors	B(–3)
Misreading	- if not serious	M(–1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

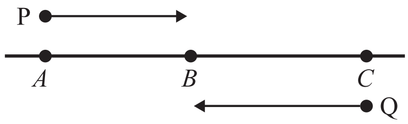
4. Number the grid on each script 1 to 9 in numerical order, not the order of answering.

5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

1. Three points, A, B and C lie on a straight level road.



Two cars P and Q are moving towards each other on the road.

Car P passes point A with a speed of 10 m s^{-1} and has a constant acceleration of 2 m s^{-2} as it heads for point C.

At the same instant, car Q passes point C heading for point A. As it passes point C, its speed is 5 m s^{-1} and its constant acceleration is 4 m s^{-2}.

P and Q pass each other at B six seconds later.
When P reaches B, it immediately starts decelerating at a constant rate and comes to rest at C.

Find **(i)** the speed of car P and the speed of car Q at B **(20)**

P: $u = 10,\ a = 2,\ t = 6$

$v = u + at$

$v_P = (10) + (2)(6)$

$v_P = 22 \text{ ms}^{-1}$ ... (10m)

Q: $u = 5,\ a = 4,\ t = 6$

$v = u + at$

$v_Q = (5) + (4)(6)$

$v_Q = 29 \text{ ms}^{-1}$ ... (10m)

(ii) $|AB|$ and $|BC|$, the distances P and Q have moved until they pass each other **(20)**

P: $s = ut + \frac{1}{2}at^2$

$s_P = (10)(6) + \frac{1}{2}(2)(6)^2$

$s_P = 96 \text{ m} = |AB|$ ... (10m)

Q: $s = ut + \frac{1}{2}at^2$

$s_Q = (5)(6) + \frac{1}{2}(4)(6)^2$

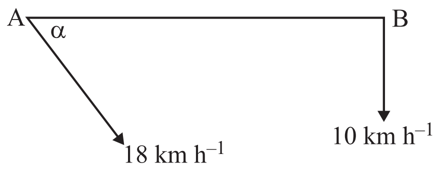
$s_Q = 102 \text{ m} = |BC|$ ... (10m)

(iii) the deceleration of P, correct to one decimal place, as it travels from B to C. **(10)**

$$P: \quad u = 22,\ v = 0,\ s = 102$$
$$v^2 = u^2 + 2as$$
$$0^2 = 22^2 + 2(a)(102)$$
$$-484 = 204a$$
$$a = -2\cdot 4$$
$$\text{deceleration} = 2\cdot 4\ \text{ms}^{-2} \qquad \text{... (10m)}$$

2. Ship A is positioned 60 km due west of ship B.

A is moving at a constant speed of 18 km h^{-1} in the direction east α south, where $\tan\alpha = \frac{4}{3}$.



B is moving due south at a constant speed of 10 km h^{-1}.

Find **(i)** the velocity of A in terms of $\vec{i}$ and $\vec{j}$ **(10)**

$\vec{v}_A = 18\cos\alpha\,\vec{i} - 18\sin\alpha\,\vec{j}$ ... (5m)

$\vec{v}_A = 10\cdot8\vec{i} - 14\cdot4\vec{j}$ ... (5m)

(ii) the velocity of B in terms of $\vec{i}$ and $\vec{j}$ **(10)**

$\vec{v}_B = -10\vec{j}$ ... (10m)

(iii) the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$ **(10)**

$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$ ... (5m)

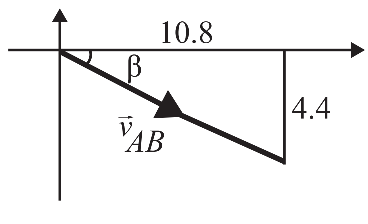
$= \left(10\cdot8\vec{i} - 14\cdot4\vec{j}\right) - \left(-10\vec{j}\right)$

$= 10\cdot8\vec{i} - 4\cdot4\vec{j}$ ... (5m)

(iv) the shortest distance between A and B in the subsequent motion. **(20)**

$\tan\beta = \frac{4\cdot4}{10\cdot8} = 0\cdot4074$ ... (5m)

$\beta = 22\cdot17°$

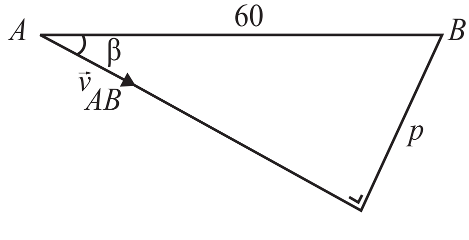


Shortest distance $= p$,

$= 60\sin\beta$ ... (5m)

$= 60\sin 22\cdot17°$ ... (5m)

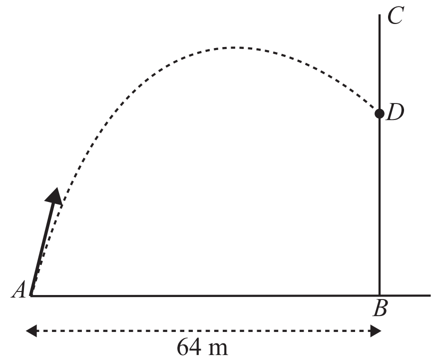
$= 22\cdot64$ km ... (5m)



3. A particle is projected with initial velocity $16\vec{i} + 25\vec{j}$ m s^{-1} from point A on horizontal ground.

$[BC]$ is a vertical wall, with B at the same horizontal ground as A and $|AB| = 64$ m.

The particle subsequently strikes the vertical wall at the point D.



(i) Find the time taken by the particle to reach the maximum height. **(10)**

At max. height: $v_y = 0$ ... (5m)

$25 - 10t = 0$

$t = 2 \cdot 5$ ... (5m)

(ii) Show that the particle strikes the vertical wall after achieving its maximum height. **(10)**

Time to strike wall: $s_x = 64$ ... (5m)

$16t = 64$

$t = 4$ s ... (5m)

as the time to striking the wall is greater than the time to max. height, the particle strikes the wall after reaching max. height.

(iii) Find $|BD|$, the height of the point on the wall at which the particle strikes. **(10)**

$|BD| = s_y$ at $t = 4$ ... (5m)

$= 25(4) - 5(4)^2$

$= 20$ m ... (5m)

(iv) Find the speed with which the particle strikes the wall. **(10)**

at $t = 4$,

$v_x = 16$ and $v_y = 25 - 10(4) = -15$ ... (5m)

Thus speed $= \sqrt{16^2 + (-15)^2}$

$= 21 \cdot 93$ ms^{-1} ... (5m)

(v) If the wall is removed and the particle fired again from A with the same initial velocity, find its range on the horizontal plane. **(10)**

Time of flight: $s_y = 0$

$25t - 5t^2 = 0$

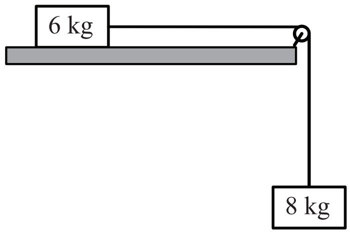
$t = 5$ ... (5m)

Then range $= s_x$ at $t = 5$

$= 16(5)$

$= 80$ m ... (5m)

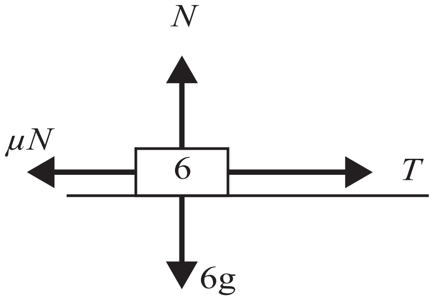
4. A particle of mass 6 kg is connected to a particle of mass 8 kg by a taut, light, inextensible string which passes over a smooth light pulley at the edge of a rough horizontal table.

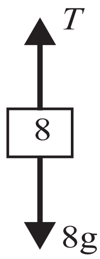


The coefficient of friction between the 6 kg mass and the table is μ.

The system is released from rest.

(i) Show on separate diagrams the forces acting on each particle. **(10)**





... (5m) +... (5m)

(ii) If $\mu = \frac{3}{4}$, find the common acceleration of the particles. **(15)**

8 kg mass: $8a = 8g - T$

$8a = 80 - T$ **... 1** ... (5m)

6 kg mass: $N = 6g$

$6a = T - \mu N$

Then $6a = T - \frac{3}{4}(6g)$

$6a = T - \frac{9}{2}g$

$6a = T - 45$ **... 2** ... (5m)

Adding 1 and 2: $14a = 35$

$a = 2\cdot5 \text{ ms}^{-2}$ ... (5m)

(iii) Find the tension in the string. **(5)**

From 2: $6(2\cdot5) = T - 45$

$T = 60 \text{ N}$ ... (5m)

(iv) Calculate the speed of the 6 kg mass as it reaches the pulley, if it starts 1 metre away from the pulley. **(5)**

From 6 kg mass: $u = 0,\ a = 2\cdot5,\ s = 1$

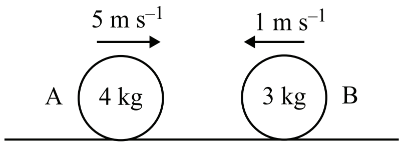
Then
$$v^2 = u^2 + 2as$$
$$v^2 = 0^2 + 2(2\cdot5)(1)$$
$$v^2 = 5$$
$$v = \sqrt{5} = 2\cdot24\ \text{ms}^{-1}$$ ... (5m)

(v) If, instead, the common acceleration of the particles is $\frac{25}{7}$ m s^{-2}, calculate the value of μ. **(15)**

8 kg particle:
$$8\left(\frac{25}{7}\right) = 80 - T$$
$$200 = 560 - 7T$$
$$7T = 360$$
$$T = \frac{360}{7}$$ ... (5m)

6 kg particle:
$$6\left(\frac{25}{7}\right) = \frac{360}{7} - \mu(60)$$ ... (5m)
$$150 = 360 - 420\mu$$
$$420\mu = 210$$
$$\mu = \frac{1}{2}$$ ... (5m)

5. A smooth sphere A, of mass 4 kg, collides directly with another smooth sphere B, of mass 3 kg, on a smooth horizontal table.



Before impact A and B are moving in opposite directions with speeds of 5 m s^{-1} and 1 m s^{-1}, respectively.

The coefficient of restitution for the collision is $\frac{1}{6}$.

Find **(i)** the speed of A and the speed of B after the collision **(30)**

PCM: $4(5)+3(-1)=4v_1+3v_2$ ... (10m)

$4v_1+3v_2=17$

NEL: $v_1-v_2=-\frac{1}{6}(5+1)$ ... (10m)

$v_1-v_2=-1$

Solving: $4v_1+3v_2=17$

$\underline{3v_1-3v_2=-3}$

$7v_1=14$

$v_1=2\text{ ms}^{-1}$

And $v_2=3\text{ ms}^{-1}$ ... (10m)

(ii) the loss in kinetic energy due to the collision **(10)**

KE before $=\frac{1}{2}(4)(5)^2+\frac{1}{2}(3)(-1)^2=51.5$ ... (5m)

KE after $=\frac{1}{2}(4)(2)^2+\frac{1}{2}(3)(3)^2=21.5$

Loss in KE = KE before – KE after

$=51\cdot5-21\cdot5=30\text{ J}$ ... (5m)

(iii) the magnitude of the impulse imparted to B due to the collision. **(10)**

$I=\left|(3)(3)-(3)(-1)\right|$

$=12\text{ Ns}$ ... (10m)

6. **(a)** Particles of weight 5 N, 2 N, 3 N and 6 N are placed at the points $(1, q)$, $(5, p)$, $(7, -5)$ and (p, p), respectively.

The co-ordinates of the centre of gravity of the system are (3, 1).

Find **(i)** the value of p **(15)**

$$3 = \frac{5(1)+2(5)+3(7)+6(p)}{16} \qquad \text{... (10m)}$$

$$48 = 36 + 6p$$

$$6p = 12$$

$$p = 2 \qquad \text{... (5m)}$$

(ii) the value of q. **(15)**

$$1 = \frac{5(q)+2(p)+3(-5)+6(p)}{16} \qquad \text{... (10m)}$$

$$16 = 5q + 8p - 15$$

$$31 = 5q + 16$$

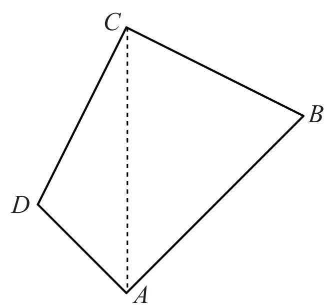
$$5q = 15$$

$$q = 3 \qquad \text{... (5m)}$$

(b) A quadrilateral lamina has vertices A, B, C and D.

The co-ordinates of the vertices are $A(2, 1)$, $B(8, 7)$, $C(2, 10)$ and $D(-1, 4)$.

Find the co-ordinates of the centre of gravity of the lamina. **(20)**



area		centre of gravity
ΔABC:	$\frac{1}{2}(9)(6) = 27$	$\left(\frac{2+8+2}{3}, \frac{1+7+10}{3}\right) = (4, 6)$ ... (5m)
ΔACD:	$\frac{1}{2}(9)(3) = 13 \cdot 5$	$\left(\frac{2+2-1}{3}, \frac{1+10+4}{3}\right) = (1, 5)$ ... (5m)

Further answers overleaf

Lamina: $40 \cdot 5$ (x, y)

Then $(40 \cdot 5)(x) = (27)(4) + (13 \cdot 5)(1)$

$x = 3$ ... (5m)

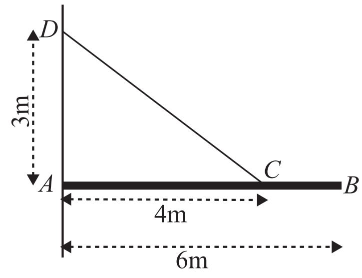
And $(40 \cdot 5)(y) = (27)(6) + (13 \cdot 5)(5)$

$y = \frac{17}{3}$ ... (5m)

The co-ordinates of the centre of gravity are $\left(3, \frac{17}{3}\right)$

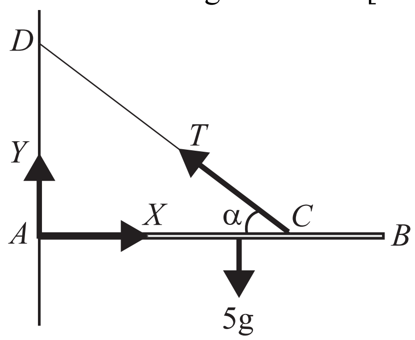
7. A uniform rod [AB], of length 6 m and mass 5 kg, is smoothly hinged at end A to a vertical wall.

The rod is held in a horizontal position by a light string attached to point C on the rod and to the point D on the wall vertically above A.



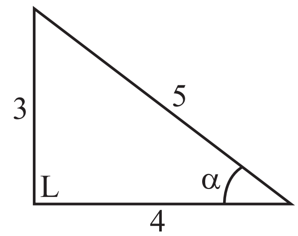
$|AC| = 4$ m and $|AD| = 3$m.

(i) Show on a diagram all the forces acting on the rod [AB]. **(10)**



... (10m)

(ii) Write down the two equations that arise from resolving the forces horizontally and vertically. **(10)**



$\uparrow = \downarrow$ $\quad T\sin\alpha + Y = 5g$ ... (5m)

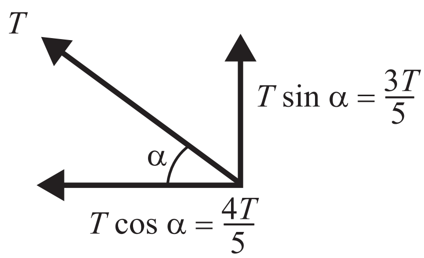
$$\frac{3T}{5} + Y = 50$$

$$3T + 5Y = 250 \quad \textbf{... 1}$$

$\leftarrow = \rightarrow$ $\quad T\cos\alpha = X$ ... (5m)

$$\frac{4T}{5} = X$$

$$4T = 5X \quad \textbf{... 2}$$



(iii) Write down the equation that arises from taking moments about the point C. **(5)**

Taking moments about C

$$5g(1) = Y(4)$$ ... (5m)

$$50 = 4Y$$

$$Y = 12\cdot 5$$

(iv) Find the tension in the string. **(10)**

From 1: $3T + 5(12 \cdot 5) = 250$ ... (5m)

$3T = 187 \cdot 5$

$T = 62 \cdot 5$ N ... (5m)

(v) Find the magnitude of the reaction at the hinge, A. **(15)**

From 2: $4(62 \cdot 5) = 5X$ ... (5m)

$250 = 5X$

$X = 50$ ... (5m)

Then $R = \sqrt{50^2 + 12 \cdot 5^2}$

$R = 51 \cdot 54$ N ... (5m)

8. **(a)** A particle describes a horizontal circle of radius r metres with uniform angular velocity ω radians per second.
Its speed is 24 m s^{-1} and its acceleration is 144 m s^{-2}.

Find **(i)** the value of ω **(15)**

Speed $= 24$

$r\omega = 24$ ... (5m)

And acceleration = 144

$r\omega^2 = 144$ ... (5m)

Then

$(r\omega)\omega = 144$

$24\omega = 144$

$\omega = 6 \text{ rads}^{-1}$ ... (5m)

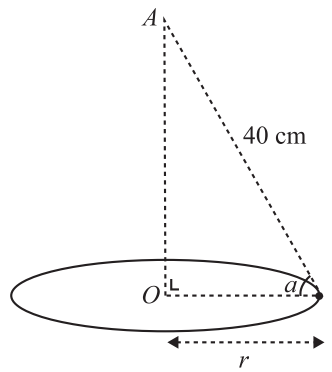
(ii) the value of r. **(5)**

Then

$r(6) = 24$

$r = 4 \text{ m}$ ... (5m)

(b) A particle of mass 2 kg describes horizontal circles with centre O and radius r on a smooth horizontal table. The particle is attached to a string of length 40 cm. The other end of the string is attached to a point A vertically above O. The string makes an angle α with the horizontal, where $\tan\alpha = \frac{4}{3}$.
The angular velocity of the particle is 3 radians per second.



Find **(i)** the value of r **(10)**

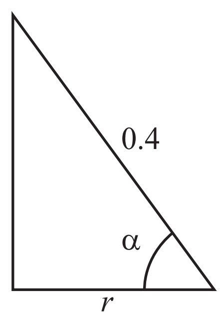
$\tan\alpha = \frac{4}{3}$

$\sin\alpha = \frac{4}{5}, \ \cos\alpha = \frac{3}{5}$ ... (5m)

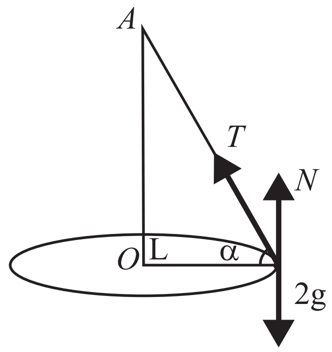
Then $r = 0\cdot 4\cos\alpha$

$r = 0\cdot 4\left(\frac{3}{5}\right)$

$r = 0\cdot 24 \text{ m}$ ... (5m)



(ii) the tension in the string **(10)**



Circular motion:

$T\cos\alpha = mr\omega^2$ ... (5m)

$T\left(\frac{3}{5}\right) = (2)(0\cdot 24)(3)^2$

$T = 7\cdot 2$ N ... (5m)

(iii) the reaction force between the table and the particle. **(10)**

Vertically:

$T\sin\alpha + N = 2g$ ... (5m)

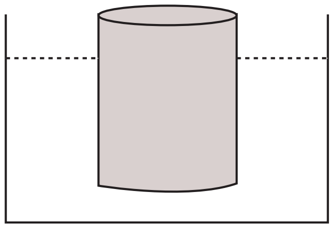
$(7\cdot 2)\left(\frac{4}{5}\right) + N = 20$

$N = 14\cdot 24$ N ... (5m)

9. **(a)** A right circular cylinder has a base of radius 8 cm and a height of 24 cm.

It floats in a liquid of relative density 0.8 with its axis vertical.

75% of the cylinder lies below the surface of the liquid.



Find the weight of the cylinder, correct to the nearest Newton. **(20)**

$W = B$ ... (5m)

$W =$ weight of displaced liquid

$W = 1000 sVg$ ... (5m)

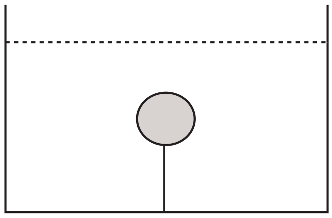
$W = 1000(0\cdot8)\left[\frac{3}{4}\pi(0\cdot08)^2(0\cdot24)\right](10)$ ... (5m)

$W = 29$ N ... (5m)

(b) A solid sphere of radius 36 cm has a relative density of 1.1.

It is completely immersed in a tank of liquid of relative density 1.5.

The sphere is held at rest by a light inextensible vertical string which is attached to the base of the tank.



Find the tension in the string.

[Density of water = 1000 kg m^{-3}]. **(30)**

$B = 1500\left[\frac{4}{3}\pi(0\cdot36)^3\right](10)$ ... (5m)

$B = 2931\cdot48$ ... (5m)

And

$W = 1100\left[\frac{4}{3}\pi(0\cdot36)^3\right](10)$ ... (5m)

$W = 2149\cdot75$ ... (5m)

Then

$W + T = B$ ... (5m)

$2149\cdot75 + T = 2931\cdot48$

$T = 781\cdot73$ N ... (5m)

Notes:

Pre-Leaving Certificate Examination, 2013

Applied Mathematics

Higher Level
Marking Scheme (300 marks)

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Blunders	- mathematical errors	B(–3)
Misreading	- if not serious	M(–1)

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- award the attempt mark only.

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3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

4. Number the grid on each script 1 to 10 in numerical order, not the order of answering.

5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

1. **(a)** A particle moving in a straight line with constant acceleration passes three points K, L and M, and has speeds u and $u\sqrt{71}$ at K and M, respectively.

(i) If L is the midpoint of $[KM]$, express the speed of the particle at L in terms of u. **(15)**

$[KL]$ $u = u$ **LM3:** $v^2 = u^2 + 2as$

$v = v$ $v^2 = u^2 + 2ax$

$a = a$ $2ax = v^2 - u^2$ **… 1** ... (5m)

$s = x$

$t = t_1$

$[LM]$ $u = v$ **LM3:** $v^2 = u^2 + 2as$

$v = u\sqrt{71}$ $(u\sqrt{71})^2 = v^2 + 2ax$

$a = a$ $71u^2 = v^2 + 2ax$

$s = x$ $2ax = 71u^2 - v^2$ **… 2** ... (5m)

$t = 7$

From **1** and **2**,

$v^2 - u^2 = 71u^2 - v^2$

$2v^2 = 72u^2$

$v^2 = 36u^2$

$v = 6u$ ... (5m)

(ii) If the time taken by the particle to travel from L to M is 7 seconds, express the time taken to travel from K to L in the form $\sqrt{a} + b$ seconds, where $a, b \in \mathbb{N}$. **(10)**

$[LM]$ **LM1:** $v = u + at$

$u\sqrt{71} = 6u + a(7)$

$7a = \left(\sqrt{71} - 6\right)u$

$a = \dfrac{\left(\sqrt{71} - 6\right)u}{7}$ ... (5m)

$[KL]$ **LM1:** $v = u + at$

$6u = u + at_1$

$\dfrac{\left(\sqrt{71} - 6\right)u}{7} t_1 = 5u$

$t_1 = \dfrac{35}{\sqrt{71} - 6} \times \dfrac{\sqrt{71} + 6}{\sqrt{71} + 6}$

$t_1 = \dfrac{35\left(\sqrt{71} + 6\right)}{35}$

$t_1 = \sqrt{71} + 6.$ ... (5m)

(b) Two particles, A and B, travelling in the same direction, pass the point P at the same instant. A is moving with velocity u_1 and retardation f_1 and B with velocity $u_2 (< u_1)$ and retardation $f_2 (< f_1)$.

(i) Show that B will overtake A after a time $\dfrac{2(u_1 - u_2)}{f_1 - f_2}$. **(10)**

The particles are level when

$$s_1 = s_2$$

$$u_1 t - \frac{1}{2} f_1 t^2 = u_2 t - \frac{1}{2} f_2 t^2 \qquad \text{... (5m)}$$

$$u_1 t - u_2 t = \frac{1}{2} f_1 t^2 - \frac{1}{2} f_2 t^2$$

$$(u_1 - u_2) t = \frac{1}{2}(f_1 - f_2) t^2$$

$$2(u_1 - u_2) = (f_1 - f_2) t$$

$$t = \frac{2(u_1 - u_2)}{f_1 - f_2}. \qquad \text{... (5m)}$$

(ii) Find, in terms of u_1, u_2, f_1 and f_2, the greatest lead that A has over B during this motion. **(15)**

The greatest lead occurs when

$$v_1 = v_2$$

$$u_1 - f_1 t = u_2 - f_2 t$$

$$(u_1 - u_2) = f_1 t - f_2 t$$

$$(u_1 - u_2) = (f_1 - f_2) t$$

$$t = \frac{u_1 - u_2}{f_1 - f_2} \qquad \text{... (5m)}$$

The greatest lead is $s_1 - s_2$ when $t = \dfrac{u_1 - u_2}{f_1 - f_2}$

$$\text{greatest lead} = \left(u_1 t - \frac{1}{2} f_1 t^2 \right) - \left(u_2 t - \frac{1}{2} f_2 t^2 \right) \qquad \text{... (5m)}$$

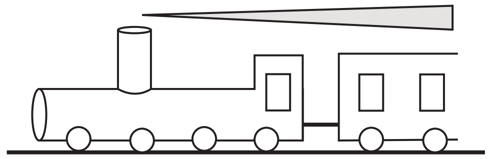
$$= (u_1 - u_2) t - \frac{1}{2}(f_1 - f_2) t^2$$

$$= (u_1 - u_2)\frac{u_1 - u_2}{f_1 - f_2} - \frac{1}{2}(f_1 - f_2)\frac{(u_1 - u_2)^2}{(f_1 - f_2)^2}$$

$$= \frac{(u_1 - u_2)^2}{f_1 - f_2} - \frac{1}{2}\frac{(u_1 - u_2)^2}{f_1 - f_2}$$

$$= \frac{(u_1 - u_2)^2}{2(f_1 - f_2)}. \qquad \text{... (5m)}$$

2. **(a)** An old train, drawn by a steam engine, is moving due north at 20 km h^{-1}. The wind is blowing at 10 km h^{-1} from the southeast.



As the smoke leaves the engine, it immediately takes the velocity of the wind.

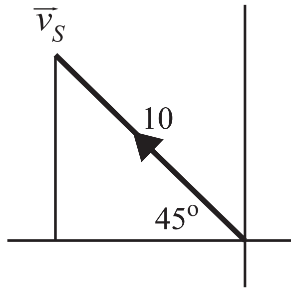
Find the angle, in degrees, that the smoke trail makes with the engine correct to two places of decimals. **(20)**

When a smoke particle, S, leaves the engine, it immediately acquires the velocity of the wind, W

Thus $\vec{v}_S = \vec{v}_W$

$\vec{v}_S = -(10\cos 45°)\vec{i} + (10\sin 45°)\vec{j}$

$\vec{v}_S = -7\cdot071\vec{i} + 7\cdot071\vec{j}$ ... (5m)



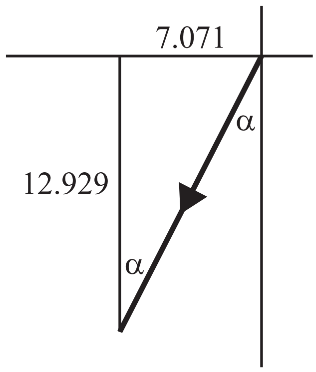
Also, the velocity of the train, $\vec{v}_T$, is given by $\vec{v}_T = 20\vec{j}$

Then, $\vec{v}_{ST}$, the velocity of the smoke relative to the train is given by $\vec{v}_{ST} = \vec{v}_S - \vec{v}_T$

$= \left(-7\cdot071\vec{i} + 7\cdot071\vec{j}\right) - (20\vec{j})$

$= -7\cdot071\vec{i} - 12\cdot929\vec{j}$ ... (5m)

The angle between the smoke trail and the train is the angle between this vector and north/south, *i.e.* the $\vec{j}$ direction. This is the angle α shown in the diagram. ... (5m)



$\tan\alpha = \dfrac{7\cdot071}{12\cdot929}$

$\alpha = \tan^{-1} 0\cdot5469$

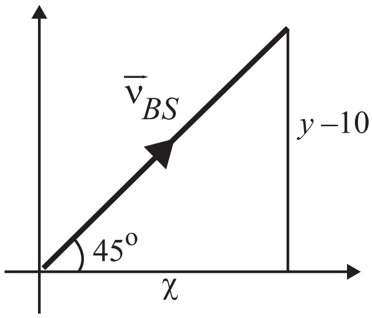
$\alpha = 28\cdot67°$ ... (5m)

in degrees, correct to two decimal places.

(b) Relative to a ship which is travelling due north at a speed of 10 km h^{-1}, the velocity of a speedboat is in the direction North 45° East. Relative to a second ship which is travelling due south at 10 km h^{-1}, the velocity of the speedboat is in the direction North 30° East. **(30)**

Find the true speed and direction of the speedboat.

Let the velocity of the speedboat be
$\vec{v}_B = x\vec{i} + y\vec{j}$.



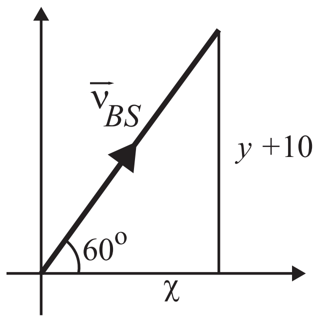
[1] $\vec{v}_S = 10\vec{j}$

$\vec{v}_{BS} = \vec{v}_B - \vec{v}_S$
$= (x\vec{i} + y\vec{j}) - (10\vec{j})$
$= x\vec{i} + (y-10)\vec{j}$ ... (5m)

This vector points north-east

$$\tan 45° = \frac{y-10}{x}$$
$$1 = \frac{y-10}{x}$$
$x = y - 10$
$x - y = -10$ **... 1** ... (5m)

[2] $\vec{v}_S = -10\vec{j}$



$\vec{v}_{BS} = \vec{v}_B - \vec{v}_S$
$= (x\vec{i} + y\vec{j}) - (-10\vec{j})$
$= x\vec{i} + (y+10)\vec{j}$ ... (5m)

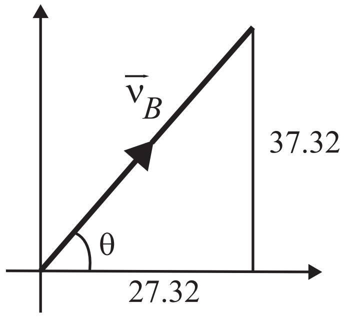
This vector points $30°$ east or north, or $60°$ north of east.

$$\tan 60° = \frac{y+10}{x}$$
$$1\cdot 732 = \frac{y+10}{x}$$
$1\cdot 732x = y + 10$
$1\cdot 732x - y = 10$ **... 2** ... (5m)

Solving **1** and **2**,

1$\times -1$: $-x + y = 10$
2: $1\cdot 732x - y = 10$
$0\cdot 732x = 20$
$x = 27\cdot 32$
1: $27\cdot 32 - y = -10$
$y = 37\cdot 32$

Thus $\vec{v}_B = 27\cdot 32\vec{i} + 37\cdot 32\vec{j}$ ... (5m)

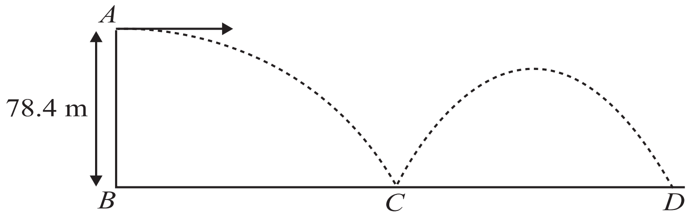


Then $|\vec{v}_B| = \sqrt{27\cdot 32^2 + 37\cdot 32^2}$
$= 46\cdot 25$ km/h

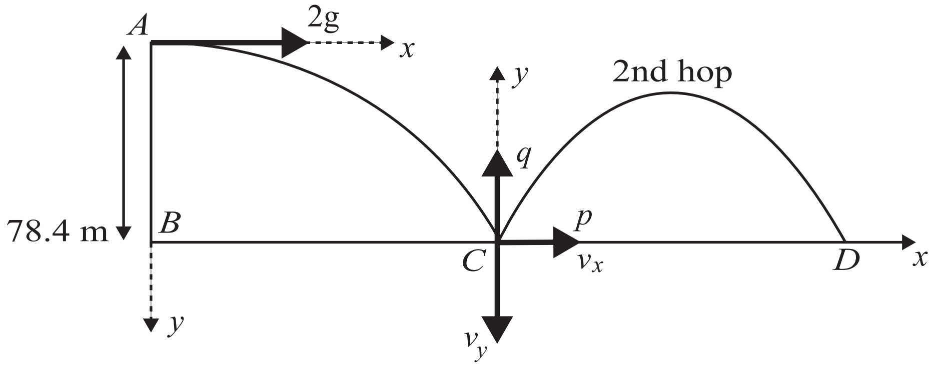
Also, $\tan\theta = \dfrac{37\cdot 32}{27\cdot 32}$
$= 1\cdot 366$
$\theta = 53\cdot 8°$

The direction is $53\cdot 8°$ north of east ... (5m)

3. **(a)** A particle is projected horizontally from a point A, which is 78.4 m vertically above a point B on horizontal ground. Its initial speed is $2g$ m s^{-1}. It first strikes the horizontal plane at C, and next strikes the horizontal plane at D.



If $|BC| = |CD|$ find the coefficient of restitution, e. **(25)**



(For the first hop, A is the origin and the x and y axes are as shown. For the second hop, C is the origin and the axes are as shown.)

1st hop:

$u \to 2g$

$\alpha \to 0°$

Equations (y direction down):

$v_x = u\cos\alpha \qquad v_y = u\sin\alpha + gt$

$v_x = (2g)\cos 0° \quad v_y = (2g)\sin 0° + gt$

$v_x = 2g \qquad v_y = gt$

And

$s_x = 2gt \qquad s_y = \frac{1}{2}gt^2$

Strikes the horizontal plane:

$s_y = 78\cdot 4$

$4\cdot 9t^2 = 78\cdot 4$ ... (5m)

$t^2 = 16$

$t = 4$

When $t = 4$:

$|BC| = s_x$ at $t = 4$

$|BC| = 2g(4)$

$|BC| = 8g$ ... (5m)

And $v_x = 2g$

$v_y = g(4) = 4g$

Collision Result:
(The fixed surface is the horizontal plane.)

$\parallel$: $p = v_x = 2g$

$\perp$: $q = -ev_y = -e(4g) = -4eg$ ... (5m)

Change y direction:

$q \to -q = 4eg$

2nd hop:

$u\cos\alpha \to p = 2g$

$u\sin\alpha \to q = 4eg$

Equations (y direction up):

$v_x = p$ $\qquad v_y = q - gt$

$v_x = 2g$ $\qquad v_y = 4eg - gt$

And

$s_x = 2gt$ $\qquad s_y = 4egt - \frac{1}{2}gt^2$

TOF: $s_y = 0$

$4egt - \frac{1}{2}gt^2 = 0$ ... (5m)

$8eg = gt$

$t = 8e$

Range: $|CD| = s_x$ at TOF

$|CD| = 2g(8e)$

$|CD| = 16ge$

Given: $|CD| = |BC|$

$16ge = 8g$

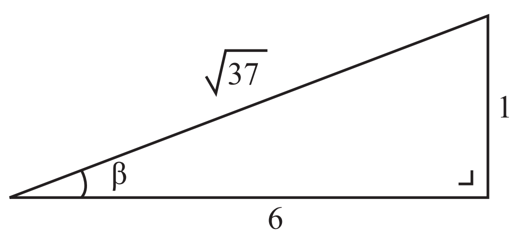
$e = \frac{1}{2}.$ ... (5m)

(b) A particle is projected from a point P, up a plane inclined at an angle $\tan^{-1}\frac{1}{6}$ to the horizontal. The direction of projection makes an angle α with the inclined plane. (The plane of projection is vertical and contains the line of greatest slope.)

If the particle strikes the inclined plane while travelling horizontally, show that $\tan\alpha = \frac{3}{19}$. **(25)**

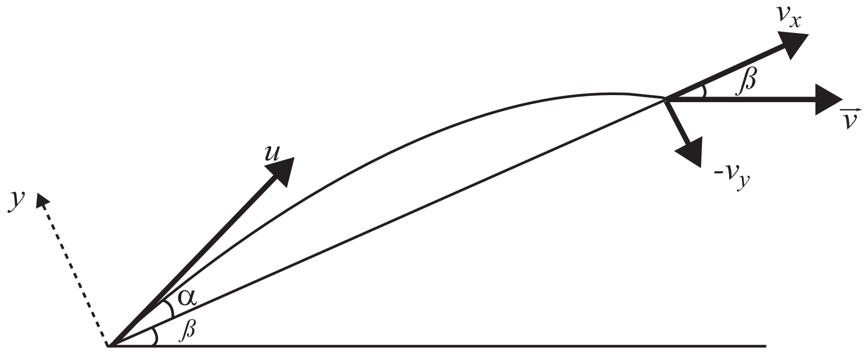
$\beta = \tan^{-1}\frac{1}{6}$

$\tan\beta = \frac{1}{6}$



$$\cos\beta = \frac{6}{\sqrt{37}}, \quad \sin\beta = \frac{1}{\sqrt{37}}$$

$$\alpha \to \alpha$$
$$u \to u$$



Equations:

$$v_x = u\cos\alpha - g\sin\beta t \qquad v_y = u\sin\alpha - g\cos\beta t$$

$$v_x = u\cos\alpha - g\left(\frac{1}{\sqrt{37}}\right)t \qquad v_y = u\sin\alpha - g\left(\frac{6}{\sqrt{37}}\right)t$$

$$v_x = u\cos\alpha - \frac{gt}{\sqrt{37}} \qquad v_y = u\sin\alpha - \frac{6gt}{\sqrt{37}}$$

And

$$s_x = u\cos\alpha t - \frac{gt^2}{2\sqrt{37}} \qquad s_y = u\sin\alpha t - \frac{3gt^2}{\sqrt{37}}$$

Two events occur at the same time:

TOF: $s_y = 0$ ... (5m)

$$u\sin\alpha t - \frac{3gt^2}{\sqrt{37}} = 0$$

$$\sqrt{37}u\sin\alpha = 3gt$$

$$t = \frac{\sqrt{37}u\sin\alpha}{3g} \quad \text{... } \mathbf{1} \qquad \text{... (5m)}$$

Travelling horizontally:

$$\tan\beta = \frac{-v_y}{v_x} \qquad \text{... (5m)}$$

$$\frac{1}{6} = \frac{-v_y}{v_x}$$

$$v_x = -6v_y$$

$$v_x + 6v_y = 0$$

$$\left(u\cos\alpha - \frac{gt}{\sqrt{37}}\right) + 6\left(u\sin\alpha - \frac{6gt}{\sqrt{37}}\right) = 0$$

$$u\cos\alpha + 6u\sin\alpha = \frac{37gt}{\sqrt{37}}$$

$$u(\cos\alpha + 6\sin\alpha) = \sqrt{37}gt$$

$$t = \frac{u(\cos\alpha + 6\sin\alpha)}{\sqrt{37}g} \qquad \text{... } \mathbf{2} \qquad \text{... (5m)}$$

Comparing 1 and 2:

$$\frac{\sqrt{37}u\sin\alpha}{3g} = \frac{u(\cos\alpha + 6\sin\alpha)}{\sqrt{37}g}$$

$$37\sin\alpha = 3(\cos\alpha + 6\sin\alpha)$$

$$37\sin\alpha = 3\cos\alpha + 18\sin\alpha$$

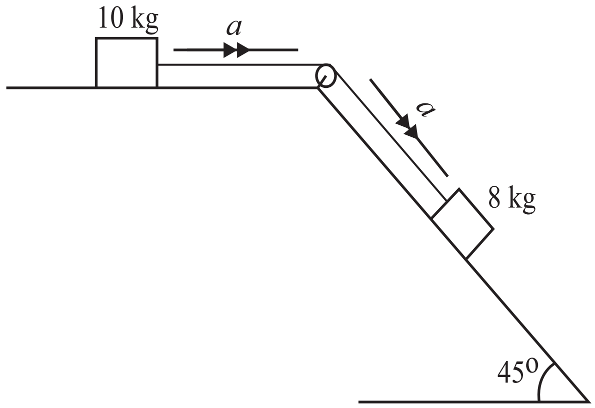
$$19\sin\alpha = 3\cos\alpha$$

$$\tan\alpha = \frac{3}{19}. \qquad \text{... (5m)}$$

4. **(a)** A particle of mass 10 kg is connected by a light inextensible string passing over a light smooth pulley at the edge to another particle of mass 8 kg on a plane inclined at 45° to the horizontal.

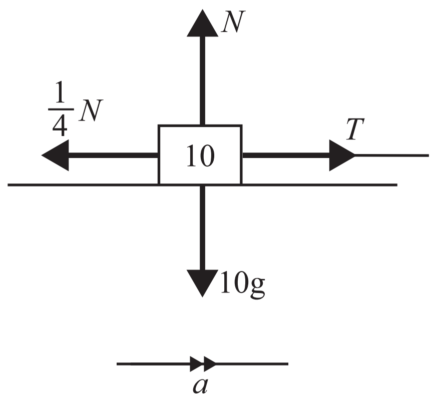
The coefficient of friction between the particles and the planes is $\frac{1}{4}$.

Find **(i)** the acceleration of the particles
(ii) the tension in the string, correct to two places of decimals. **(25)**



Let the common acceleration of the particles be a.

10 kg mass:



$\updownarrow$: Statics

$N = 10g$ ... **1**

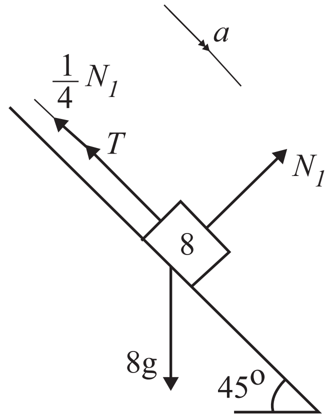
$\leftrightarrow$: Newton's 2nd law

$$10a = T - \frac{1}{4}N \qquad \text{... (5m)}$$

1 : $10a = T - \frac{1}{4}(10g)$

$$10a = T - \frac{5}{2}g \qquad \ldots \mathbf{2} \qquad \text{... (5m)}$$

8 kg mass:



(Resolve the forces parallel and perpendicular to the inclined plane.)

$\nearrow\!\!\!\swarrow$ Statics

$N_1 = 4\sqrt{2}g$ … **3**

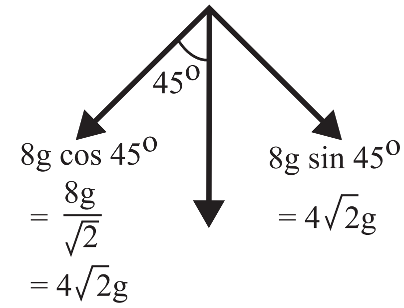
Newton's 2nd law

$$8a = 4\sqrt{2}g - T - \frac{1}{4}N_1 \qquad \text{... (5m)}$$

3 : $8a = 4\sqrt{2}g - T - \frac{1}{4}(4\sqrt{2}g)$

$8a = 4\sqrt{2}g - T - \sqrt{2}g$

$8a = 3\sqrt{2}g - T$ … **4** ... (5m)



Solving **2** and **4**:

2 : $10a = T - \frac{5}{2}g$

4 : $8a = 3\sqrt{2}g - T$

$18a = 3\sqrt{2}g - \frac{5}{2}g$

$18a = 17 \cdot 08$

$a = 0 \cdot 95 \text{ m/s}^2$

4 : $8(0 \cdot 95) = 3\sqrt{2}g - T$

$T = 3\sqrt{2}g - 7 \cdot 6$

$T = 33 \cdot 98$ N. ... (5m)

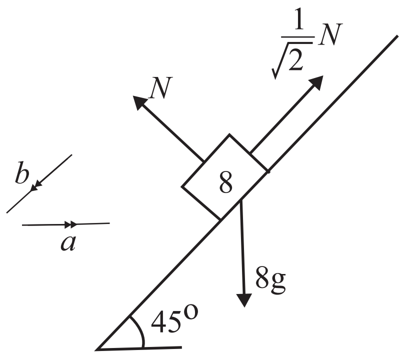
(b) A wedge of mass 10 kg sits on a smooth horizontal plane. A particle of mass 8 kg sits on a face of the wedge inclined at an angle of 45° to the horizontal. The coefficient of friction between the particle and the wedge is $\frac{1}{\sqrt{2}}$.

The system is released from rest.

Find **(i)** the acceleration of the wedge
(ii) the acceleration of the particle relative to the wedge. **(25)**

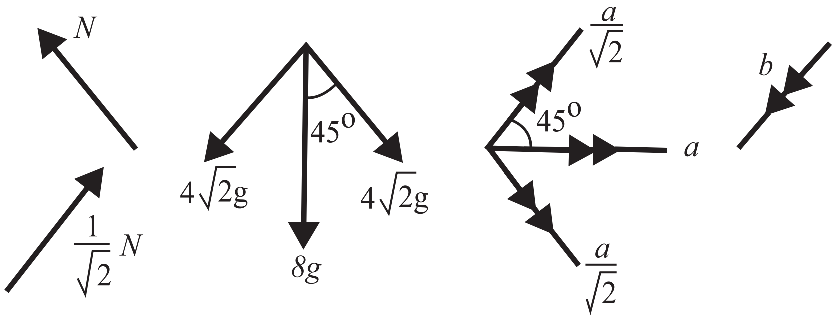
8 kg particle

(We must remember to include the friction force, $\frac{1}{\sqrt{2}}N$, acting up the inclined face of the wedge.)



Further answers overleaf

Forces Accelerations



⤡ Newton's 2nd Law

$$8\left(\frac{a}{\sqrt{2}}\right) = 4\sqrt{2}g - N \qquad \text{... (5m)}$$

$$4\sqrt{2}a = 4\sqrt{2}g - N \qquad \textbf{... 1}$$

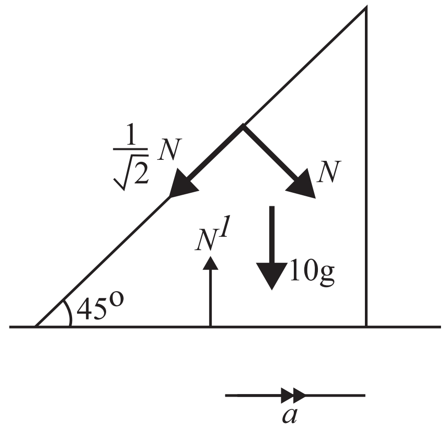
⤢ Newton's 2nd Law

$$8\left(b - \frac{a}{\sqrt{2}}\right) = 4\sqrt{2}g - \frac{1}{\sqrt{2}}N \qquad \text{... (5m)}$$

$$8b - 4\sqrt{2}a = 4\sqrt{2}g - \frac{1}{\sqrt{2}}N \qquad \textbf{... 2}$$

Wedge

(The contact between the wedge and the ground is smooth. Hence we only need the horizontal equation.)

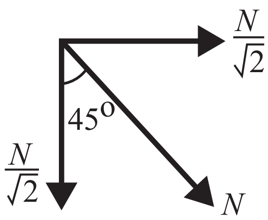


↔ : Newton's 2nd law

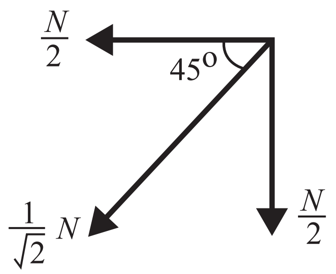
$$10a = \frac{N}{\sqrt{2}} - \frac{N}{2} \qquad \text{... (5m)}$$

$$10a = \frac{\sqrt{2}N}{2} - \frac{N}{2}$$

$$10a = \frac{(\sqrt{2}-1)}{2}N \qquad \textbf{... 3}$$



3 : $N = \dfrac{20}{\sqrt{2}-1}a$



1 : $4\sqrt{2}a = 4\sqrt{2}g - \dfrac{20}{\sqrt{2}-1}a$

$4\sqrt{2}\left(\sqrt{2}-1\right)a = 4\sqrt{2}\left(\sqrt{2}-1\right)g - 20a$

$\sqrt{2}\left(\sqrt{2}-1\right)a = \sqrt{2}\left(\sqrt{2}-1\right)g - 5a$

$\left(2-\sqrt{2}+5\right)a = \left(2-\sqrt{2}\right)g$

$a = \dfrac{2-\sqrt{2}}{7-\sqrt{2}}g$ ... (5m)

$a = 1\cdot 03 \text{ m/s}^2$

3 : $N = \dfrac{20}{\sqrt{2}-1}(1\cdot 03)$

$N = 49\cdot 73$

2 : $8b - 5\cdot 66(1\cdot 03) = 5\cdot 66(9\cdot 8) - 0\cdot 71(49\cdot 73)$

$8b - 5\cdot 83 = 20\cdot 16$

$8b = 25\cdot 99$

$b = 3\cdot 25 \text{ m/s}^2.$ ... (5m)

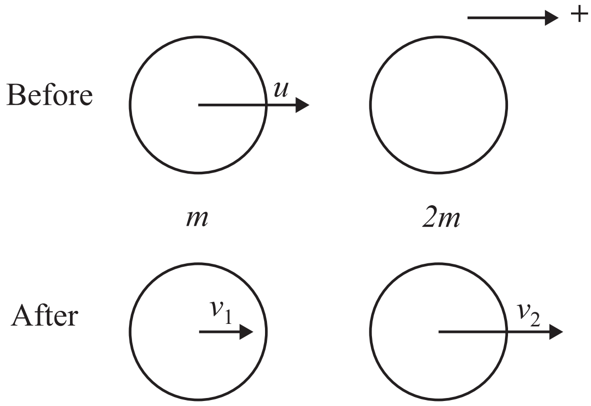
5. **(a)** A smooth sphere A of mass m collides directly with a smooth sphere B of mass $2m$ which is at rest. During the impact, $\frac{10}{27}$ of the original kinetic energy is lost.

Find the coefficient of restitution between the spheres. **(25)**

PCM: $mv_1 + 2mv_2 = mu + 2m(0)$

$v_1 + 2v_2 = u$ **… 1** ... (5m)

NEL: $v_1 - v_2 = -eu$ **… 2** ... (5m)



Solving **1** and **2**,

1 : $v_1 + 2v_2 = u$

2×2 : $2v_1 - 2v_2 = -2eu$

$3v_1 = (1-2e)u$

$v_1 = \frac{(1-2e)u}{3}$

Also,

1 : $v_1 + 2v_2 = u$

2×−1 : $-v_1 + v_2 = eu$

$3v_2 = (1+e)u$

$v_2 = \frac{(1+e)u}{3}$ ... (5m)

Then

KE before $= \frac{1}{2}mu^2$

KE after $= \frac{1}{2}mv_1^2 + \frac{1}{2}(2m)v_2^2$

$= \frac{1}{2}m\left[\frac{(1-2e)u}{3}\right]^2 + m\left[\frac{(1+e)u}{3}\right]^2$

$= \frac{m}{2}.\frac{(1-2e)^2u^2}{9} + m.\frac{(1+e)^2u^2}{9}$

$= \frac{mu^2}{18}\left[(1-2e)^2 + 2(1+e)^2\right]$

$= \frac{mu^2}{18}\left[1 - 4e + 4e^2 + 2 + 4e + 2e^2\right]$

$= \frac{mu^2}{18}\left[3 + 6e^2\right]$

$= \frac{mu^2\left(1+2e^2\right)}{6}$

Given: KE after $= \frac{17}{27} \times$ KE before

$$\frac{mu^2(1+2e^2)}{6} = \frac{17}{27} \times \frac{mu^2}{2} \qquad \text{... (5m)}$$

$$1+2e^2 = \frac{17}{9}$$

$$9+18e^2 = 17$$

$$18e^2 = 8$$

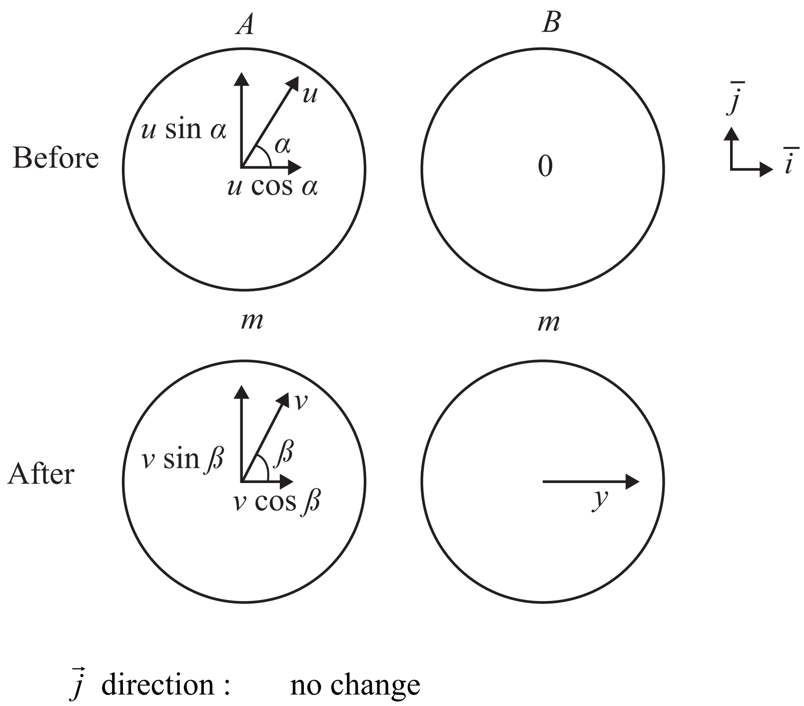
$$e^2 = \frac{4}{9}$$

$$e = \frac{2}{3}. \qquad \text{... (5m)}$$

(b) A smooth sphere A collides with an equal smooth sphere B which is at rest. The directions of motion of A before and after impact make angles α and β, respectively with the line of centres.

(i) Express β in terms of α and e, the coefficient of restitution.

(ii) If $\tan\alpha = \frac{1}{2}$ and $e = \frac{1}{2}$ show that A is deflected through the angle $\tan^{-1} = \frac{3}{4}$ by the collision. **(25)**



$\vec{j}$ direction : no change

$$v\sin\beta = u\sin\alpha \qquad \textbf{... 1}$$

Further answers overleaf

PCM ($\vec{i}$): $mv\cos\beta + my = mu\cos\alpha + m(0)$

$v\cos\beta + y = u\cos\alpha$ **… 2** ... (5m)

NEL ($\vec{i}$): $v\cos\beta - y = -e(u\cos\alpha)$ **… 3** ... (5m)

2, 3 : $2v\cos\beta = (1-e)u\cos\alpha$

$v\cos\beta = \frac{1-e}{2}u\cos\alpha$ **… 4**

1, 4 : $\frac{v\sin\beta}{v\cos\beta} = \frac{u\sin\alpha}{\frac{1-e}{2}u\cos\alpha}$

$\tan\beta = \frac{2}{1-e}\tan\alpha$

$\beta = \tan^{-1}\left[\frac{2}{1-e}\tan\alpha\right]$ ... (5m)

(ii) If $\tan\alpha = \frac{1}{2}$ and $e = \frac{1}{2}$, then $\tan\beta = \frac{2}{1-\frac{1}{2}}\left(\frac{1}{2}\right)$

$\tan\beta = 2$ ... (5m)

If θ is the deviation of A after the collision, then

$\theta = \beta - \alpha$

$\tan\theta = \tan(\beta - \alpha)$

$\tan\theta = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$

$\tan\theta = \frac{2 - \frac{1}{2}}{1 + (2)\left(\frac{1}{2}\right)}$

$\tan\theta = \frac{3}{4}$

$\theta = \tan^{-1}\frac{3}{4}.$ ... (5m)

6. **(a)** The displacement, x metres, of a moving particle at any time t seconds is given by the equation

$$x = 8\sin 3t + 6\cos 3t\,.$$

(i) Show that the motion is simple harmonic motion. **(10)**

$$x = 8\sin 3t + 6\cos 3t$$

$$\frac{dx}{dt} = 24\cos 3t - 18\sin 3t \qquad \text{... (5m)}$$

$$\frac{d^2x}{dt^2} = -72\sin 3t - 54\cos 3t$$

$$= -9(8\sin 3t + 6\cos 3t)$$

$$= -9x$$

As this equation is in the form $\frac{d^2x}{dt^2} = -\omega^2 x$,

the motion is simple harmonic. ... (5m)

(ii) By considering the velocity of the particle, calculate the amplitude of the motion. **(10)**

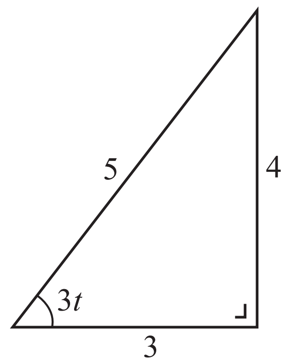
(To find the amplitude, we put $\frac{dx}{dt} = 0$.*)*

$$24\cos 3t - 18\sin 3t = 0 \qquad \text{... (5m)}$$

$$24\cos 3t = 18\sin 3t$$

$$\frac{4}{3} = \tan 3t$$

(Construct the triangle shown opposite.)



The amplitude is the maximum value of x.

$$x = 8\left(\frac{4}{5}\right) + 6\left(\frac{3}{5}\right)$$

$$x = \frac{50}{5}$$

$$x = 10$$

The amplitude is 10. ... (5m)

(iii) Find the first time that the particle is at the centre of the motion. **(5)**

When $x=0$:

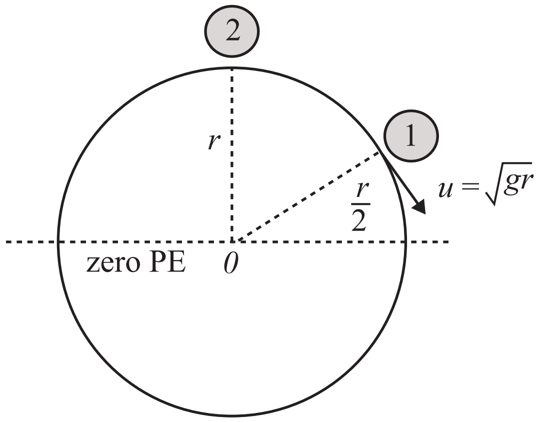
$$8\sin 3t + 6\cos 3t = 0$$
$$8\sin 3t = -6\cos 3t$$
$$\tan 3t = -\frac{3}{4}$$
$$3t = 2\cdot 498 \text{ (radians)}$$
$$t = 0\cdot 83 \text{ seconds} \qquad \text{... (5m)}$$

(b) A small ring of mass m can slide on a smooth circular wire of radius r and centre O which is fixed in a vertical plane. From a point on the wire at a vertical distance $\frac{r}{2}$ above O, the ring is given a velocity $\sqrt{gr}$ along the downward tangent to the wire.

(i) Show that the particle will just reach the highest point of the wire. **(10)**

(Let v be the speed of the particle at the top of the circle.)

$$KE_1 = \frac{1}{2}mu^2$$
$$= \frac{1}{2}m(gr)$$
$$= \frac{1}{2}mgr$$
$$PE_1 = mg\left(\frac{r}{2}\right) = \frac{1}{2}mgr$$
$$KE_2 = \frac{1}{2}mv^2$$
$$PE_2 = mgr$$



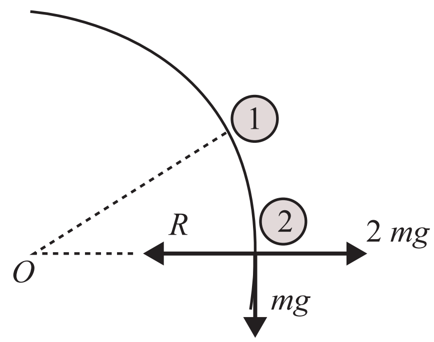
PCE: $KE_2 + PE_2 = KE_1 + PE_1$

$$\frac{1}{2}mv^2 + mgr = \frac{1}{2}mgr + \frac{1}{2}mgr \qquad \text{... (5m)}$$
$$\frac{1}{2}mv^2 = 0$$
$$v^2 = 0$$
$$v = 0 \qquad \text{... (5m)}$$

Thus the particle just reaches the highest point of the wire.

(ii) Find the reaction between the ring and the wire when the ring is at the same horizontal level as O. **(15)**

(We need to find the speed, v, of the particle when it is at the same horizontal level as O.)



$$KE_1 = \frac{1}{2}mgr$$

$$PE_1 = \frac{1}{2}mgr$$

$$KE_2 = \frac{1}{2}mv^2$$

$$PE_2 = 0$$

PCE: $KE_2 + PE_2 = KE_1 + PE_1$

$$\frac{1}{2}mv^2 + 0 = \frac{1}{2}mgr + \frac{1}{2}mgr \qquad \text{... (5m)}$$

$$\frac{1}{2}mv^2 = mgr$$

$$v^2 = 2gr$$

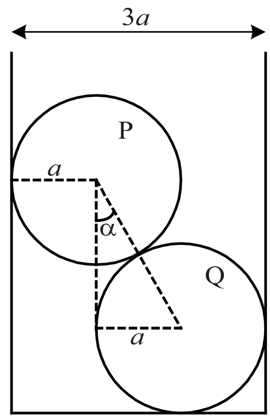
$$v = \sqrt{2gr}$$

The circular motion force is then

$$\frac{mv^2}{r} = \frac{m(2gr)}{r} = 2mg \qquad \text{... (5m)}$$

In = Out : $R = 2mg$. ... (5m)

7. **(a)** The diagram shows two smooth vertical walls at a distance $3a$ apart, with two smooth cylinders, P and Q, each of weight W and radius a. The axes of the cylinders are horizontal and parallel to the walls. Q rests on smooth horizontal ground. α is the angle between the line of centres and the vertical.



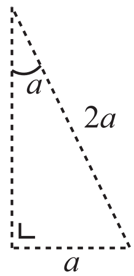
(i) Find α. **(5)**

From the triangle shown opposite:

$$\sin\alpha = \frac{a}{2a}$$

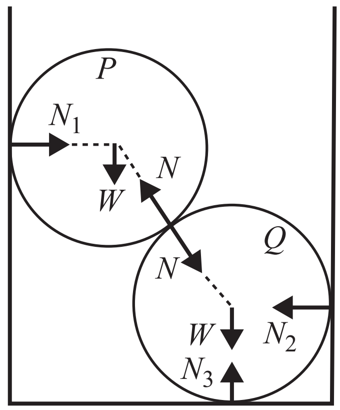
$$\sin\alpha = \frac{1}{2}$$

$$\alpha = 30°$$



... (5m)

(ii) Find, in terms of W, the magnitude of the reaction between the cylinders. **(15)**



Overall structure

The overall structure is in equilibrium.

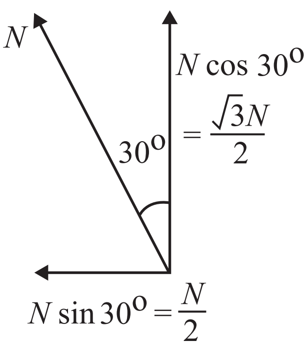
$\uparrow = \downarrow :$ $\quad N_3 = 2W$

(We don't need this equation here.)

$\leftarrow = \rightarrow :$ $\quad N_2 = N_1$ **… 1**

... (5m)

(The reaction forces at the two walls are equal.)



Sphere P

Sphere P is in equilibrium.

$\uparrow = \downarrow :$ $\quad \frac{\sqrt{3}N}{2} = W$ **… 2**

$\leftarrow = \rightarrow :$ $\quad \frac{N}{2} = N_1$ **… 3**

... (5m)

2 : $\quad N = \frac{2W}{\sqrt{3}}$

This is the reaction between the two spheres.

... (5m)

(iii) Find, in terms of W, the magnitude of the reaction between each cylinder and the corresponding wall. **(5)**

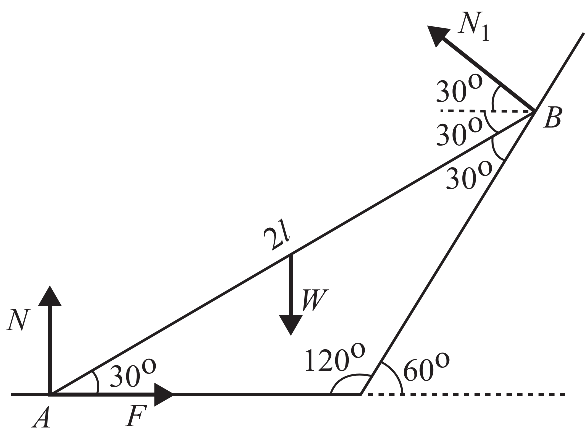
3 : $N_1 = \frac{1}{2}\left(\frac{2W}{\sqrt{3}}\right)$

$N_1 = \frac{W}{\sqrt{3}}$

1 : $N_2 = \frac{W}{\sqrt{3}}$ ... (5m)

$\frac{W}{\sqrt{3}}$ is the reaction between each sphere and the corresponding wall.

(b) A thin uniform rod rests with one end on a smooth plane inclined at 60° to the horizontal and the other end on rough horizontal ground. If the rod is about to slip when inclined at 30° to the horizontal, show that the coefficient of friction between the rod and the ground is $\frac{1}{\sqrt{3}}$. **(25)**



... (5m)

The rod is in equilibrium with limiting friction.

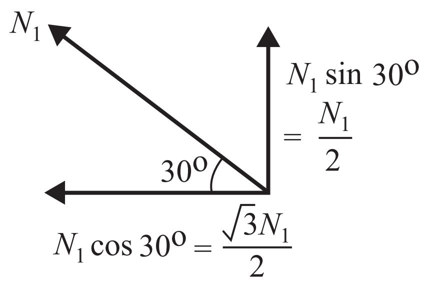
$\uparrow = \downarrow$: $N + \frac{N_1}{2} = W$ **... 1** ... (5m)

$\leftarrow = \rightarrow$: $\frac{\sqrt{3}N_1}{2} = F$ **... 2** ... (5m)

Further answers overleaf

Limiting friction:

$$F = \mu N \qquad \textbf{... 3}$$



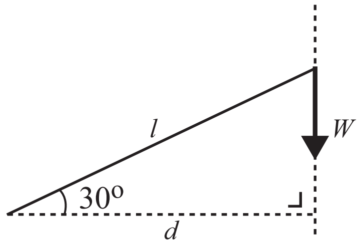
Moments about A:

Anticlockwise Moments = Clockwise Moments

$$N_1.2l \sin 60^\circ = W.l \cos 30^\circ$$

$$2N_1\left(\frac{\sqrt{3}}{2}\right) = W\left(\frac{\sqrt{3}}{2}\right)$$

$$N_1 = \frac{W}{2} \qquad \textbf{... 4}$$



... (5m)

Solving **1** to **4**:

1, **4 :** $N + \frac{W}{4} = W$

$$N = \frac{3W}{4}$$

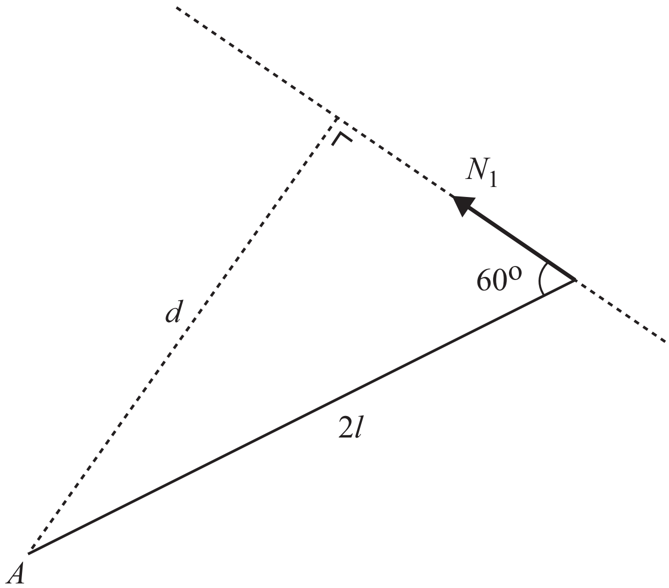
2 : $\frac{\sqrt{3}}{2}\left(\frac{W}{2}\right) = F$

$$F = \frac{\sqrt{3}W}{4}$$

3 : $\frac{\sqrt{3}W}{4} = \mu \frac{3W}{4}$

$$\mu = \frac{\sqrt{3}}{3}$$

$$\mu = \frac{1}{\sqrt{3}}.$$



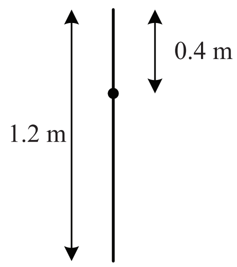
... (5m)

8. **(a)** Prove that the moment of inertia of a uniform rod of mass m and length $2l$ about an axis through its centre perpendicular to the rod is $\frac{1}{3}ml^2$. **(20)**

Standard Proof

Moment of mass element	... (5m)
Moment of body	... (5m)
Integral	... (5m)
Deduce	... (5m)

(b) A uniform rod of mass m and of length 1.2 m swings in a vertical plane about a horizontal axis through the rod at a distance of 0.4 m from its upper end.



(i) If v m s^{-1} is the velocity of the lower end when the rod is vertical, prove that the rod will make a complete revolution if $v \geq 5.6$ m s^{-1}. **(15)**

For the rod,

$$h = 0 \cdot 2$$

$$I = \frac{1}{3}ml^2 + mb^2$$

$$= \frac{1}{3}m(0 \cdot 6)^2 + m(0 \cdot 2)^2$$

$$= 0 \cdot 16m \qquad \text{... (5m)}$$

Let the position of zero PE be when the centre of mass of the rod is below the axis of rotation.

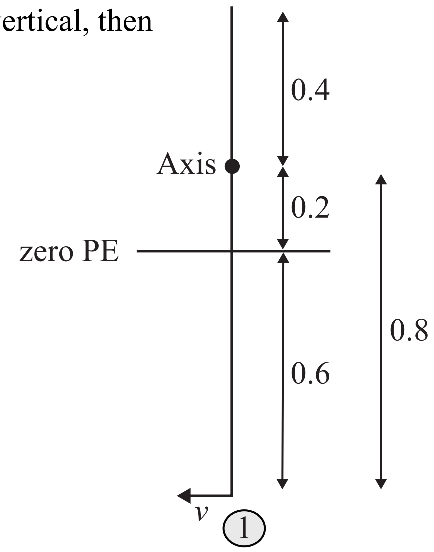
If ω is the angular velocity when the rod is vertical, then

$$v = r\omega$$

$$v = 0 \cdot 8\omega$$

$$\omega = 1 \cdot 25v$$

Position 1: Centre of mass vertically below the axis



$$KE_1 = \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}(0 \cdot 16m)(1 \cdot 25v)^2$$

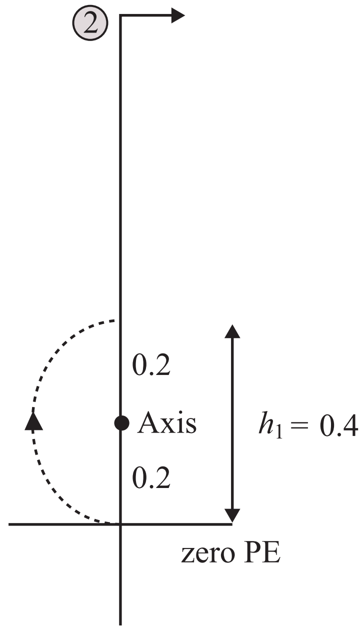
$$= 0 \cdot 125mv^2$$

$$PE_1 = 0$$

Further answers overleaf

Position 2: Centre of mass vertically above the axis.

Let ω_1 be the angular velocity at the top point. The rod will make a complete revolution if $\omega_1 > 0$.



$$KE_2 = \frac{1}{2}I\omega^2$$
$$= \frac{1}{2}(0\cdot 16m)\omega_1^2$$
$$= 0\cdot 08m\omega_1^2$$
$$PE_2 = mgh_1$$
$$= mg(0\cdot 4)$$
$$= 0\cdot 4mg$$

PCE: $KE_2 + PE_2 = KE_1 + PE_1$

$0\cdot 08m\omega_1^2 + 0\cdot 4mg = 0\cdot 125mv^2 + 0$ ... (5m)

$0\cdot 08\omega_1^2 = 0\cdot 125v^2 - 0\cdot 4g$

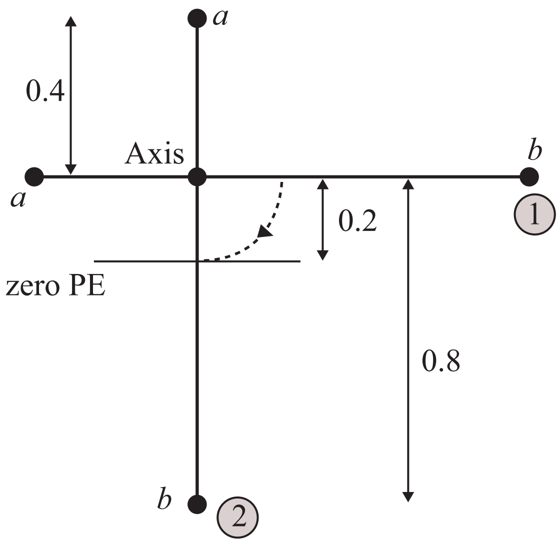
If $\omega_1^2 \geq 0$, $0\cdot 125v^2 - 0\cdot 4g \geq 0$

$$v^2 \geq \frac{0\cdot 4g}{0\cdot 125}$$
$$v^2 \geq 31\cdot 36$$

$v \geq 5\cdot 6$ m/s. ... (5m)

(ii) If a mass *m* is attached to each end of the rod and the compound body is released from rest in a horizontal position, find the angular speed when the rod is next vertical. **(15)**

Let the zero PE position for the rod and the two masses be when the rod is vertical, *i.e.* in Position 2. *(Note that the PE of the mass at a will be negative in Position 1.)*



Position 1: Horizontal

$$KE_1 = 0$$
$$PE_1 = PE_{\text{rod}} + PE_a + PE_b$$
$$= mg(0\cdot 2)$$
$$+mg(-0\cdot 4)$$
$$+mg(0\cdot 8)$$
$$= 0\cdot 6mg$$

Position 2: Vertical

$$
\begin{aligned}
I &= I_{\text{rod}} + I_a + I_b \\
&= 0\cdot 16m + m(0\cdot 4)^2 + m(0\cdot 8)^2 \\
&= 0\cdot 96m
\end{aligned}
$$ ... (5m)

Let ω_2 be the angular velocity in this position.

$$
\begin{aligned}
KE_2 &= \frac{1}{2} I\omega^2 \\
&= \frac{1}{2}(0\cdot 96m)\omega_2^2 \\
&= 0\cdot 48m\omega_2^2 \\
PE_2 &= 0
\end{aligned}
$$

PCE: $KE_2 + PE_2 = KE_1 + PE_1$

$0\cdot 48m\omega_2^2 + 0 = 0 + 0\cdot 6mg$ ... (5m)

$\omega_2^2 = 1\cdot 25g$

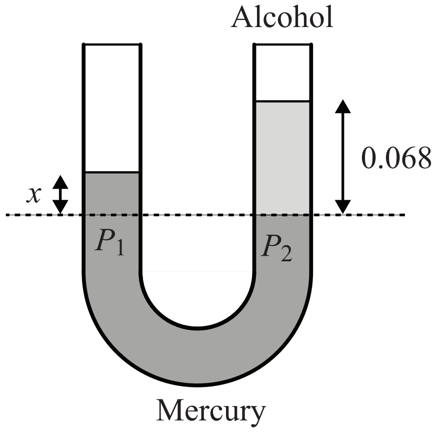
$\omega_2^2 = 12\cdot 25$

$\omega_2 = 3\cdot 5$ rad/s. ... (5m)

9. (a) Mercury, of relative density 13.6, is poured into a vertical U-tube.
Alcohol, of relative density 0.8, is then poured into one limb of the U-tube until it occupies a height of 6.8 cm.

(i) What is the difference in the free surface levels of the mercury and the alcohol? **(10)**

(Let x be the height of the mercury above the common surface of the alcohol and mercury.)



$p_1 = p_2$

$1000(13 \cdot 6)g(x)$

$= 1000(0 \cdot 8)g(0 \cdot 068)$ ... (5m)

$13 \cdot 6x = 0 \cdot 0544$

$x = 0 \cdot 004$ m

$x = 0 \cdot 4$ cm

The difference in the free surface levels

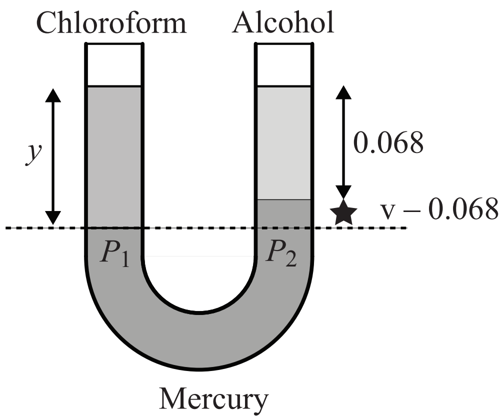
$= 0 \cdot 068 - 0 \cdot 004$

$= 0 \cdot 064$ m

$= 6 \cdot 4$ cm. ... (5m)

(ii) If chloroform, of relative density 1.5, is then poured into the other limb until the free surfaces of the chloroform and the alcohol are at the same level, what is the height of the chloroform in the tube? **(15)**

(Let y be the height of the chloroform. Because the chloroform is more dense than the alcohol, the mercury will rise up in the alcohol column.)



$p_1 = p_2$

$1000(1 \cdot 5)g(y)$

$= 1000(0 \cdot 8)g(0 \cdot 068)$

$+ 1000(13 \cdot 6)g(y - 0 \cdot 068)$

... (5m + 5m)

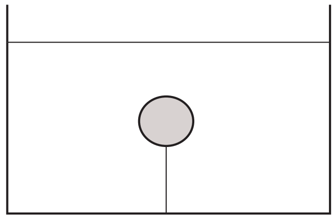
$1 \cdot 5y = 0 \cdot 0544 + 13 \cdot 6y - 0 \cdot 9248$

$0 \cdot 8704 = 12 \cdot 1y$

$y = 0 \cdot 072$ m

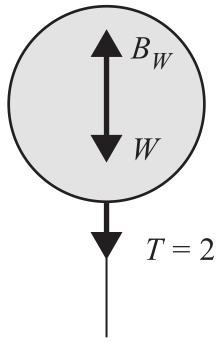
$y = 7 \cdot 2$ cm. ... (5m)

(b) A uniform sphere is held completely immersed in water by means of a string tied to a point on the base of the container. The tension in the string is 2.94 N. When the water is replaced by a liquid of relative density 1.2, the tension in the string is 5.88 N.



Find the mass of the sphere. **(25)**

Let m be the mass of the sphere and W be its weight. Let s be its relative density and B_W be the buoyancy in water.



Equilibrium:

$B_W = W + T$ ... ... (5m)

$\frac{W}{s} = W + 2 \cdot 94$ **... 1** ... (5m)

Let $s_L = 1 \cdot 2$ be the relative density of the liquid, and let B_L be the buoyancy in the liquid.

Equilibrium:

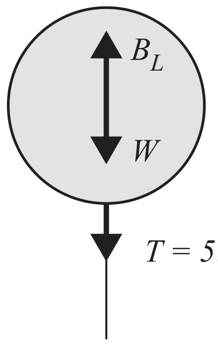
$B_L = W + T$

and

$s_L B_W = W + 5 \cdot 88$ ... (5m)

and

$1 \cdot 2 \frac{W}{s} = W + 5 \cdot 88$ **... 2** ... (5m)



Solving,

1, **2**: $1 \cdot 2(W + 2 \cdot 94) = W + 5 \cdot 88$

$1 \cdot 2W + 3 \cdot 528 = W + 5 \cdot 88$

$0 \cdot 2W = 2 \cdot 352$

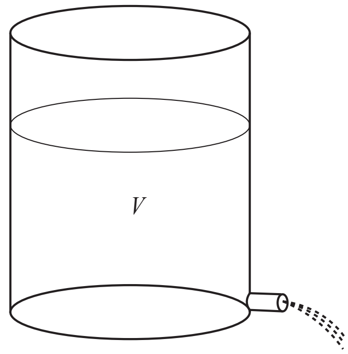
$W = 11 \cdot 76$

Then

$mg = 11 \cdot 76$

$m = 1 \cdot 2$ kg. ... (5m)

10. **(a)** The diagram shows water flowing out of a container through a tap. It can be shown that the volume V litres after a time t minutes is given by $\frac{dV}{dt} = -k\sqrt{V}$.
Initially, the container holds 100 litres of water. After 2 minutes, the volume of water is 96 litres. Find



(i) the value of k, correct to three places of decimals. **(15)**

$$\frac{dV}{dt} = -k\sqrt{V}$$
$$\frac{dV}{\sqrt{V}} = -k\,dt$$
$$\int V^{-\frac{1}{2}}dV = -k\int dt$$
$$2\sqrt{V} = -kt + c \qquad \text{... (5m)}$$

I.C.: $t = 0,\ V = 100$
$$2\sqrt{100} = c$$
$$c = 20 \qquad \text{... (5m)}$$

Unique solution:
$$2\sqrt{V} = -kt + 20$$

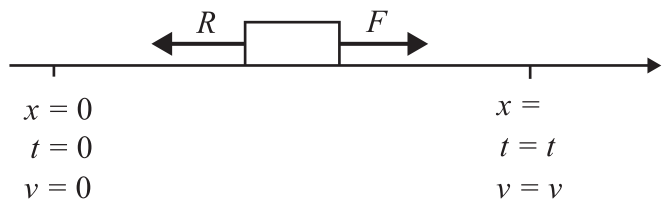
I.C.: $t = 2,\ V = 96$
$$2\sqrt{96} = -2k + 20$$
$$k = 10 - \sqrt{96}$$
$$k = 0\cdot 202 \qquad \text{... (5m)}$$

(ii) the volume of water left in the container after 10 minutes. **(5)**

$$2\sqrt{V} = -0\cdot 202t + 20$$
$$\sqrt{V} = -0\cdot 101t + 10$$
$$t = 10:$$
$$\sqrt{V} = -0\cdot 101(10) + 10$$
$$\sqrt{V} = 8\cdot 99$$
$$V = 80\cdot 82 \text{ litres.} \qquad \text{... (5m)}$$

(b) A car of mass m moving on a straight horizontal road is subject to a constant resistance R. The engine is working at a constant rate kR.

(i) Given that in time t the car accelerates from rest to a speed v, where $v < k$ find an expression for t in terms of m, R, k and v. **(20)**



(We start by using the power to calculate the force acting when the speed is v.)

Power $= F \times v$

$$kR = Fv$$

$$F = \frac{kR}{v} \qquad \text{... (5m)}$$

Newton's 2nd law:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{kR}{v} - R$$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{kR - vR}{v}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{R}{m}.\frac{k-v}{v}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{R}{m}.\frac{v-k}{v}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{R}{m}.\frac{v-k}{v}$$

$$\frac{v\,\mathrm{d}v}{v-k} = -\frac{R}{m}\mathrm{d}t$$

$$\int \frac{v\,\mathrm{d}v}{v-k} = -\frac{R}{m}\int \mathrm{d}t \qquad \textbf{... 1} \qquad \text{... (5m)}$$

Let $I = \displaystyle\int \frac{v\,\mathrm{d}v}{v-k}$

$$\begin{array}{r} 1 \\ v-k\,\overline{\big)\,v\quad} \\ \underline{v-k} \\ k \end{array}$$

$$\frac{v}{v-k} = 1 + \frac{k}{v-k}$$

$$I = \int \left(1 + \frac{k}{v-k}\right)\mathrm{d}v$$

$$I = v + k\ln|\,v-k\,|$$

Further answers overleaf

1 : $v + k\ln|v-k| = -\frac{R}{m}t + c$ **… 2** ... (5m)

I.C.: $t = 0,\ v = 0$

$k\ln k = c$

Unique solution:

$$v + k\ln|v-k| = -\frac{R}{m}t + k\ln k$$

$$\frac{R}{m}t = k\ln k - k\ln|v-k| - v$$

$$t = \frac{m}{R}\left(k\ln\frac{k}{|v-k|} - v\right).$$ ... (5m)

(ii) Find the time taken by the car to increase its speed from $\frac{1}{3}k$ to $\frac{2}{3}k$. **(10)**

2 : $v + k\ln|v-k| = -\frac{R}{m}t + c$ I.C.: $t = 0,\ v = \frac{1}{3}k$

$\frac{1}{3}k + k\ln\frac{2}{3}k = c$

Unique solution:

$$v + k\ln|v-k| = -\frac{R}{m}t + \frac{1}{3}k + k\ln\frac{2k}{3}$$ ... (5m)

When $v = \frac{2}{3}k$:

$$\frac{2}{3}k + k\ln\frac{k}{3} = -\frac{R}{m}t + \frac{1}{3}k + k\ln\frac{2k}{3}$$

$$\frac{R}{m}t = -\frac{1}{3}k + k\ln\frac{2k}{3} - k\ln\frac{k}{3}$$

$$\frac{R}{m}t = -\frac{1}{3}k + k\ln\frac{\frac{2k}{3}}{\frac{k}{3}}$$

$$\frac{R}{m}t = -\frac{1}{3}k + k\ln 2$$

$$t = \frac{mk}{R}\left(-\frac{1}{3} + \ln 2\right).$$ ... (5m)