
L.41/42

Pre-Leaving Certificate Examination, 2012

# Applied Mathematics

## Marking Scheme




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**Pre-Leaving Certificate Examination, 2012**

# Applied Mathematics

## Ordinary Level
## Marking Scheme (300 marks)

**General Instructions**

**1.** Penalties of three types are applied to students' work as follows:

| | | |
|---|---|---|
| Slips | - numerical slips | S(–1) |
| Blunders | - mathematical errors | B(–3) |
| Misreading | - if not serious | M(–1) |

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

**2.** Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

**3.** Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

**4.** Number the grid on each script 1 to 9 in numerical order, not the order of answering.

**5.** Scrutinise **all** pages of the answer book.

**6.** The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

**Six questions to be answered. All questions carry equal marks. (6 × 50m)**

**1.** A car, travelling with uniform acceleration, passes four points, $P$, $Q$, $R$ and $S$, on a straight level road.
Its speed at $P$ is 10 m s$^{-1}$ and 5 seconds later it passes $Q$ with a speed of 15 m s$^{-1}$.
The time taken by the car to travel from $Q$ to $R$ is equal to the time it takes to travel from $R$ to $S$. Its speed at $S$ is 30 m s$^{-1}$.

Find

**(i)** the uniform acceleration of the car **(10)**

$v = u + at$
$15 = 10 + a(5)$
$5 = 5a$
$a = 1$ m s$^{-2}$ ... (10m)

**(ii)** $|PQ|$, the distance from $P$ to $Q$. **(10)**

$$s = ut + \frac{1}{2}at^2$$
$$s = (10)(5) + \frac{1}{2}(1)(5)^2$$
$$s = 50 + \frac{25}{2}$$
$$s = \frac{125}{2} = 62.5 \text{ m}$$ ... (10m)

**(iii)** $|QS|$, the distance from $Q$ to $S$. **(10)**

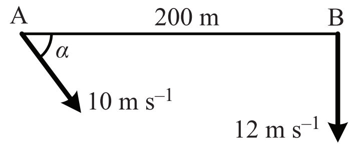
$v^2 = u^2 + 2as$
$30^2 = 15^2 + 2(1)s$
$675 = 2s$
$s = 337.5$ m ... (10m)

**(iv)** $|QR|$, the distance from $Q$ to $R$, correct to the nearest metre. **(20)**

For $[QS]$,
$v = u + at$
$30 = 15 + (1)t$
$t = 15$

For $[QR]$, $t = 7.5$ ... (10m)
$$s = ut + \frac{1}{2}at^2$$
$$s = (15)(7.5) + \frac{1}{2}(1)(7.5)^2$$
$s = 140.625$
$s = 141$ m ... (10m)

**2.** At a certain instant, a boat A is 200 m due west of another boat B.
A is travelling at a constant speed of $10 \text{ m s}^{-1}$ in the direction $\alpha°$ south of east, where $\tan \alpha = \frac{4}{3}$.
B is travelling due south at a constant speed of $12 \text{ m s}^{-1}$.




Find

**(i)** the velocity of A in terms of $\vec{i}$ and $\vec{j}$ **(10)**

$\vec{v}_{\mathbf{A}} = (10\cos\alpha)\vec{i} - (10\sin\alpha)\vec{j}$ ... (5m)

$= 6\vec{i} - 8\vec{j}$ ... (5m)

**(ii)** the velocity of B in terms of $\vec{i}$ and $\vec{j}$ **(10)**

$\vec{v}_{\text{B}} = -12\vec{j}$ ... (10m)

**(iii)** the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$ **(10)**

$\vec{v}_{\text{AB}} = \vec{v}_{\text{A}} - \vec{v}_{\text{B}}$ ... (5m)

$= (6\vec{i} - 8\vec{j}) - (-12\vec{j})$

$= 6\vec{i} + 4\vec{j}$ ... (5m)

**(iv)** the magnitude and direction of the velocity of A relative to B **(10)**

$|\vec{v}_{\text{AB}}| = \sqrt{6^2 + 4^2}$

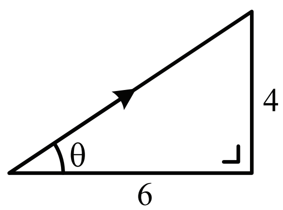
$= \sqrt{52}$ or $7.2 \text{ m s}^{-1}$ ... (5m)

Also,

$\tan\theta = \frac{4}{6} = \frac{2}{3}$

$\theta = \tan^{-1}\frac{2}{3} = 33.7°$

direction: E 33.7° N ... (5m)

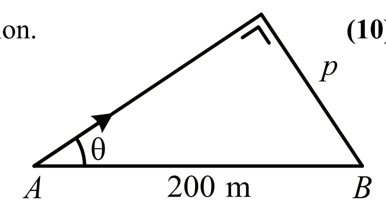



**(v)** the shortest distance between them in their subsequent motion. **(10)**

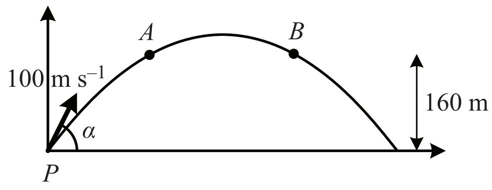
shortest distance $= p$ ... (5m)

$= 200\sin\theta$

$= 110.9$ m ... (5m)



**3.** A particle is projected from a point $P$ on horizontal ground with an initial speed of 100 m $s^{-1}$ at an angle $\alpha$ to the horizontal, where $\tan\alpha = \frac{4}{3}$. The height of the particle at $A$ and at $B$ is 160 m.




**(i)** Find the initial velocity of the particle in terms of $\vec{i}$ and $\vec{j}$. **(10)**

$\vec{v} = (100\cos\alpha)\vec{i} + (100\sin\alpha)\vec{j}$ ... (5m)

$\vec{v} = 80\vec{i} + 60\vec{j}$ ... (5m)

**(ii)** Find the two times at which the height of the particle is 160 m. **(20)**

$s = ut + \frac{1}{2}at^2$

$160 = 60t + \frac{1}{2}(-10)t^2$ ... (10m)

$160 = 60t - 5t^2$

$5t^2 - 60t + 160 = 0$

$t^2 - 12t + 32 = 0$

$(t-4)(t-8) = 0$

$t - 4 = 0$ or $t - 8 = 0$

$t = 4$ s or $t = 8$ s

The particle is at $A$ after 4 seconds

and at $B$ after 8 seconds ... (10m)

**(iii)** Find the speed of the particle at $A$. **(10)**

At $A$, $t = 4$ and

$\vec{v} = (80)\vec{i} + (60-40)\vec{j}$

$= 80\vec{i} + 20\vec{j}$ ... (5m)

Speed $= |\vec{v}| = \sqrt{80^2 + 20^2} = 20\sqrt{17}$ m $s^{-1}$ ... (5m)

**(iv)** Show that the speed of the particle at $B$ is the same as its speed at $A$. **(10)**

At $B$, $t = 8$ and

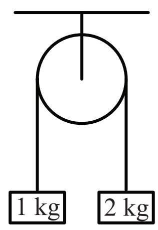
$\vec{v} = 80\vec{i} + (60-80)\vec{j}$ ... (5m)

$\vec{v} = 80\vec{i} - 20\vec{j}$

Speed $|\vec{v}| = \sqrt{80^2 + 20^2} = 20\sqrt{17}$ m $s^{-1}$

Thus the speed at $B$ is the same as the speed at $A$ ... (5m)

**4.** **(a)** Two particles of masses 1 kg and 2 kg are connected by a taut, light, inextensible string which passes over a smooth light pulley.




The system is released from rest.

Find

**(i)** the common acceleration of the particles **(15)**

$T - g = a$ ... (5m)

$2g - T = 2a$ ... (5m)

$g = 3a$

$a = \frac{10}{3}$ m s$^{-2}$ ... (5m)

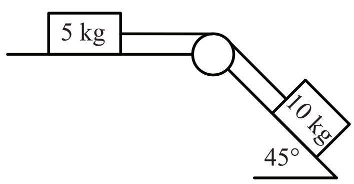
**(ii)** the tension in the string. **(5)**

$T = g + a$

$T = 10 + \frac{10}{3}$

$T = \frac{40}{3}$ N ... (5m)

**(b)** Masses of 5 kg and 10 kg are connected by a taut, light, inextensible string which passes over a smooth light pulley as shown in the diagram.



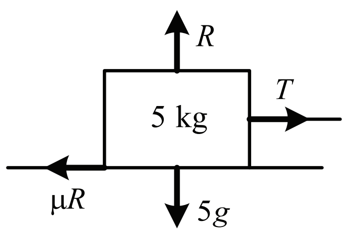

The 5 kg mass lies on a rough horizontal plane and the coefficient of friction between the 5 kg mass and the plane is $\frac{3}{5}$.

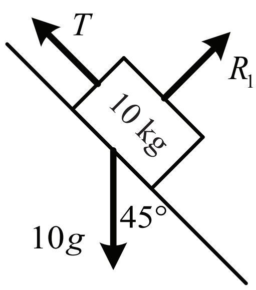
The 10 kg mass lies on a smooth plane which is inclined at 45° to the horizontal.

The system is released from rest.

**(i)** Show on separate diagrams the forces acting on each particle. **(10)**

❶ ... (5m) ❷ ... (5m)



**4(ii)** Find the common acceleration of the masses, correct to two decimal places. **(15)**

$T - \mu R = 5a$

$T - \frac{3}{5}(5g) = 5a$ ... (5m)

$T - 30 = 5a$

Also:

$10g\cos 45° - T = 10a$ ... (5m)

$50\sqrt{2} - T = 10a$

Then

$50\sqrt{2} - 30 = 15a$

$15a = 40.71$

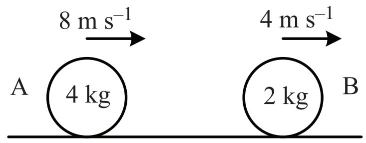
$a = 2.71 \text{ m s}^{-2}$ ... (5m)

**(iii)** Find the tension in the string. **(5)**

$T = 30 + 5(2.71)$

$T = 43.57 \text{ N}$ ... (5m)

**5.** A smooth sphere A, of mass 4 kg, collides directly with another smooth sphere B, of mass 2 kg, on a smooth horizontal surface.




Before impact A and B are moving in the same direction with speeds of 8 m s$^{-1}$ and 4 m s$^{-1}$, respectively.

The coefficient of restitution for the collision is $\frac{1}{2}$.

Find

**(i)** the speed of A and the speed of B after the collision **(30)**

PCM: $(4)(8) + (2)(4) = (4)v_1 + (2)v_2$ ... (10m)

$40 = 4v_1 + 2v_2$

$2v_1 + v_2 = 20$

NEL: $v_1 - v_2 = -\frac{1}{2}(8 - 4)$ ... (10m)

$v_1 - v_2 = -2$

Thus

$3v_1 = 18$

$v_1 = 6$ m s$^{-1}$

and $v_2 = 8$ m s$^{-1}$ ... (10m)

**(ii)** the change in the kinetic energy of A due to the collision **(15)**

KE of A before collision = $\frac{1}{2}(4)(8)^2 = 128$ ... (5m)

KE of A after collision = $\frac{1}{2}(4)(6)^2 = 72$ ... (5m)

Change in KE of A = 128 – 72 = 56 J ... (5m)

**(iii)** the magnitude of the impulse imparted to B due to the collision. **(5)**

Impulse $= |(2)(8) - (2)(4)|$

$= 8$ Ns ... (5m)

**6.** **(a)** Particles of weight 3 N, 2 N, 5 N and 2 N are placed at the points (1, 2), (3, $p$), ($p$, $q$) and (8, $q$), respectively.

The co-ordinates of the centre of gravity of the system are (5, $q$).

Find

**(i)** the value of $p$ **(15)**

$$5 = \frac{3(1)+2(3)+5(p)+2(8)}{12} \qquad \text{... (10m)}$$

$60 = 25 + 5p$

$p = 7$ ... (5m)

**(ii)** the value of $q$.

$$q = \frac{3(2)+2(7)+5(q)+2(q)}{12} \qquad \text{... (10m)}$$

$12q = 20 + 7q$

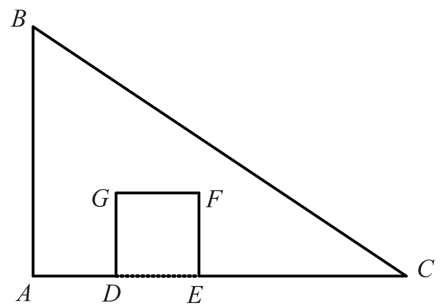
$5q = 20$

$q = 4$ ... (5m)

**(b)** A triangular lamina with vertices $A$, $B$ and $C$ has the square portion with diagonal [$DF$] removed.

The co-ordinates of the points are $A(0, 0)$, $B(0, 6)$, $C(9, 0)$, $D(2, 0)$ and $F(4, 2)$.

Find the co-ordinates of the centre of gravity of the remaining lamina. **(20)**




| | area | centre of gravity | |
|---|---|---|---|
| $\Delta ABC$ | $\frac{1}{2}(9)(6) = 27$ | (3, 2) | ... (5m) |
| $\square DEFG$ | (2)(2) = 4 | (3, 3) | |
| lamina | 23 | ($x$, $y$) | ... (5m) |

Then

$(23)(x) = 27(3) - 4(3)$

$x = 3$ ... (5m)

and $(23)(y) = 27(2) - 4(3)$

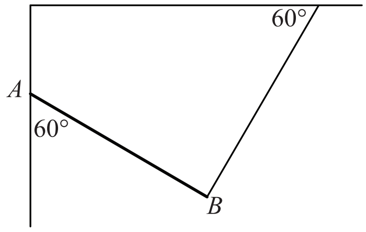
$y = \frac{42}{23}$ ... (5m)

The co-ordinates of the centre of gravity are $\left(3, \frac{42}{23}\right)$

**7.** A uniform rod, $AB$, of length 4 m and weight 50 N is smoothly hinged at end $A$ to a vertical wall.

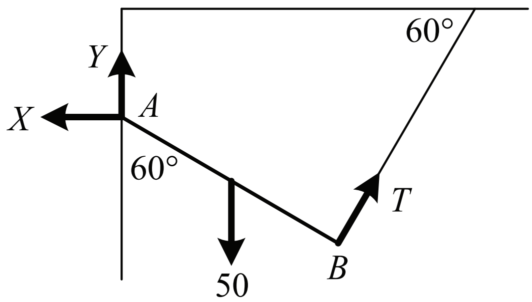
One end of a light inelastic string is attached to $B$ and the other end is attached to a horizontal ceiling.

The string makes an angle of 60° with the ceiling and the rod makes an angle of 60° with the wall, as shown in the diagram.




The rod is in equilibrium.

**(i)** Show on a diagram all the forces acting on the rod $AB$. **(10)**




... (10m)

**(ii)** Write down two equations that arise from resolving the forces horizontally and vertically. **(10)**

Horizontal:

$$T\cos 60° = X \qquad \text{... (5m)}$$

$$\frac{1}{2}T = X$$

Vertical:

$$Y + T\sin 60° = 50 \qquad \text{... (5m)}$$

$$Y + \frac{\sqrt{3}}{2}T = 50$$

**(iii)** Write down the equation that arises from taking moments about point $A$. **(10)**

Taking moments about $A$

$$T(4) = 50(2\sin 60°) \qquad \text{... (10m)}$$

**(iv)** Find the tension in the string. **(5)**

$$4T = 50\sqrt{3}$$

$$T = \frac{25\sqrt{3}}{4}\text{ N} \qquad \text{... (5m)}$$

**7(v)** Find the magnitude of the reaction at the hinge, $A$. **(15)**

$$X=\frac{25\sqrt{3}}{4}$$ ... (5m)

$$Y+\frac{\sqrt{3}}{2}\left(\frac{25\sqrt{3}}{2}\right)=50$$

$$Y=50-\frac{75}{4}$$

$$Y=\frac{25}{4}$$ ... (5m)

Then

$$R=\sqrt{\left(\frac{25\sqrt{3}}{4}\right)^2+\left(\frac{25}{4}\right)^2}$$

$R = 12.5$ N ... (5m)

**8.** **(a)** A particle describes a horizontal circle of radius $r$ metres with uniform angular velocity $\omega$ radians per second.
Its speed and acceleration are 6 m s$^{-1}$ and 9 m s$^{-2}$, respectively.

Find

**(i)** the value of $\omega$ **(10)**

$v = r\omega$
$6 = r\omega$
and
acc $= r\omega^2$
$9 = (r\omega)\omega$
$9 = 6\omega$
$\omega = \dfrac{3}{2}$ rad s$^{-1}$ ... (10m)

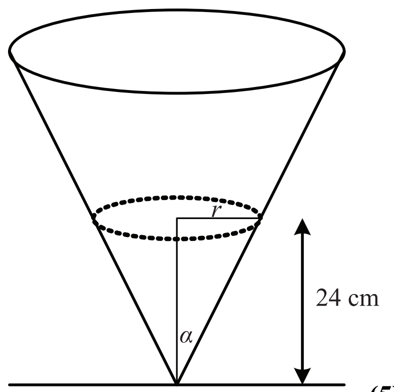
**(ii)** the value of $r$. **(10)**

$6 = r\left(\dfrac{3}{2}\right)$
$r = 4$ m ... (10m)

**(b)** A right circular cone is fixed to a horizontal surface. Its semi vertical angle is $\alpha$, where $\tan\alpha = \dfrac{5}{12}$ and its axis is vertical.

A smooth particle of mass 2 kg describes a horizontal circle of radius $r$ cm on the smooth inside surface of the cone.

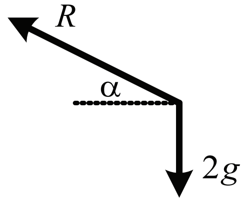
The plane of the circular motion is 24 cm above the horizontal surface.




**(i)** Find the value of $r$. **(5)**

$\dfrac{r}{24} = \tan\alpha = \dfrac{5}{12}$
$r = 10$ cm $= 0.1$ m ... (5m)

**(ii)** Show on a diagram all the forces acting on the particle. **(5)**




... (5m)

**(iii)** Find the reaction force between the particle and the surface of the cone. **(10)**

$$R\sin\alpha = 2g$$
$$R\left(\frac{5}{13}\right) = 20$$
$$R = 52 \text{ N}$$
... (10m)

**(iv)** Calculate the angular velocity of the particle.

$$R\cos\alpha = mr\omega^2$$
$$\frac{12}{13}R = 2(0.1)\,\omega^2$$
$$\frac{12}{13}(52) = 0.2\omega^2$$
$$\omega^2 = 240$$
$$\omega = \sqrt{240}\ \text{rad s}^{-1}$$
... (10m)

**9. (a)** State the Principle of Archimedes. **(10)**

– when a body is wholly or partly immersed in a liquid, ... (5m)
– it suffers an upthrust or buoyancy equal in magnitude to the weight of the liquid displaced ... (5m)

A solid piece of metal has a weight of 1700 N. When it is completely immersed in a liquid of relative density 1.2, the metal weighs 1100 N.

Find

**(i)** the volume of the metal **(10)**

$B$ = weight of displaced liquid
$600 = \rho Vg$
$600 = (1200)V(10)$ ... (5m)
$V = 0.05 \text{ m}^3$ ... (5m)

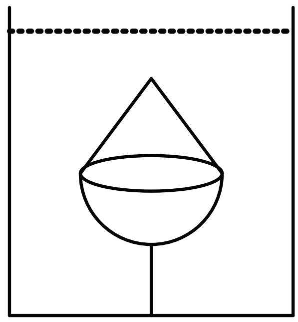
**(ii)** the relative density of the metal. **(10)**

Weight of metal $= \rho Vg$
$1700 = \rho(0.05)(10)$ ... (5m)
$\rho = 3400$
$s = 3.4$ ... (5m)

**9(b)** An object consists of a hemisphere of radius 6 cm surmounted by a cone of radius 6 cm and height 8 cm.

The relative density of the object is 0.6 and it is completely immersed in a tank of liquid of relative density 1.1.

The object is held at rest with its axis vertical by a light inextensible vertical string which is attached to the base of the tank.



Find the tension in the string. **(20)**

[Density of water = 1000 kg m$^{-3}$]

$$B = 1100\left\{\frac{2}{3}\pi(0.06)^3 + \frac{1}{3}\pi(0.06)^2(0.08)\right\}(10)$$

$B = 8.29$ (5m)

$$W = 600\left\{\frac{2}{3}\pi(0.06)^3 + \frac{1}{3}\pi(0.06)^2(0.08)\right\}(10)$$

$W = 4.52$ (5m)
$T + W = B$ (5m)
$T = 8.29 - 4.52$
$T = 3.77$ N (5m)

## Pre-Leaving Certificate Examination, 2012

# Applied Mathematics

## Higher Level
## Marking Scheme (300 marks)

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Serious blunder or omission or misreading which oversimplifies:
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**5.** Scrutinise **all** pages of the answer book.

**6.** The marking scheme shows one correct solution to each question. In many cases there are other equally valid methods.

**Six questions to be answered. All questions carry equal marks. (6 × 50m)**

**1.** **(a)** A particle falls from rest from a point $X$ under gravity. After it has fallen a distance $a$ metres, another particle is given a downward speed of $\sqrt{8ga}$ m s$^{-1}$ from the same starting point $X$.

**(i)** How long is the first particle moving before the second particle starts moving? **(10)**

Let $t_1$ be the time the second particle starts after the first particle.

$$s = ut + \frac{1}{2}at^2$$

$$a = (0)(t_1) + \frac{1}{2}g(t_1)^2 \qquad \text{... (5m)}$$

$$\frac{2a}{g} = (t_1)^2$$

$$t_1 = \sqrt{\frac{2a}{g}} \text{ s} \qquad \text{... (5m)}$$

**(ii)** If the particles collide $t$ seconds after the second particle starts moving, express $t$ in terms of $a$ and $g$ **(10)**

1st: $s_1 = (0)(t + t_1) + \frac{1}{2}g(t + t_1)^2$

$$= \frac{1}{2}g(t^2 + 2tt_1 + (t_1)^2)$$

$$= \frac{1}{2}gt^2 + gtt_1 + \frac{1}{2}g(t_1)^2$$

2nd: $s_2 = \left(\sqrt{8ga}\right)t + \frac{1}{2}gt^2$

Collision: $s_1 = s_2$

$$\frac{1}{2}gt^2 + gt\sqrt{\frac{2a}{g}} + \frac{g}{2}\left(\frac{2a}{g}\right) = \left(2\sqrt{2ga}\right)t + \frac{1}{2}gt^2 \qquad \text{... (5m)}$$

$$\left(\sqrt{2ga}\right)t + a = \left(2\sqrt{2ga}\right)t$$

$$t = \frac{a}{\sqrt{2ga}} = \sqrt{\frac{a^2}{2ga}} = \sqrt{\frac{a}{2g}} \qquad \text{... (5m)}$$

**(iii)** Find, in terms of $a$, the distance from $X$ to the point where they collide. **(5)**

Distance from $X$:

$$s_2 = 2\sqrt{2ga}\left(\sqrt{\frac{a}{2g}}\right) + \frac{g}{2}\left(\frac{a}{2g}\right)$$

$$= 2a + \frac{a}{4} + \frac{9a}{4} \text{ m.} \qquad \text{... (5m)}$$

**1(b)** Two trains, S and T, travel on parallel rails in the same direction with uniform accelerations of $f$ and $\frac{3}{2}f$ , respectively.
At the same instant, both trains pass a signal box $P$ with speeds $u$ and $\frac{1}{2}u$ , respectively.
The trains are once again level when they pass the next signal box $Q$.

**(i)** Show that the greatest distance that S is ahead of T is $\dfrac{u^2}{4f}$. **(15)**

S: $v_1 = u + ft$

T: $v_2 = \frac{1}{2}u + \frac{3}{2}ft$

Greatest lead when: $v_1 = v_2$ ... (5m)

$u + ft = \frac{1}{2}u + \frac{3}{2}ft$

$2u + 2ft = u + 3ft$

$t = \dfrac{u}{f}$ ... (5m)

Greatest lead $= s_1 - s_2$

$$= \left[u\left(\frac{u}{f}\right) + \frac{1}{2}f\left(\frac{u^2}{f^2}\right)\right] - \left[\frac{1}{2}u\left(\frac{u}{f}\right) + \frac{1}{2}\left(\frac{3}{2}f\right)\left(\frac{u^2}{f^2}\right)\right]$$

$$= \frac{u^2}{f} + \frac{1}{2}\frac{u^2}{f} - \frac{1}{2}\frac{u^2}{f} - \frac{3}{4}\frac{u^2}{f}$$

$$= \frac{u^2}{4f}$$ ... (5m)

**(ii)** Find, in terms of $u$, the speed of each train as it passes $Q$. **(10)**

At $Q$: $s_1 = s_2$

$ut + \frac{1}{2}ft^2 = \frac{1}{2}ut + \frac{3}{4}ft^2$ ... (5m)

$\frac{1}{2}ut = \frac{1}{4}ft^2$

$t = \dfrac{2u}{f}$

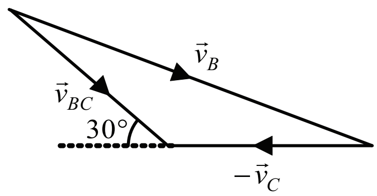
and $v_1 = u + f\left(\dfrac{2u}{f}\right) = 3u$

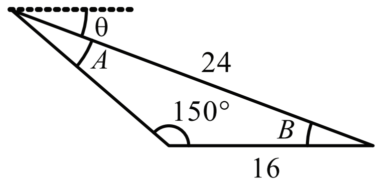
$v_2 = \frac{1}{2}u + \frac{3}{2}f\left(\dfrac{2u}{f}\right) = \dfrac{7u}{2}$. ... (5m)

**2.** **(a)** A battleship, whose top speed is 24 km/h, sets out to intercept a convoy which is sailing due east at 16 km/h. The convoy is 29 km away in the direction 30° south of east, and it maintains its course.

Find

**(i)** the shortest time in which the battleship can reach the convoy **(20)**




...(5m)

$$\vec{v}_{BC} = \vec{v}_B - \vec{v}_C$$
$$\vec{v}_{BC} = \vec{v}_B + (-\vec{v}_C)$$

Then: $\dfrac{\sin A}{16} = \dfrac{\sin 150°}{24}$ ... (5m)

$A = 19.47°$

Then:

$B = 180° - (150° + 19.47°)$
$= 10.53°$ ... (5m)

Thus:

$$\frac{|\vec{v}_{BC}|}{\sin 10.53°} = \frac{24}{\sin 150°}$$
$|\vec{v}_{BC}| = 8.772$

Thus, time to intercept

$= \dfrac{29}{8.772}$
$= 3.306$ hours
$=3$ hours and 18 minutes ... (5m)

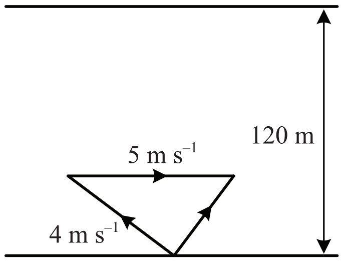
**(ii)** the direction in which the battleship should travel, correct to the nearest degree. **(5)**

Also,

$\theta = B = 10.53° = 11°$

Direction of $B$ is E11° S. ... (5m)

**2(b)** A man can swim at 4 m s$^{-1}$ in still water. He swims across a river of width 120 metres. The river flows with a constant speed of 5 m s$^{-1}$ parallel to the straight banks.
He wishes to cross by the shortest path.



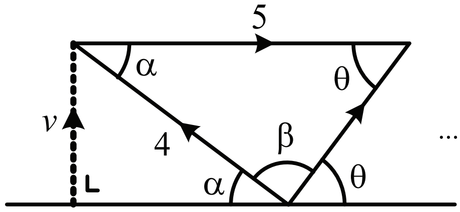

Find

**(i)** the direction he should take **(15)**

Shortest path is when $\theta$ is a max. by Sine Rule,

$$\frac{\sin\theta}{4} = \frac{\sin\beta}{5}$$

$$\sin\ \theta = \frac{4\sin\beta}{5}$$




...(5m)

For max $\theta$, and sin $\theta$,
$\sin\beta = 1$
$\beta = 90°$ ... (5m)

Thus

$$\cos\alpha = \frac{4}{5}$$

$$\alpha = \cos^{-1}\frac{4}{5} = 36.87°$$ ... (5m)

He should swim at $\cos^{-1}\frac{4}{5}$ to the upstream direction

**(ii)** the time taken to cross the river by the shortest path. **(10)**

$$\cos\alpha = \frac{4}{5} \text{ and so } \sin\alpha = \frac{3}{5}$$

From diagram,

$$\frac{v}{4} = \sin\alpha = \frac{3}{5}$$ ... (5m)

$v = 2.4$ m s$^{-1}$

Time to cross river $= \dfrac{120}{v}$

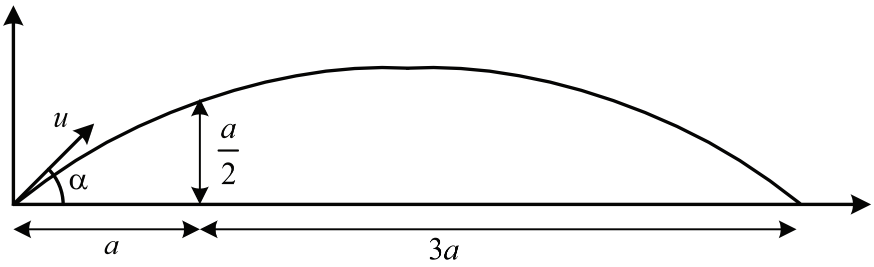
$= \dfrac{120}{2.4}$

$= 50$ s ... (5m)

**3.** **(a)** A particle is projected from a point on a horizontal plane so that it just clears a vertical wall of height $\frac{a}{2}$ metres at a horizontal distance of $a$ from the point of projection.

The particle strikes the plane at a horizontal distance of $3a$ metres beyond the wall.

Show that the angle of projection, measured to the horizontal, is $\tan^{-1}\frac{2}{3}$. **(25)**




[1] Range = $4a$
TOF: $s_y = 0$

$$u\sin\alpha t - \frac{1}{2}gt^2 = 0 \qquad \text{... (5m)}$$

$$t = \frac{2u\sin\alpha}{g}$$

$$s_x = 4a:\ u\cos\alpha\left(\frac{2u\sin\alpha}{g}\right) = 4a \qquad \text{... (5m)}$$

$$u^2 = \frac{2ga}{\sin\alpha\cos\alpha} \qquad \textbf{... 1}$$

[2] Contains the point $\left(a, \frac{a}{2}\right)$:

$$u\cos\alpha t = a,\ u\sin\alpha t - \frac{1}{2}gt^2 = \frac{a}{2} \qquad \text{... (5m)}$$

$$t = \frac{a}{u\cos\alpha}$$

Thus $$u\sin\alpha\left(\frac{a}{u\cos\alpha}\right) - \frac{g}{2}\left(\frac{a^2}{u^2\cos^2\alpha}\right) = \frac{a}{2} \qquad \text{... (5m)}$$

$$a\tan\alpha - \frac{ga^2}{2\cos^2\alpha}\left(\frac{\sin\alpha\cos\alpha}{2ga}\right) = \frac{a}{2} \qquad \text{… by } \mathbf{1}$$

$$a\tan\alpha - \frac{1}{4}a\tan\alpha = \frac{a}{2}$$

$$\frac{3}{4}\tan\alpha = \frac{1}{2}$$

$$\tan\alpha = \frac{2}{3}$$

$$\alpha = \tan^{-1}\frac{2}{3} \qquad \text{... (5m)}$$

**3(b)** A particle is projected up a plane which is inclined at an angle $\beta$ to the horizontal. The angle of projection, measured to the inclined plane, is $\alpha$. The plane of projection is vertical and contains the line of greatest slope. The particle strikes the plane after $t$ seconds.

Show that the component of the velocity parallel to the inclined plane after $t$ seconds is negative if $2\tan\alpha\tan\beta > 1$. **(25)**

TOF: $s_y = 0$

$u\sin\alpha t - \frac{1}{2}g\cos\beta t^2 = 0$ ... (5m)

$2u\sin\alpha = g\cos\beta t$

$t = \dfrac{2u\sin\alpha}{g\cos\beta}$

At TOF: $v_x = u\cos\alpha - g\sin\beta\left(\dfrac{2u\sin\alpha}{g\cos\beta}\right)$ ... (5m)

$v_x = u\cos\alpha - \dfrac{2u\sin\alpha\sin\beta}{\cos\beta}$

$v_x = u\cos\alpha - \dfrac{u(\cos\alpha\cos\beta - 2\sin\alpha\sin\beta)}{\cos\beta}$ ... (5m)

If $v_x < 0$:

$\dfrac{u(\cos\alpha\cos\beta - 2\sin\alpha\sin\beta)}{\cos\beta} < 0$ ... (5m)

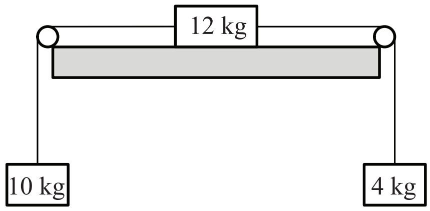
$\cos\alpha\cos\beta - 2\sin\alpha\sin\beta < 0$

$\cos\alpha\cos\beta < 2\sin\alpha\sin\beta$

$1 > 2\dfrac{\sin\alpha}{\cos\alpha}\dfrac{\sin\beta}{\cos\beta}$

$2\tan\alpha\tan\beta > 1$ ... (5m)

**4.** **(a)** A particle of mass 12 kg rests on a rough horizontal table. It is attached by two horizontal inelastic strings to particles of masses of 10 kg and 4 kg which hang freely over smooth pulleys at opposite edges of the table.

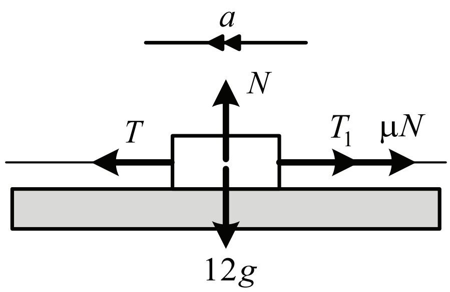



The coefficient of friction between the 12 kg mass and the table is $\mu$.

**(i)** Show that if $\mu = \frac{1}{4}$, the common acceleration of the particles is $\frac{3g}{26}$. **(20)**

Let the common acceleration be $a$.

**12 kg particle:**



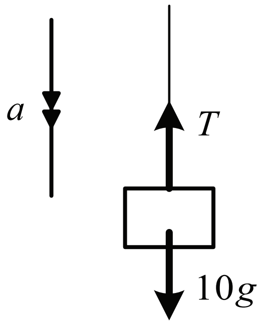

$\updownarrow$ : $N = 12g$

$\leftrightarrow$ : $12a = T - T_1 - \mu N$

$12a = T - T_1 - 12\mu g$ ... (5m)

... **1**

**10 kg particle:**



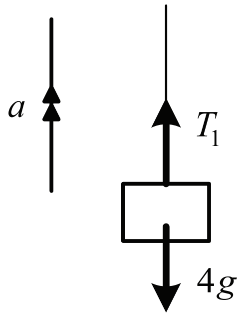

$\updownarrow$ : $10a = 10g - T$

... **2**

**4 kg particle:**

$\updownarrow$ : $4a = T_1 - 4g$ ... (5m)

... **3**




**2:** $T = 10g - 10a$

**3:** $T_1 = 4g + 4a$

**1:** $12a = (10g - 10a) - (4g + 4a) - 12\mu g$

$26a = 6g - 12\mu g$

$13a = 3g - 6\mu g$

$a = \dfrac{3g(1 - 2\mu)}{13}$ ... (5m)

If $\mu = \dfrac{1}{4}$ then

$a = \dfrac{3g\left(1 - \frac{1}{2}\right)}{13} = \dfrac{3g}{26}$ ... (5m)

**4(a) (ii)** Find the least value of $\mu$ for which the particles will not move. **(5)**

The particles will not move if $a \le 0$.

$$\frac{3g(1-2\mu)}{13} \le 0$$

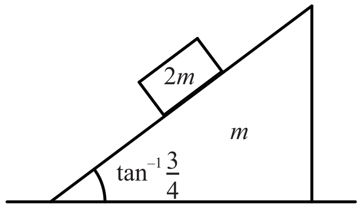
$$1 - 2\mu \le 0$$

$$1 \le 2\mu$$

$$\mu \ge \frac{1}{2} \qquad \text{... (5m)}$$

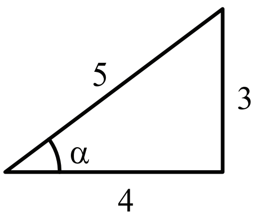
The particles will not move if $\mu \ge \frac{1}{2}$. Thus $\frac{1}{2}$ is the least value of $\mu$ for which the particles will not move.

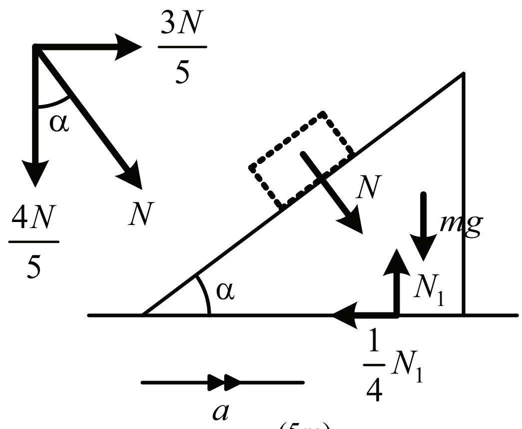
**4(b)** A smooth particle of mass $2m$ rests on the smooth face of a wedge of mass $m$ and angle $\tan^{-1}\frac{3}{4}$ The wedge is free to move on a rough horizontal table, the coefficient of friction being $\frac{1}{4}$. The system is released from rest.



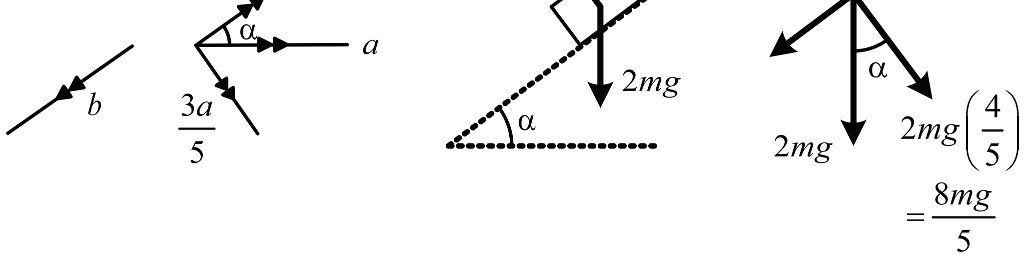

**(i)** Show, on separate diagrams, the forces acting on the wedge and the particle. **(10)**

$\tan\ \alpha = \frac{3}{4}$




... (5m)




... (5m)

**4(b) (ii)** Show that the acceleration of the wedge is $\frac{39g}{148}$. **(15)**

**Wedge:**

$\leftrightarrow: \quad ma = \frac{3N}{5} - \frac{1}{4}N_1$

$\updownarrow: \quad N_1 = mg + \frac{4N}{5}$

Then: $ma = \frac{3N}{5} - \frac{1}{4}\left(mg + \frac{4N}{5}\right)$

$ma = \frac{3N}{5} - \frac{1}{4}mg - \frac{N}{5}$ (× 20)

$20ma = 8N - 5mg$ **… 1** ... (5m)

**Particle:**

$\nwarrow\!\!\searrow: \quad 2m = \left(\frac{3a}{5}\right) = \frac{8mg}{5} - N$ (× 5) ... (5m)

$6ma = 8mg - 5N$ **… 2**

Then:

**1** × 5: $100ma = 40N - 25mg$

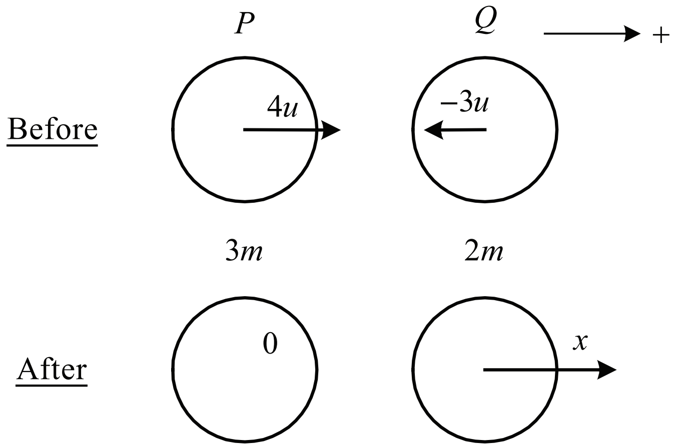
**2** × 8: $\underline{48ma = 64mg - 40N}$

$148ma = 39mg$

$a = \frac{39g}{148}$ ... (5m)

**5.** **(a)** A smooth sphere P, of mass $3m$ and moving with speed $4u$, collides directly with another smooth sphere Q, of mass $2m$ and moving in the opposite direction with a speed $3u$. Sphere P is brought to rest by the collision.

**(i)** Show that the velocity of Q is reversed in direction but unchanged in magnitude by the collision. **(10)**




**PCM:** $2mx = 3m(4u) + 2m(-3u)$ ... (5m)

$2x = 12u - 6u$

$2x = 6u$

$x = 3u$

Thus Q has the direction of its velocity reversed, but its speed is maintained. ... (5m)

**(ii)** Find $e$, the coefficient of restitution. **(5)**

**NEL:** $0 - x = -e(4u + 3u)$

$-3u = -7eu$

$7e = 3$

$e = \frac{3}{7}$ ... (5m)

**(iii)** Determine the fraction of the total kinetic energy that was lost during the impact. **(10)**

KE before $= \frac{1}{2}(3m)(4u)^2 + \frac{1}{2}(2m)(-3u)^2$

$= 24mu^2 + 9mu^2 = 33mu^2$

KE after $= \frac{1}{2}(2m)(3u)^2 = 9mu^2$

Loss in KE $= 33mu^2 - 9mu^2 = 24mu^2$ ... (5m)

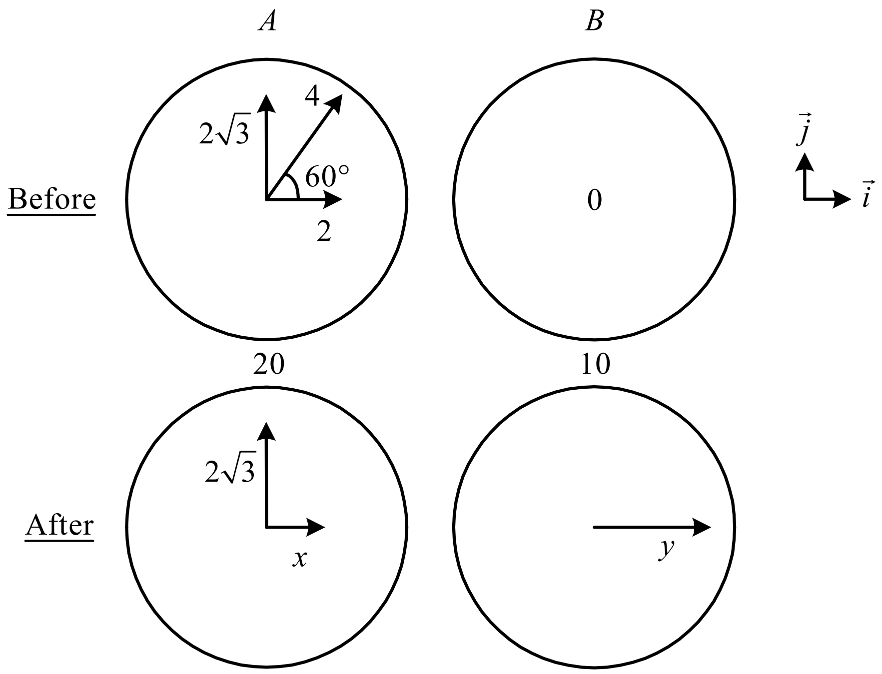
Fraction loss in KE $= \frac{24mu^2}{33mu^2} = \frac{8}{11}$ ... (5m)

**5(b)** Two smooth spheres, A and B, of equal radii but of masses 20 kg and 10 kg, respectively, lie on a smooth horizontal table. B is at rest and A, moving at 4 m $s^{-1}$, strikes B obliquely, the direction of A at the moment of impact making an angle of 60° with the line of centres.

The coefficient of restitution between the spheres is $\frac{1}{2}$.

Find

**(i)** the speed of A and the speed of B directly after impact **(15)**




**PCMi:** $20x + 10y = 20(2)$

$2x + y = 4$ **… 1**

**NELi:** $x - y = -\frac{1}{2}(2 - 0)$

$x - y = -1$ **… 2** ... (5m)

**1:** $2x + y = 4$

**2:** $\underline{x - y = -1}$

$3x = 3$

$x = 1$

**1:** $2 + y = 4$

$y = 2$

After impact,

$|\vec{v}_A = \vec{i} + 2\sqrt{3}\vec{j}$ ... (5m)

$|\vec{v}_A| = \sqrt{1^2 + (2\sqrt{3})^2} = \sqrt{13}$ m $s^{-1}$

and

$\vec{v}_B = 2\vec{i}$

$|\vec{v}_B| = 2$ m $s^{-1}$ ... (5m)

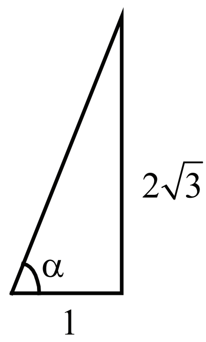
**5(b) (ii)** the angle between their directions of motion after impact. **(5)**

If A's direction of motion after impact makes an angle $\alpha$ with the line of centres, then

$\tan\ \alpha = \frac{2\sqrt{3}}{1} = 2\sqrt{3}$

Thus $\alpha = \tan^{-1} 2\sqrt{3}\ = 73.9°$

is the angle between their directions of motion after impact.




... (5m)

**(iii)** the loss in kinetic energy due to the impact. **(5)**

$KE_{before} = \frac{1}{2}(20)(4^2)$

$= 160$ J

$KE_{after} = \frac{1}{2}(20)(\sqrt{13}\ )^2 + \frac{1}{2}(10)(2)^2$

$= 130 + 20$

$= 150$ J

$KE_{loss} = 160 - 150 = 10$ J ... (5m)

**OR**

KEi before $= \frac{1}{2}(20)(2^2)$

$= 40$ J

KEi after $= \frac{1}{2}(20)(1)^2 + \frac{1}{2}(10)(2)^2$

$= 10 + 20$

$= 30$ J

Loss in KE $= 40 - 30$

$= 10$ J

**6.** **(a)** A particle moves with simple harmonic motion and performs two complete oscillations per second. Its speed when it is $\frac{\sqrt{3}}{4}$ metres from the centre of the motion is half the maximum speed.

**(i)** Find the amplitude of the motion. **(15)**

Two complete oscillations per second:

$$\frac{1}{f} = \frac{2\pi}{\omega}$$

$\omega = 2\pi(2) = 4\pi$ ... (5m)

Maximum speed = $\omega a = 4\pi a$

$x = \frac{\sqrt{3}}{4}$, $v = 2\pi a$: $2\pi a = = 4\pi\sqrt{a^2 - \frac{3}{16}}$ ... (5m)

$$\frac{a}{2} = \sqrt{a^2 - \frac{3}{16}}$$

$$\frac{a^2}{4} = a^2 - \frac{3}{16}$$

$$\frac{3}{16} = \frac{3}{4}a^2$$

$$a^2 = \frac{1}{4}$$

$a = \frac{1}{2}$ ... (5m)

**(ii)** The speed of the particle is three quarters of the maximum speed at two points. Find the distance between these two points. **(10)**

$\frac{3}{4}\omega a = \omega\sqrt{a^2 - x^2}$ ... (5m)

$$\frac{9a^2}{16} = a^2 - x^2$$

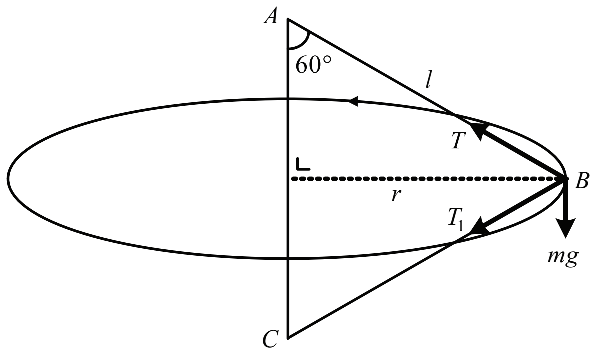
$$x^2 = \frac{7a^2}{16} = \frac{7}{16}\left(\frac{1}{4}\right) = \frac{7}{64}$$

$$x = \pm\frac{\sqrt{7a}}{8}$$

Distance between points $= \left(\frac{\sqrt{7}}{8}\right) - \left(-\frac{\sqrt{7}}{8}\right) = \frac{\sqrt{7}}{4}$ m ... (5m)

**6(b)** Two light, inextensible strings AB and BC, each of length $l$, are attached to a particle of mass $m$ at B. The other ends, A and C, are fixed to two points in a vertical line such that A is distant $l$ above C. The particle describes a horizontal circle with constant angular velocity.

Find the greatest periodic time for which both strings will be taut. **(25)**




From the equilateral triangle, $r = \frac{\sqrt{3}}{2}l$

The circular motion force is $m\omega^2 r = m\omega^2\left(\frac{\sqrt{3}}{2}l\right) = \frac{\sqrt{3}}{2}m\omega^2 l$

$\uparrow = \downarrow$ : $\frac{T}{2} = \frac{T_1}{2} + mg$ ... (5m)

$T = T_1 + 2mg$ **… 1**

Circular : $\frac{\sqrt{3}}{2}T + \frac{\sqrt{3}}{2}T_1 = \frac{\sqrt{3}}{2}m\omega^2 l$ ... (5m)

$T + T_1 = m\omega^2 l$ **… 2**

**1 :** $-T + T_1 = -2mg$

**2 :** $\underline{T + T_1 = m\omega^2 l}$

$2T_1 = m(\omega^2 l - 2g)$

$T_1 = \frac{m}{2}(\omega^2 l - 2g)$

The upper string will always be taut.
The lower string will be taut if

$T_1 \geq 0$

$\frac{m}{2}(\omega^2 l - 2g) \geq 0$ ... (5m)

$\omega^2 l \geq 2g$

$\omega^2 \geq \frac{2g}{l}$

$\omega \geq \sqrt{\frac{2g}{l}}$ ... (5m)

Thus $\frac{1}{\omega} \leq \sqrt{\frac{l}{2g}}$

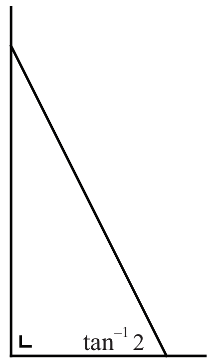
$\frac{2\pi}{\omega} \leq 2\pi\sqrt{\frac{l}{2g}}$

Periodic time, $T \leq 2\pi\sqrt{\frac{l}{2g}}$ ... (5m)

Thus the greatest periodic time for which both strings are taut is $2\pi\sqrt{\frac{l}{2g}}$

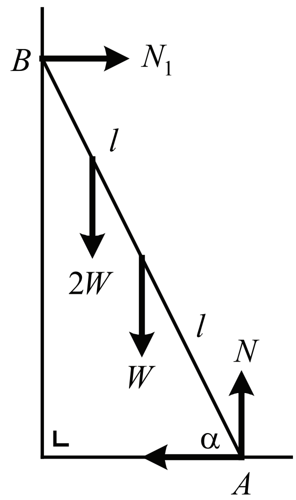
**7.** **(a)** A uniform ladder, of weight $W$ and length $2l$, rests with one end on rough horizontal ground and the other end against a smooth vertical wall. The ladder makes an angle of $\tan^{-1} 2$ with the ground and is in a vertical plane perpendicular to the wall.
The coefficient of friction between the ladder and the ground is $\frac{1}{3}$.

Find the maximum distance up the ladder that a boy of weight $2W$ can climb before the ladder begins to slip.



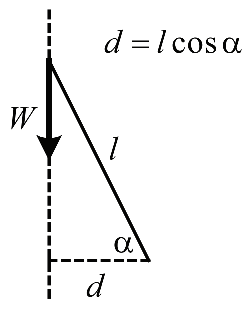

**(20)**

Let $x$ be the distance up the rod of the boy when it is on the point of slipping.
Let $\alpha = \tan^{-1}2$
Then $\tan \alpha = 2$
Rod $AB$:
$\uparrow = \downarrow$: $N = 3W$
$\leftarrow = \rightarrow$: $N_1 = F$ ... (5m)

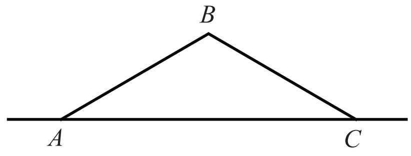



Limiting friction:
$$F = \frac{1}{3}N$$
$$F = W$$
$$N_1 = W$$ ... (5m)

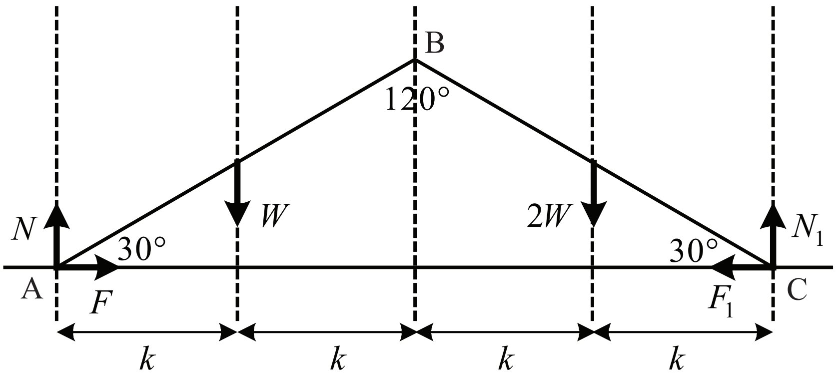
Moments about $A$:
$Wl \cos \alpha + 2Wx \cos \alpha$
$= N_1 2l \sin \alpha$ ... (5m)
$Wl \cos \alpha + 2Wx \cos \alpha$
$= 2Wl \sin \alpha$
$l + 2x = 2l \tan \alpha$
$l + 2x = 2l\,(2)$
$2x = 3l$
$$x = \frac{3l}{2}$$ ... (5m)



**7(b)** Two equal uniform rods *AB* and *BC*, of weights *W* and 2*W*, respectively, are freely jointed at *B*. They rest in a vertical plane with *A* and *C* on rough horizontal ground. The coefficient of friction at both *A* and *C* is $\mu$. When the angle *ABC* is 120° slipping is about to occur.




**(i)** Explain why slipping occurs at *A* before *C*. **(25)**




Structure *ABC*:

$N + N_1 = 3W$ ... (5m)

$F = F_1$

Moments about *A*:

$Wk + 2W3k = N_1 4k$

$7W = 4N_1$

$N_1 = \frac{7W}{4}$

$N = \frac{5W}{4}$ ... (5m)

Rod *AB*:

Moments about *B*:

$Wl\cos 30° + F2l\sin 30°$

$= N2l\cos 30°$ ... (5m)

$W + 2F\tan 30° = 2N$

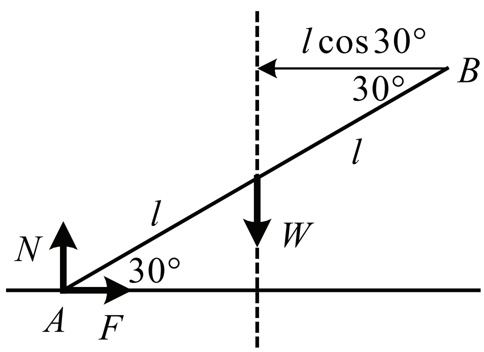
$W = 2F\left(\frac{1}{\sqrt{3}}\right) = 2\left(\frac{5W}{4}\right)$

$\frac{2F}{\sqrt{3}} = \frac{3W}{2}$

$F = \frac{3\sqrt{3}W}{4} = F_1$

At *A*,

$\frac{F}{N} = \frac{\frac{3\sqrt{3}W}{4}}{\frac{5W}{4}} = \frac{3\sqrt{3}}{5}$




... (5m)

At $C$,

$$\frac{F_1}{N_1} = \frac{\frac{3\sqrt{3}W}{4}}{\frac{7W}{4}} = \frac{3\sqrt{3}}{7}$$

As $\frac{F}{N} > \frac{F_1}{N_1}$ slipping occurs first at $A$. ... (5m)

**7(b) (ii)** Find the value of $\mu$. **(5)**

As slipping is about to occur at $A$,

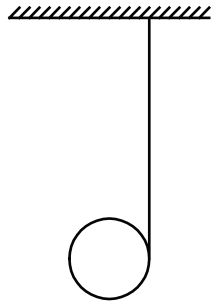
$$\mu = \frac{F}{N} = \frac{3\sqrt{3}}{5}$$ ... (5m)

**8.** **(a)** Prove that the moment of inertia of a uniform circular disc of mass $m$ and radius $r$ about an axis through its centre perpendicular to its plane is $\frac{1}{2}mr^2$. **(20)**

Standard Proof

Moment of mass element ... (5m)
Moment of body ... (5m)
Integral ... (5m)
Deduce ... (5m)

**8(b)** A light string is wound around the rim of a uniform disc of radius $2r$ and mass $2m$. One end of the string is attached to the rim of the disc and the other end is attached to a fixed point above the disc, with the plane of the disc vertical.
When the disc is released from rest, it falls vertically and the string unwinds.



Find

**(i)** the vertical acceleration of the disc **(20)**

For the disc,

$I = \frac{1}{2}mr^2$

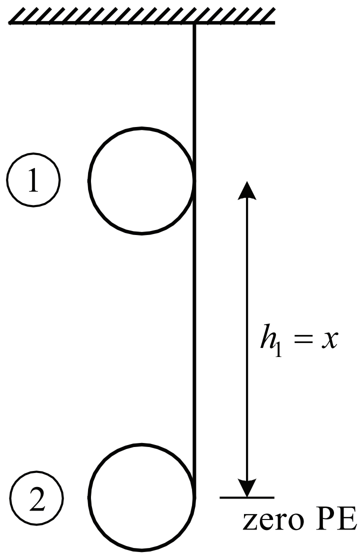
$= \frac{1}{2}(2m)(2r)^2$

$=4mr^2$ ... (5m)

Rolling disc:

$v = r\omega$

$\omega = \frac{v}{2r}$ ... (5m)




**Position 1:** Starts from rest.

$KE_1 = 0$

$PE_1 = mgh_1$

$= (2m)gx$

$= 2mgx$

**Position 2:** After falling $x$ m.

$KE_2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$= \frac{1}{2}(2m)v^2 + \frac{1}{2}(4mr^2)\left(\frac{v}{2r}\right)^2$

$= mv^2 + \frac{1}{2}mv^2$

$= \frac{3}{2}mv^2$

$PE_2 = 0$

**PCE:** $KE_2 + PE_2 = KE_1 + PE_1$

$$\frac{3}{2}mv^2 + 0 = 0 + 2mgx \qquad \text{... (5m)}$$

$$v^2 = \frac{4gx}{3}$$

To find the acceleration

$$v^2 = u^2 + 2as$$

$$\frac{4gx}{3} = 0 + 2ax$$

$$a = \frac{2g}{3} \text{ m s}^{-2} \qquad \text{... (5m)}$$

**8(b) (ii)** the tension in the string. **(5)**

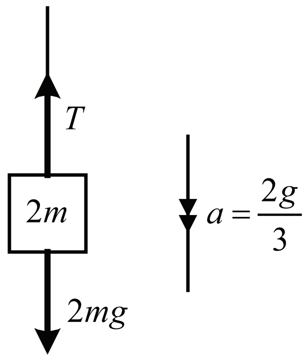
To find the tension in the string:

↕ : Newton's 2nd Law:

$$(2m)\left(\frac{2g}{3}\right) = 2mg - T$$

$$T = 2mg - \left(\frac{4mg}{3}\right)$$

$$T = \left(\frac{2mg}{3}\right) \text{ N} \qquad \text{... (5m)}$$




**(iii)** the time taken for the disc to fall a distance $r$. **(5)**

To find the time taken to fall a distance $r$:

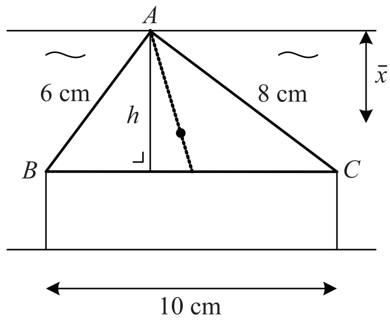
$$s = ut + \frac{1}{2}at^2$$

$$r = (0)t + \frac{1}{2}\left(\frac{2g}{3}\right)t^2$$

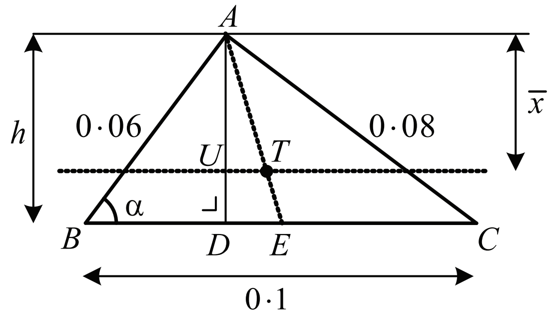
$$t^2 = \frac{3r}{g}$$

$$t = \sqrt{\frac{3r}{g}} \text{ s} \qquad \text{... (5m)}$$

**9.** **(a)** The triangle $ABC$ is right angled at $A$, with $|AB| = 6$ cm, $|AC| = 8$ cm and $|BC| = 10$ cm. This triangular lamina is held submerged in water with $A$ at the surface of the water and $[BC]$ horizontal. $h$ is the height of the triangle, as shown. $\overline{x}$ is the depth of the centre of mass of the triangle below the water surface.




**(i)** Show that $\overline{x} = \frac{2}{3}h$ and hence find $\overline{x}$ . **(10)**




If $E$ is the midpoint of $[BC]$ and $T \in [AE]$ such that $|AT| : |TE| = 2{:}1$, then $T$ is the centre of mass of the triangle.
As $UT$ is parallel to $BC$, it divides $[AD]$ and $[AE]$ in the same ratio.

Thus $|AU| = \frac{2}{3}|AD|$

*i.e.* $\overline{x} = \frac{2}{3}h$ ... (5m)

Also $\sin\ \alpha = \frac{0.08}{0.1} = \frac{h}{0.06}$

$h = 0.048$

and $\overline{x} = \frac{2}{3}h$

$\frac{2}{3}(0.048)$

$= 0.032$ m ... (5m)

**(ii)** Find the thrust on the triangular lamina. **(10)**

Let $A$ be the area of the lamina.

$A = \frac{1}{2}(0.06)(0.08)$

$= 0.0024\ \text{m}^2$ ... (5m)

Let $T$ be the thrust on the lamina.

$T = \rho g \overline{x} A$

$T = (1000)(9.8)(0.032)(0.0024)$

$T = 0.75264$ N ... (5m)

**9(b)** A compound object is made of some plastic of relative density 2.5 and of some metal of relative density 3.
The compound object weighs 12.544 N in water and 9.7216 N in a liquid of relative density 1.4.

Find the volume of the plastic and the volume of the metal in the object. **(30)**

Object: mass $M$, volume $V$
Plastic: mass $m_1$, volume $V_1$, $s_1 = 2.5$
Metal: mass $m_2$, volume $V_2$, $s_2 = 3$
Then

$W = Mg$
$= (m_1 + m_2)g$
$= (1000s_1V_1 + 1000s_2V_2)g$
$= (1000(2.5)V_1 + 1000(3)V_2)g$ ... (5m)
$= (2500V_1 + 3000V_2)g$

**Water:**
Buoyancy:
$B_1 = 1000(1)(V_1 + V_2)g$
$=(1000V_1 + 1000V_2)g$ ... (5m)

Apparent weight:
$W - B_1 = 12.544$
$(2500V_1 + 3000V_2)g - (1000V_1 + 1000V_2)g = 12.544$ ... (5m)
$1500V_1 + 2000V_2 = 1.28$
$3V_1 + 4V_2 = 0.00256$ **... 1**

**Liquid:**
Buoyancy:
$B_2 = 1000(1.4)(V_1 + V_2)g$ ... (5m)
$=(1400V_1 + 1400V_2)g$

Apparent weight:
$W - B_2 = 9.7216$
$(2500V_1 + 3000V_2)g - (1400V_1 + 1400V_2)g = 9.7216$ ... (5m)
$1100V_1 + 1600V_2 = 0.0992$
$11V_1 + 16V_2 = 0.00992$ **... 2**

Solving:
**1** × –4: $-12V_1 - 16V_2 = -0.01024$
**2**: $\underline{11V_1 + 16V_2 = 0.00992}$
$-V_1 = -0.00032$
$V_1 = 0.00032 \text{ m}^3$
$= 320 \text{ cm}^3$
**1**: $3(0.00032) + 4V_2 = 0.00256$
$4V_2 = 0.0016$
$V_2 = 0.0004 \text{ m}^3$
$V_2 = 400 \text{ cm}^3$ ... (5m)

**10.** **(a)** Solve the differential equation

$$x\frac{dy}{dx} = y(3-x)$$

given that $y = \frac{2}{e}$ when $x = 1$. **(20)**

$x\frac{dy}{dx} = y(3-x)$

$xdy = y(3-x)dx$

$\frac{dy}{y} = \frac{3-x}{x}dx$

$\int\frac{dy}{y} = \int\left(\frac{3}{x}-1\right)dx$ ... (5m)

$\ln|y| = 3\ln|x| - x + \ln c$ ... (5m)

$\ln|y| = \ln|x^3| + \ln e^{-x} + \ln c$

$\ln|y| = \ln ce^{-x}|x^3|$

$y = cx^3e^{-x}$ ... (5m)

IC: $x = 1, y = \frac{2}{e}$

$\frac{2}{e} = c(1)e^{-1}$

$\frac{2}{e} = \frac{c}{e}$

$2 = c$

Unique solution:

$y = 2x^3e^{-x}$ ... (5m)

**OR**

$x\frac{dy}{dx} = y(3-x)$

$\frac{dy}{y} = \frac{(3-x)}{x}dx$

$\int\frac{dy}{y} = \int\frac{3-x}{x}dx$ ... (5m)

$\int\frac{dy}{y} = \int\left(\frac{3}{x}-1\right)dx$

$\ln|y| = 3\ln|x| - x + \ln c$ ... (5m)

$x = 1, y = \frac{2}{e}$

$\ln\frac{2}{e} = 3\ln 1 - 1 + \ln c$

$\ln 2 - \ln e = 3\ln 1 - 1 + \ln c$

$\ln 2 - 1 = 3(0) - 1 + \ln c$

$\ln 2 = \ln c$

$c = 2$ ... (5m)

$\ln|y| = 3\ln|x| - x + \ln 2$

$\ln|y| = \ln|x^3| + \ln e^{-x} + \ln 2$

$\ln|y| = \ln 2x^3e^{-x}$

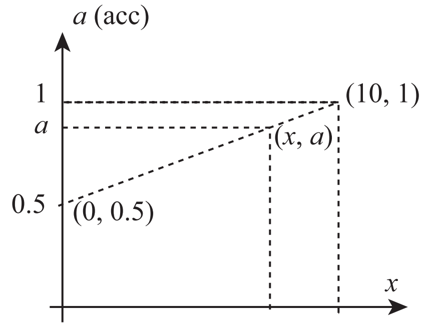
$y = 2x^3e^{-x}$ ... (5m)

**10(b)** A particle starts with a speed of 15 m s$^{-1}$ and moves in a straight line. The particle is subjected to a force which produces an acceleration which is initially 0.5 m s$^{-2}$ and which increases uniformly with the distance moved, having a value of 1 m s$^{-2}$ when the particle has moved a distance of 10 metres.

If $v$ m s$^{-1}$ is the speed of the particle when it has moved a distance of $x$ metres,

**(i)** prove that while the particle is in motion, **(10)**

$$v\frac{dv}{dx} = \frac{x+10}{20}$$




Slope of line:

$$m = \frac{1-0.5}{10-0} = \frac{1}{20}$$

Equation of line:

$$(y-0.5) = \frac{1}{20}(x-0)$$

$$20y - 10 = x$$

Line contains $(x, a)$, where $a$ is the acceleration.

$$20a - 10 = x$$

$$20a = x + 10$$

$$a = \frac{1x+10}{20}$$ ... (5m)

Thus

$$v\frac{dv}{dx} = \frac{x+10}{20}$$ ... (5m)

**(ii)** calculate the distance moved by the particle while its speed increases to $5\sqrt{33}$ m s$^{-1}$. **(20)**

$$vdv = \frac{1}{20}(x+10)dx$$

$$\int vdv = \frac{1}{20}\int (x+20)dx$$ ... (5m)

$$\frac{1}{2}v^2 = \frac{1}{20}\left(\frac{1}{2}x^2 + 10x\right) + c$$ ... (5m)

IC: $v = 15, x = 0$

$$\frac{225}{2} = c$$

Unique solution:

$$\frac{1}{2}v^2 = \frac{1}{40}x^2 + x + 225$$ ... (5m)

$$v^2 = \frac{1}{20}x^2 + x + 225$$

$$v = 5\sqrt{33}$$

$$825 = \frac{1}{20}x^2 + x + 225$$

$$600 = \frac{1}{20}x^2 + x$$

$x^2 + 20x - 12000 = 0$
$(x - 100)(x + 120) = 0$
$x - 100 = 0$ or $x + 120 = 0$
$x = 100$ or $x = -120$ (impossible)
Answer: $x = 100$ m ... (5m)