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Pre-Leaving Certificate Examination, 2011

Applied Mathematics

Marking Scheme

Ordinary	Pg. 2
Higher	Pg. 21

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Pre-Leaving Certificate Examination, 2011

Applied Mathematics

Ordinary Level
Marking Scheme (300 marks)

General Instructions

1. Penalties of three types are applied to students' work as follows:

Slips	- numerical slips	S(–1)
Blunders	- mathematical errors	B(–3)
Misreading	- if not serious	M(–1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

4. Number the grid on each script 1 to 9 in numerical order, not the order of answering.

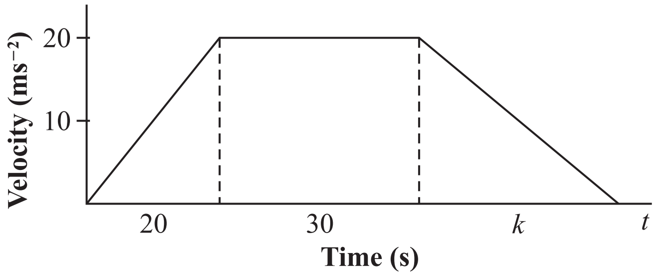
5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

1. A car travels along a straight level road.
It starts from rest at a point P and accelerates uniformly for 20 seconds to a speed of 40 ms^{-1}.
It then moves at a constant speed of 40 ms^{-1} for 30 seconds.
Finally the car decelerates uniformly from 40 ms^{-1} to rest at Q in k seconds.
The distance from P to Q is 2,200 m.

(i) Draw a speed-time graph of the motion of the car from P to Q. **(10)**



... (10m)

(ii) Find the uniform acceleration of the car. **(10)**

$$v = u + at$$
$$u_1 = 0$$
$$v_1 = 40$$
$$t_1 = 20$$

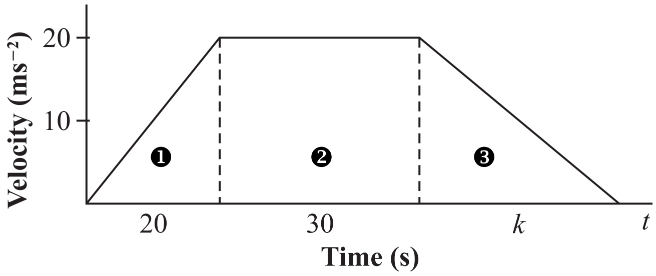
$$\Rightarrow 40 = 0 + a(20)$$ ... (5m)
$$= 20a$$
$$\Rightarrow a_1 = \frac{40}{20}$$
$$= 2 \text{ m/s}$$ ... (5m)

(iii) Find the distance travelled while the car is accelerating. **(5)**

$$s = ut + \tfrac{1}{2}at^2$$
$$u_1 = 0$$
$$a_1 = 2$$
$$t_1 = 20$$

$$\Rightarrow s_1 = 0(20) + \frac{1}{2}(2)(20)^2$$
$$= 0 + 400$$
$$= 400 \text{ m}$$ ... (5m)

1(iv) Find the distance travelled while the car is decelerating. **(15)**



s_3 (area 3)	=	Total distance – area 1 + area 2	... (5m)
	=	$2{,}200 - \frac{1}{2}(20)(40) + (30)(40)$	... (5m)
	=	2,200 – 400 + 1,200	
	=	600 m	... (5m)

<u>or</u>

$$s = \left(\frac{u+v}{2}\right)t$$

$u_1 = 0$
$v_1 = 40$
$t_1 = 20$
$t_2 = 30$

<u>Distance accelerating</u>

$$s_1 = \left(\frac{0+40}{2}\right)(20) = \frac{1}{2}(40)(20) = 400 \text{ m}$$

<u>Distance at constant speed</u>

$$s_2 = \left(\frac{40+40}{2}\right)(30) = (40)(30) = 1{,}200 \text{ m}$$

<u>Distance decelerating</u>

	d (total distance)	=	$s_1 + s_2 + s_3$	
⇒	s_3	=	$d - s_1 - s_2$	... (5m)
		=	$2{,}200 - \frac{1}{2}(40)(20) - (40)(30)$	... (5m)
		=	2,200 – 400 – 1,200	
		=	600 m	... (5m)

1(v) Find the value of k. **(10)**

$$
\begin{aligned}
s_3 &= \text{area } 3 \\
&= 600 \\
u_3 &= 40 \\
t_3 &= k \\
\Rightarrow \quad 40k &= 600 && \text{... (5m)} \\
\Rightarrow \quad k &= \frac{600}{40} \\
&= 15 \text{ s} && \text{... (5m)}
\end{aligned}
$$

<u>or</u>

$$
\begin{aligned}
v^2 &= u^2 + 2as \\
u_3 &= 40 \\
v_3 &= 0 \\
s_3 &= 600 \\
\Rightarrow \quad 0 &= (40)^2 + 2a(600) \\
\Rightarrow \quad 1{,}200a &= -1{,}600 \\
\Rightarrow \quad a &= -\frac{1{,}600}{1{,}200} \\
&= -\frac{4}{3}
\end{aligned}
$$

$$
\begin{aligned}
v &= u + at \\
u_3 &= 40 \\
v_3 &= 0 \\
a_3 &= -\frac{4}{3} \\
\Rightarrow \quad 0 &= 40 - \frac{4}{3}k && \text{... (5m)} \\
\Rightarrow \quad \frac{4}{3}k &= 40 \\
\Rightarrow \quad k &= \frac{(40)(3)}{4} \\
&= 30 \text{ s} && \text{... (5m)}
\end{aligned}
$$

2. A river is 48 m wide and is flowing with a speed of 1.2 ms^{-1} parallel to the straight banks. The speed of a boat in still water is 2 ms^{-1}.

(i) Find the total time it takes the boat to travel up river (against the flow) for 800 m and then back the same distance (with the flow). **(20)**

$$t_{\text{up river}} = \frac{d}{v_{\text{boat}} - v_{\text{river}}}$$

$$t_{\text{down river}} = \frac{d}{v_{\text{boat}} + v_{\text{river}}}$$

$$d = 800$$

$$v_{\text{boat}} = 2$$

$$v_{\text{river}} = 1.2$$

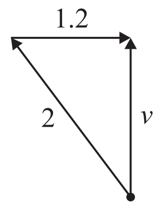
$$\Rightarrow \quad t_{\text{total}} = \frac{800}{2.0 - 1.2} + \frac{800}{2.0 + 1.2} \quad \text{... (10m)}$$

$$= \frac{800}{0.8} + \frac{800}{3.2}$$

$$= 1{,}000 + 250$$

$$= 1{,}250 \text{ s} \quad \text{... (10m)}$$

(ii) How long does it take the boat to cross the river by the shortest path? **(15)**



$$v^2 = (v_{\text{boat}})^2 - (v_{\text{river}})^2$$

$$= (2)^2 - (1.2)^2 \quad \text{... (5m)}$$

$$= 4 - 1.44$$

$$= 2.56$$

$$\Rightarrow \quad v = 1.6 \quad \text{... (5m)}$$

$$t = \frac{d}{v}$$

$$\Rightarrow \quad t = \frac{48}{1.6}$$

$$= 30 \text{ s} \quad \text{... (5m)}$$

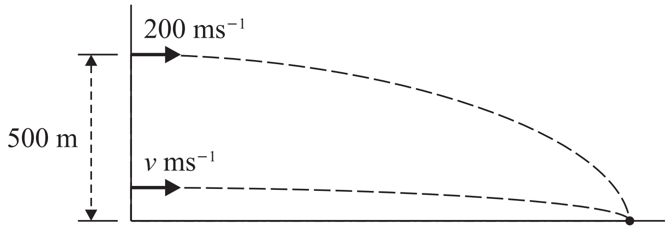
(iii) What is the shortest time it takes the boat to cross the river? **(15)**

$$t = \frac{d}{v_{\text{boat}}}$$

$$= \frac{48}{2} \quad \text{... (10m)}$$

$$= 24 \text{ s} \quad \text{... (5m)}$$

3. A straight vertical cliff is 500 m high. A particle is fired horizontally directly out to sea from the top of the cliff with an initial speed of 200 ms^{-1}.



(i) Find the distance from the foot of the cliff that the particle strikes the sea. **(30)**

$$s_y = u_y t + \tfrac{1}{2} a_y t^2$$
$$s_y = -500$$
$$u_y = 0$$
$$a_y = g = -10$$

$\Rightarrow -500 = 0(t) + \frac{1}{2}(-10)t^2$ ... (10)

$\Rightarrow -500 = -5t^2$

$\Rightarrow 5t^2 = 500$

$\Rightarrow t^2 = 100$

$\Rightarrow t = 10$ s ... (5m)

$$s_x = u_x t$$
$$u_x = 200$$
$$t = 10$$

$\Rightarrow s_x = 200(10)$ ... (10m)

$= 2,000$ m ... (5m)

A second particle is fired horizontally directly out to sea with an initial speed of v ms^{-1} from a point 80 m above the foot of the cliff. This particle strikes the sea at the same point as the first particle.

(ii) Find the value of v. **(20)**

$$s_y = u_y t + \tfrac{1}{2} a_y t^2$$
$$s_y = -80$$
$$u_y = 0$$
$$a_y = g = -10$$

$\Rightarrow -80 = 0(t) + \frac{1}{2}(-10)t^2$ ... (5m)

$\Rightarrow -80 = -5t^2$

$\Rightarrow 5t^2 = 80$

$\Rightarrow t^2 = 16$

$\Rightarrow t = 4$ s ... (5m)

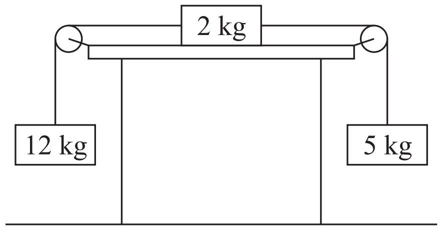
$$s_x = u_x t$$
$$s_x = 2,000$$
$$t = 4$$

$\Rightarrow 2,000 = v(4)$ ... (5m)

$\Rightarrow v = \frac{2,000}{4}$

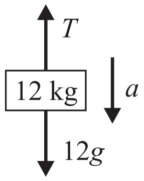
$= 500$ ms^{-1} ... (5m)

4. A particle, of mass 2 kg, rests on a smooth horizontal table. It is connected by two taut, light, inextensible strings which pass over smooth light pulleys at the edges of the table to particles, of masses 5 kg and 12 kg, which hang freely under gravity.

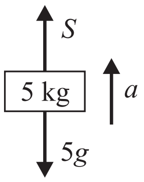


The system is released from rest.

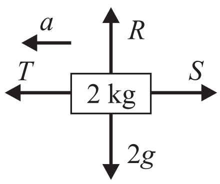
(i) Show on separate diagrams the forces acting on each particle. **(15)**

❶ 

... (5m)

❷ 

... (5m)

❸ 

... (5m)

(ii) Write down the equation of motion for each particle. **(15)**

❶ $12g - T = 12a$ ... (5m)

❷ $S - 5g = 5a$ ... (5m)

❸ $T - S = 2a$ ... (5m)

❹
$$\begin{aligned} R &= 2g \\ &= 2(10) \\ &= 20 \text{ N} \end{aligned}$$

(iii) Find the common acceleration of the particles. **(10)**

❶
$$\begin{aligned} 12g - T &= 12a \\ \Rightarrow 12(10) - T &= 12a \\ \Rightarrow 120 - T &= 12a \end{aligned}$$

❷
$$\begin{aligned} S - 5g &= 5a \\ \Rightarrow S - 5(10) &= 5a \\ \Rightarrow -50 + S &= 5a \end{aligned}$$

$$\begin{aligned} ❶ \quad 120 - T &= 12a \\ ❷ \quad -50 + S &= 5a \\ ❸ \quad T - S &= 2a \\ \hline \Rightarrow 70 &= 19a \\ \Rightarrow a &= \frac{70}{19} \text{ ms}^{-2} \ / \ 3.684 \text{ m/s}^2 \end{aligned}$$

... (10m)

4(iv) Find the tension in each string. **(10)**

❶ $12g - T = 12a$

$\Rightarrow T = 12g - 12a$

$= 12(10) - 12(\frac{70}{19})$

$= 120 - \frac{840}{19}$

$= \frac{1,440}{19}$ N / 75.789 N ... (5m)

❷ $S - 5g = 5a$

$\Rightarrow S = 5a - 5g$

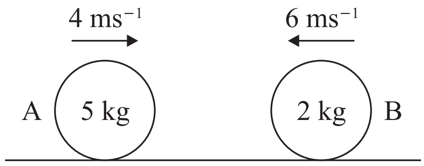
$= 5(\frac{70}{19}) - 5(10)$

$= \frac{350}{19} - 50$

$= \frac{350}{19} - 50$

$= -\frac{600}{19}$ N / –31.579 N ... (5m)

5. A smooth sphere A, of mass 5 kg, collides directly with another smooth sphere B, of mass 2 kg, on a smooth horizontal table.



Before impact A and B are moving in opposite directions with speeds of 4 ms^{-1} and 6 ms^{-1} respectively.

The coefficient of restitution for the collision is $\frac{1}{4}$.

Find

(i) the speed of A and the speed of B after the collision **(30)**

Sphere	Mass	Speed before	Speed after
A	5 kg	4	p
B	2 kg	–6	q

COM:

$m_A u_A + m_B u_B = m_A v_A + m_A v_A$

$\Rightarrow 5(4) + 2(-6) = 5(p) + 2(q)$ ... (5m)

❶ $\Rightarrow 8 = 5p + 2q$ ... (5m)

NEL:

$\frac{v_A - v_B}{u_A - u_B} = -e$

$\frac{p - q}{4 - (-6)} = -\frac{1}{4}$ ... (5m)

$\Rightarrow 4(p - q) = -1(4 + 6)$

$= -10$

$\Rightarrow 4p - 4q = -10$ ... (5m)

❷ $\Rightarrow 2p - 2q = -5$

❶ $5p + 2q = 8 \ (\times 2)$

❷ $2p - 2q = -5 \ (\times 1)$

$\Rightarrow 7p = 3$

$\Rightarrow p = \frac{3}{7}$ ms^{-1} ... (5m)

❶ $5p + 2q = 8$

$\Rightarrow 2q = 8 - 5p$

$= 8 - 5(\frac{3}{7})$

$= \frac{41}{7}$

$\Rightarrow q = \frac{41}{14}$ ms^{-1} ... (5m)

5(ii) the loss in kinetic energy due to the collision **(15)**

K.E. before collision

$$\text{K.E.}_{\text{before}} = \frac{1}{2}m_A u_A^2 + \frac{1}{2}m_B u_B^2$$

$$= \frac{1}{2}(5)(4)^2 + \frac{1}{2}(2)(-6)^2$$

$$= \frac{1}{2}(80) + \frac{1}{2}(72)$$

$$= 40 + 36$$

$$= 76 \text{ J}$$ ... (5m)

K.E. after collision

$$\text{K.E.}_{\text{after}} = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$

$$= \frac{1}{2}(5)(\frac{3}{7})^2 + \frac{1}{2}(2)(\frac{41}{14})^2$$

$$= \frac{45}{98} + \frac{1{,}681}{196}$$

$$= \frac{1{,}771}{196} \text{ J} \ / \ 9.036 \text{ J}$$ ... (5m)

$\Rightarrow$ $$\text{K.E.}_{\text{loss}} = \text{K.E.}_{\text{before}} - \text{K.E.}_{\text{after}}$$

$$= 76 - \frac{1{,}771}{196}$$

$$= \frac{13{,}125}{196} \text{ J} \ / \ 66.964 \text{ J}$$ ... (5m)

(iii) the magnitude of the impulse imparted to A due to the collision. **(5)**

$$I_{\text{A}} = m_{\text{A}}v_{\text{A}} - m_{\text{A}}u_{\text{A}}$$

$$= 5(\frac{3}{7}) - 5(4)$$

$$= \frac{15}{7} - 20$$

$$= -\frac{125}{7} \text{ Ns} \ / \ 17.857 \text{ N}$$ ... (5m)

6. **(a)** Particles of weights 3 N, 2 N and 5 N are placed at the points (11, –10), (1, 5) and (5, –6), respectively.

Find the co-ordinates of the centre of gravity of the system. **(20)**

$$3\text{ N} \ @\ (11, -10) + 2\text{ N} \ @\ (1, 5) + 5\text{ N} \ @\ (5, -6) = 10\text{ N} \ @\ (x, y)$$

$$\Rightarrow \quad 3(11) + 2(1) + 5(5) = 10\bar{x}$$

$$\Rightarrow \quad \bar{x} = \frac{3(11) + 2(1) + 5(5)}{10} \qquad \text{... (5m)}$$

$$= \frac{33 + 2 + 25}{10}$$

$$= \frac{60}{10}$$

$$= 6 \qquad \text{... (5m)}$$

$$\Rightarrow \quad 3(-10) + 2(5) + 5(-6) = 10\bar{y}$$

$$\Rightarrow \quad \bar{y} = \frac{3(-10) + 2(5) + 5(-6)}{10} \qquad \text{... (5m)}$$

$$= \frac{-30 + 10 - 30}{10}$$

$$= \frac{-50}{10}$$

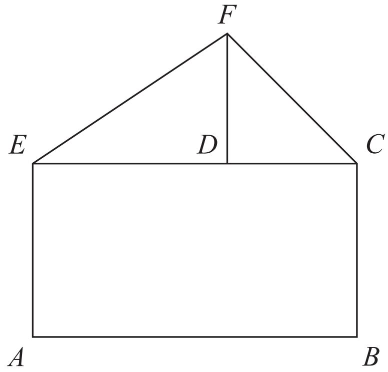
$$= -5 \qquad \text{... (5m)}$$

$$\Rightarrow \quad cog = (6, -5)$$

6(b) A uniform lamina *ABCFE* consists of a rectangle *ABCE* and two right-angled triangles *EDF* and *DCF*.

The co-ordinates of the points are $A(0, 0)$, $B(15, 0)$, $C(15, 8)$, $D(9, 8)$, $E(0, 8)$ and $F(9, 14)$.

Find the co-ordinates of the centre of gravity of the lamina.



(30)

<u>Rectangle *ABCE*</u>

	Area of rectangle *ABCE*	=	$\lvert(15-0)(8-0)\rvert$
		=	$\lvert(15(8)\rvert$
		=	120 units^2 ... (5m)
	Centres of gravity of ΔECF		
	g_1	=	$\left(\frac{0+15+15+0}{4}, \frac{0+0+8+8}{4}\right)$
⇒	*cog* ΔECF	=	(7.5, 4) ... (5m)

<u>Triangle *ECF*</u>

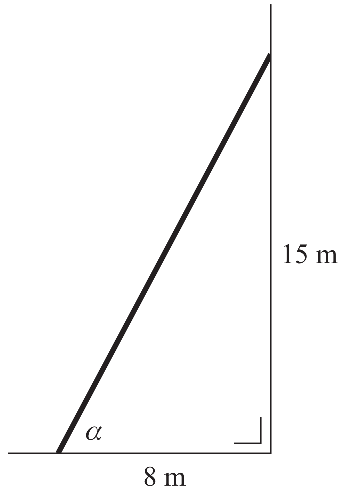
	$E(0, 8)$	→	(0, 0)
	$C(15, 8)$	→	(15, 0)
	$F(9, 14)$	→	(9, 6)
⇒	Area of ΔECF	=	$\frac{1}{2}\lvert(15)(6)-(9)(0)\rvert$
		=	$\frac{1}{2}\lvert 90\rvert$
		=	45 units^2 ... (5m)
	Centres of gravity of ΔECF		
	g_2	=	$\left(\frac{0+15+9}{3}, \frac{8+8+14}{3}\right)$
⇒	*cog* ΔECF	=	(8, 10) ... (5m)

<u>Co-ordinates of the centre of gravity of the lamina</u>

	120 @ (7.5, 4) + 45 @ (8, 10)	=	(120 + 45) @ (x, y)
⇒	120(7.5) + 45(8)	=	$165(x)$
⇒	900 + 360	=	$165x$
⇒	1,260	=	$165x$
⇒	x	=	$\frac{1{,}260}{165}$
		=	$\frac{252}{33}$ / 7.636 ... (5m)
⇒	120(4) + 45(10)	=	$165(y)$
⇒	480 + 450	=	$165y$
⇒	930	=	$165y$
⇒	y	=	$\frac{930}{165}$
		=	$\frac{186}{33}$ / 5.636... ... (5m)
⇒	*cog* of lamina	=	$(\frac{252}{33}, \frac{186}{33})$ / (7.636, 5.636)

7. A uniform ladder, of weight W, rests on rough horizontal ground and leans against a smooth vertical wall.

The foot of the ladder is 8 m from the wall and the top of the ladder is 15 m above the ground.



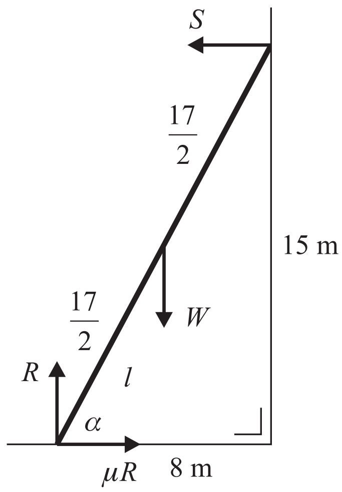
(i) Find the length of the ladder. **(5)**

$$l^2 = 15^2 + 8^2 = 225 + 64 = 289$$

$$\Rightarrow l = 17 \text{ m}$$ ... (5m)

The coefficient of friction between the ladder and the ground is μ.

(ii) Show on a diagram all of the forces acting on the ladder. **(10)**



... (10m)

(iii) Show that the value of $\mu > \frac{4}{15}$. **(20)**

❶ $R = W$

❷ $S = \mu R$ ... (5m)

$\Rightarrow S = \mu W$

Assume that the ladder is on the point of slipping
Taking moments @ foot of the ladder

❸ $(15)S = (\frac{8}{2})W$ ... (5m)

$\Rightarrow S = \frac{(8)}{(15)(2)}W$

$= \frac{4}{15}W$ ... (5m)

7(iii) (Continued)

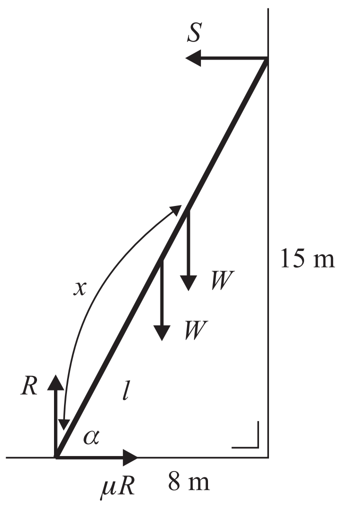
$$S = \mu W = \frac{4}{15}W$$

$$\Rightarrow \quad \mu W = \frac{4}{15}W$$

$$\Rightarrow \quad \mu = \frac{4}{15}$$

$\Rightarrow$ if $\mu \geq \frac{4}{15}$, then the ladder will not slip ... (5m)

(iv) A person, whose weight is the same as the ladder, starts to climb the ladder.
If the coefficient of friction between the ladder and the ground is $\frac{1}{3}$,
how far can the person safely climb the ladder before it begins to slip? **(15)**



❶ $$R = W + W = 2W$$

❷ $$S = \mu R = \frac{1}{3}(2W) = \frac{2}{3}W$$ ... (5m)

❸ $$S(15) = W(4) + 2W(x\sin\alpha)$$ ... (5m)

$$\Rightarrow \quad 15(\frac{2}{3}W) = 4W + 2Wx(\frac{8}{17})$$

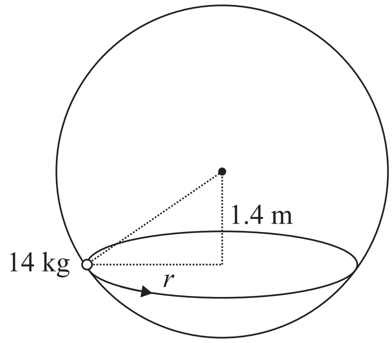
$$\Rightarrow \quad 2Wx(\frac{8}{17}) = 10W - 4W = 6W$$

$$\Rightarrow \quad x = (\frac{17}{8})(\frac{1}{2})(6) = \frac{102}{16} = 6.375 \text{ m}$$ ... (5m)

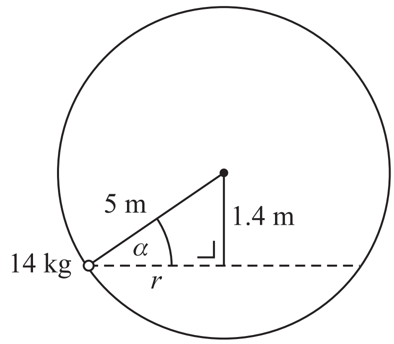
8. A sphere of diameter 10 m is fixed to a horizontal surface.

A smooth particle of mass 14 kg describes a horizontal circle of radius r m on the smooth inside surface of a sphere.

The plane of the circular motion is 1.4 m below the centre of the sphere.



(i) Find the value of r. **(10)**



$r^2 + (1.4)^2 = (5)^2$ ... (5m)

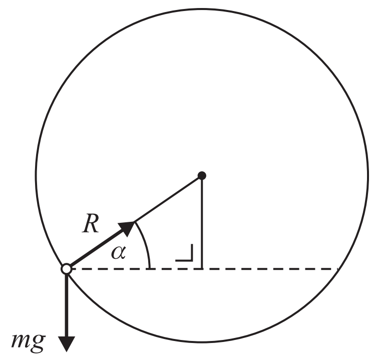
$\Rightarrow r^2 + 1.96 = 25$

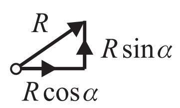
$\Rightarrow r^2 = 25 - 1.96$

$= 23.04$

$\Rightarrow r = 4.8$ m ... (5m)

(ii) Find the normal reaction between the particle and the surface of the sphere. **(20)**





$\sin\alpha = \dfrac{1.4}{5}$

$= \dfrac{7}{25}$ ... (5m)

$R\sin\alpha = 14g$ ... (10m)

$\Rightarrow R\left(\dfrac{7}{25}\right) = 14(10)$

$\Rightarrow R = \dfrac{(140)(25)}{7}$

$= 500$ N ... (5m)

8(iii) Calculate the angular velocity of the particle.
Give your answer correct to one decimal place. **(20)**

$$\cos\alpha = \frac{4.8}{5}$$

$$= \frac{24}{25} \qquad \text{... (5m)}$$

$$R\cos\alpha = m\omega^2 r$$

$$\Rightarrow \quad 500\left(\frac{24}{25}\right) = (14)(\omega^2)(4.8) \qquad \text{... (10m)}$$

$$\Rightarrow \quad 480 = 67.2\omega^2$$

$$\Rightarrow \quad \omega^2 = \frac{480}{67.2}$$

$$= 7.142857...$$

$$\Rightarrow \quad \omega = \sqrt{7.142857...}$$

$$= 2.672612...$$

$$\cong 2.7 \text{ rad/s} \qquad \text{... (5m)}$$

9. **(a)** State the Principle of Archimedes. **(10)**

– when a body is wholly or partly immersed in a liquid, ... (5m)
– it suffers an upthrust or buoyancy equal in magnitude to the weight of the liquid displaced ... (5m)

A solid piece of metal, of relative density 5, weighs 20 N in air.

(i) How much does the metal weigh in water? **(10)**

$$B_w = \frac{W}{s} = \frac{20}{5} = 4 \qquad \text{... (5m)}$$

Apparent weight

$$= 20 - 4 = 16 \text{ N} \qquad \text{... (5m)}$$

(ii) How much does the metal weigh in a liquid of relative density 0.8? **(10)**

$$B_l = \frac{s_l W}{s} = \frac{0.8(20)}{5} = 3.2 \qquad \text{... (5m)}$$

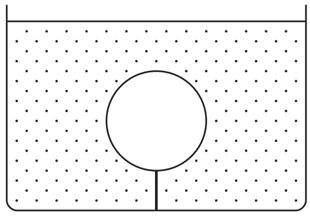
Apparent weight

$$= 20 - 3.2 = 16.8 \text{ N} \qquad \text{... (5m)}$$

9(b) A sphere has a radius of 1.25 m.

The relative density of the sphere is 0.8 and it is completely immersed in a tank of liquid of relative density 0.85.

The sphere is held at rest by a light inextensible string which is attached to the base of the tank.



(i) Find the weight of the sphere, correct to two decimal places. **(10)**

$V = \frac{4}{3}\pi r^3$

$= \frac{4}{3}(3.14...)(1.25)^3$

$= 8.181023...$

$\cong 8.18 \text{ m}^3$ ... (5m)

$W = \rho_l V g$

$= (0.8 \times 1{,}000)(8.18)(10)$

$= 65{,}440$ N ... (5m)

(ii) Find the tension in the string.
[Density of water = 1,000 kg m^{-3}.] **(10)**

$B = \rho_l V g$

$= (0.85 \times 1{,}000)(8.18)(10)$

$= 69{,}530$ N ... (5m)

$B = W + T$

$\Rightarrow 69{,}530 = 65{,}440 + T$

$\Rightarrow T = 69{,}530 - 65{,}440$

$= 4{,}090$ N ... (5m)

Notes:

Pre-Leaving Certificate Examination, 2011

Applied Mathematics

Higher Level
Marking Scheme (300 marks)

General Instructions

1. Penalties of three types are applied to students' work as follows:

Slips	- numerical slips	S(–1)
Blunders	- mathematical errors	B(–3)
Misreading	- if not serious	M(–1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

4. Number the grid on each script 1 to 10 in numerical order, not the order of answering.

5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

1. **(a)** Two people are running straight towards each other.
At a certain instant they are 57.5 m apart.
The first person is moving at 3 ms^{-1} and constant acceleration of 0.1 ms^{-2} and the other person is moving at 2 ms^{-1} and constant acceleration of 0.05 ms^{-2}.

(i) How long does it take for them to meet? **(10)**

$$s = ut + \tfrac{1}{2}at^2$$
$$s_{1\text{st person}} + s_{2\text{nd person}} = 57.5$$

$u_1 = 3$ $\quad$ $u_1 = 2$
$a_1 = 0.1$ $\quad$ $a_1 = 0.05$
$t_1 = t$ $\quad$ $t_1 = t$

$\Rightarrow (3t + \frac{1}{2}(0.1)t^2) + (2t + \frac{1}{2}(0.05)t^2) = 57.5$ ... (5m)
$\Rightarrow 3t + 0.05t^2 + 2t + 0.025t^2 = 57.5$
$\Rightarrow 5t + 0.075t^2 = 57.5$
$\Rightarrow 3t^2 + 200t - 2,300 = 0$
$\Rightarrow (3t + 230)(t - 10) = 0$
$\Rightarrow 3t + 230 = 0$
$\Rightarrow t = -\frac{230}{3}$

Solution not valid

$\Rightarrow t - 10 = 0$
$\Rightarrow t = 10$ ... (5m)

(ii) Find the speed of the two people when they meet. **(10)**

$$v = u + at$$

$u_1 = 3$
$a_1 = 0.1$
$t_1 = 10$

$\Rightarrow v_{1\text{st person}} = 3 + (0.1)(10)$
$= 4\ ms^{-1}$ ... (5m)

$$v = u + at$$

$u_1 = 2$
$a_1 = 0.05$
$t_1 = 10$

$\Rightarrow v_{2\text{nd person}} = 2 + (0.05)(10)$
$= 2.5\ ms^{-1}$ ... (5m)

1(b) A cyclist, who is travelling at a constant speed of 8 ms^{-1} on a straight road, observes a bus 50 m ahead. The bus starts to accelerate from rest, with a constant acceleration. If the cyclist continues at a constant speed of 8 ms^{-1}, he can just catch up with the bus.

(i) Find the constant acceleration of the bus. **(15)**

$$v^2 = u^2 + 2as$$

<u>Relative to the bus</u>

$u_{cb} = u_c - u_b$
$= 8 - 0$
$= 8 \text{ ms}^{-1}$ ... (5m)

$a_{cb} = a_c - a_b$
$= 0 - a$
$= -a$

$s_{cb} = 50$

$v_{cb} = 0$ ... (5m)

$\Rightarrow 0 = (8)^2 + 2(-a)(50)$
$\Rightarrow 100a = 64$
$\Rightarrow a = \dfrac{64}{100}$
$= 0.64 \text{ ms}^{-2}$ ... (5m)

(ii) How near would the cyclist get to the bus if he can only manage to cycle at 6 ms^{-1}? **(15)**

<u>Relative to the bus</u>

$u_{cb} = u_c - u_b$
$= 6 - 0$
$= 8 \text{ ms}^{-1}$

Minimum gap when their speeds are equal

$v_c = v_b$

$v_b = u_b + a_b t$
$v_b = 6$
$u_b = 0$
$a_b = 0.64$

$\Rightarrow 6 = 0 + 0.64t$
$\Rightarrow t = \dfrac{0.64}{6}$
$= 9.375 \text{ s}$ ... (5m)

at $t = 9.375$ s

$s_c = 6(9.375)$
$= 56.25 \text{ m}$

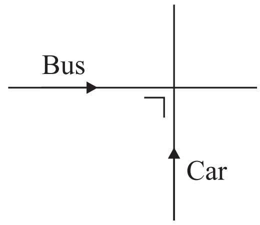
$s_b = ut + \dfrac{1}{2}at^2$
$= 0 + \dfrac{1}{2}(0.64)(9.375)^2$
$= 28.125$ ... (5m)

$\therefore$ Rel. displacement $= 56.25 - 28.125$
$= 28.125$

$\therefore$ Minimum gap $= 50 - 28.125$
$= 21.875 \text{ m}$ ... (5m)

2. **(a)** Two straight roads intersect at right angles.
A car is travelling north on one road at 4 ms^{-1}.
A bus is travelling east on the other road at 3 ms^{-1}.

The bus is 50 m away from the intersection as the car passes through the intersection.



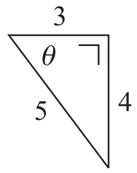
(i) Find the velocity of the bus relative to the car. **(10)**

$$v_C = 4\vec{j}$$
$$v_B = 3\vec{i}$$
$$\Rightarrow v_{BC} = v_B - v_C$$
$$= 3\vec{i} - 4\vec{j} \quad \text{... (5m)}$$

$$\Rightarrow |v_{BC}| = \sqrt{(3)^2 + (4)^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= 5 \quad \text{... (5m)}$$

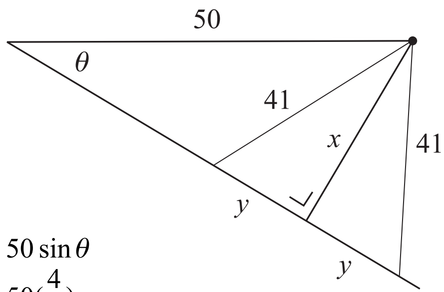
(ii) Find the shortest distance between the car and the bus and the time at which this occurs. **(10)**

Direction



$$\tan\theta = \frac{4}{3}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$$
$$= \tan^{-1}(1.333333...)$$
$$= 53.130102...^\circ$$
$$\cong 53.13^\circ$$

Relative path



$$x = 50\sin\theta$$
$$= 50\left(\frac{4}{5}\right)$$
$$= 40 \text{ m} \quad \text{... (5m)}$$

Time

$$t = \frac{x}{|v_{BC}|}$$
$$= \frac{40}{5}$$
$$= 8 \text{ s} \quad \text{... (5m)}$$

2(a) **(iii)** Calculate the length of time for which the car and the bus are less than or equal to 41 m apart. **(10)**

$$y^2 + x^2 = 41^2$$
$$\Rightarrow \quad y^2 + 40^2 = 1{,}681$$
$$\Rightarrow \quad y^2 = 1{,}681 - 1{,}600$$
$$= 81$$
$$\Rightarrow \quad y = 9 \qquad \text{... (5m)}$$

$$\text{Time} = \frac{2y}{|v_{BC}|}$$
$$= \frac{2(9)}{5}$$
$$= 3.6 \text{ s} \qquad \text{... (5m)}$$

2(b) The wind is blowing with a constant speed of 5 ms^{-1}. To a man running at 1 ms^{-1} in a southerly direction, the wind appears to blow from the north-west.

(i) Find the true velocity of the wind. **(10)**

$$\text{Let} \quad v_w = x\vec{i} + y\vec{j}$$
$$|v_w| = 5$$
$$\Rightarrow \quad \sqrt{x^2 + y^2} = 5$$
$$\Rightarrow \quad x^2 + y^2 = 25$$

$$v_m = 0\vec{i} - 1\vec{j}$$
$$\Rightarrow \quad v_{wm} = v_w - v_m$$
$$= (x\vec{i} + y\vec{j}) - (0\vec{i} - 1\vec{j})$$
$$= x\vec{i} + (y+1)\vec{j} \quad \text{from NW}$$
$$\Rightarrow \quad x = -(y+1) \qquad \text{... (5m)}$$

By substitution:

$$\Rightarrow \quad [-(y+1)]^2 + y^2 = 25$$
$$\Rightarrow \quad y^2 + 2y + 1 + y^2 = 25$$
$$\Rightarrow \quad 2y^2 + 2y - 24 = 0$$
$$\Rightarrow \quad y^2 + y - 12 = 0$$
$$\Rightarrow \quad (y-3)(y+4) = 0$$
$$\Rightarrow \quad y - 3 = 0 \qquad \Rightarrow \quad y + 4 = 0$$
$$\Rightarrow \quad y = 3 \qquad \Rightarrow \quad y = -4$$

$$\Rightarrow \quad x = -(3+1) \qquad \Rightarrow \quad x = -(-4+1)$$
$$= -4 \qquad = 3$$

$$\text{If} \quad v_w = -4\vec{i} + 3\vec{j}$$
$$\Rightarrow \quad v_{wm} = -4\vec{i} + 4\vec{j} \quad \text{to NW} \qquad \text{... (reject)}$$

$$\text{If} \quad v_w = 3\vec{i} - 4\vec{j}$$
$$\Rightarrow \quad v_{wm} = 3\vec{i} - 3\vec{j} \quad \text{from NW}$$
$$\Rightarrow \quad v_w = 3\vec{i} - 4\vec{j} \qquad \text{... (5m)}$$

2(b) (ii) From what direction would the wind appear to blow if the man reversed direction but continued to run at the same speed? **(10)**

$$v_m = 0\vec{i} + 1\vec{j}$$

$$\Rightarrow v_{wm} = (3\vec{i} - 4\vec{j}) - (0\vec{i} + 1\vec{j})$$

$$= 3\vec{i} - 5\vec{j}$$ ... (5m)

$$\Rightarrow \tan\theta = \frac{5}{3}$$

$$\Rightarrow \theta = \tan^{-1} 1.666666...$$

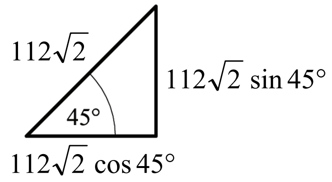
$$= 59.036243...^\circ$$

$$\cong 59^\circ \text{ South of East}$$ ... (5m)

3. **(a)** A particle is projected with a velocity of $112\sqrt{2}$ ms^{-1} at an angle of 45° to the horizontal plane.

Find

(i) the maximum height of the particle above the plane **(10)**



$$\vec{u} = 112\sqrt{2}\cos 45°\vec{i} + 56\sqrt{2}\sin 45°\vec{j}$$
$$= 112(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right)\vec{i} + 112(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right)\vec{j}$$
$$= 112\vec{i} + 112\vec{j} \quad \text{... (5m)}$$

Maximum height:

$$u_y = 112\vec{j}$$
$$a_y = -g$$
$$v_y = 0$$

$$v^2 = (112)^2 + 2(-g)h_{\max}$$
$$\Rightarrow 0 = 12{,}544 - 2(9.8)h_{\max}$$
$$\Rightarrow 19.6h_{\max} = 12{,}544$$
$$\Rightarrow h_{\max} = \frac{12{,}544}{19.6}$$
$$= 640 \text{ m} \quad \text{... (5m)}$$

(ii) the velocity of the particle after 4 seconds. **(10)**

$$v_x = 112\vec{i}$$
$$\Rightarrow v_y = (u_y + a_y t)\vec{j}$$
$$= (112 - 9.8(4))\vec{j}$$
$$= (112 - 39.2)\vec{j}$$
$$= 72.8\vec{j} \quad \text{... (5m)}$$

$$\Rightarrow \text{Speed} = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{(112)^2 + (72.8)^2}$$
$$= \sqrt{12{,}544 + 5{,}299.84}$$
$$= \sqrt{17{,}843.84}$$
$$= 133.580836...$$
$$\cong 133.6 \text{ m/s} \quad \text{... (5m)}$$

3(b) Two particles X and Y are fired at the same time and with the same velocity of 98 ms^{-1} from a point O on horizontal ground. X is fired at an angle 45° to the horizontal. Y is fired at an angle $\tan^{-1} 3$ to the horizontal. Both particles hit the same stationary target, but not at the same time.

(i) Write down the position vector of X after time t_1 and the position vector of Y after time t_2. (Take O as the origin.) **(10)**

$$s_x = u\cos\theta\, t$$

$$s_y = u\sin\theta\, t - \frac{1}{2}gt^2$$

$$\vec{r}_X = 98\left(\frac{1}{\sqrt{2}}\right)t_1\vec{i} + \left((98)\left(\frac{1}{\sqrt{2}}\right)t_1 - \frac{1}{2}(9.8)t_1^2\right)\vec{j}$$

$$= \frac{98}{\sqrt{2}}t_1\vec{i} + \left(\frac{98}{\sqrt{2}}t_1 - 4.9t_1^2\right)\vec{j} \qquad \text{... (5m)}$$

$$\vec{r}_Y = 98\left(\frac{1}{\sqrt{10}}\right)t_2\vec{i} + \left((98)\left(\frac{3}{\sqrt{10}}\right)t_2 - \frac{1}{2}(9.8)t_2^2\right)\vec{j}$$

$$= \frac{98}{\sqrt{10}}t_2\vec{i} + \left(\frac{294}{\sqrt{10}}t_2 - 4.9t_2^2\right)\vec{j} \qquad \text{... (5m)}$$

(ii) Show that $t_2 = \sqrt{5}\,t_1$. **(5)**

$$\vec{r}_Y = \vec{r}_X$$

$$\Rightarrow \frac{98}{\sqrt{10}}t_2\vec{i} + \left(\frac{294}{\sqrt{10}}t_2 - 4.9t_2^2\right)\vec{j} = \frac{98}{\sqrt{2}}t_1\vec{i} + \left(\frac{98}{\sqrt{2}}t_1 - 4.9t_1^2\right)\vec{j}$$

$$\Rightarrow \frac{98}{\sqrt{10}}t_2 = \frac{98}{\sqrt{2}}t_1$$

$$\Rightarrow t_2 = \frac{\sqrt{10}}{\sqrt{2}}t_1$$

$$= \frac{\sqrt{2}\sqrt{5}}{\sqrt{2}}t_1$$

$$= \sqrt{5}\,t_1 \qquad \text{... (5)}$$

3(b)(iii) Show that the position of the target is $490\vec{i}+245\vec{j}$. **(15)**

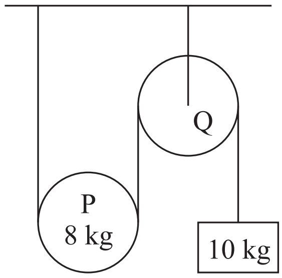
$$
\begin{aligned}
& & \frac{98}{\sqrt{2}}t_1 - 4.9t_1^2 &= \frac{294}{\sqrt{10}}t_2 - 4.9t_2^2 & \text{... (5m)}\\
& & &= \frac{294}{\sqrt{10}}(\sqrt{5}t_1) - 4.9(\sqrt{5}t_1)^2 &\\
& & &= \frac{294}{\sqrt{2}}t_1 - 24.5t_1^2 &\\
&\Rightarrow & -4.9t_1^2 + 24.5t_1^2 &= \frac{294}{\sqrt{2}}t_1 - \frac{98}{\sqrt{2}}t_1 &\\
&\Rightarrow & 19.6t_1^2 &= \frac{196}{\sqrt{2}}t_1 &\\
&\Rightarrow & t_1^2 &= \frac{196}{(19.6)\sqrt{2}}t_1 &\\
&\Rightarrow & t_1^2 &= \frac{10}{\sqrt{2}}t_1 &\\
&\Rightarrow & t_1 &= \frac{10}{\sqrt{2}} & \text{... (5m)}
\end{aligned}
$$

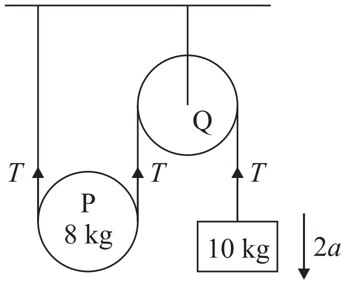
$$
\begin{aligned}
\vec{r}_X &= \frac{98}{\sqrt{2}}t_1\vec{i} + \left(\frac{98}{\sqrt{2}}t_1 - 4.9t_1^2\right)\vec{j} &\\
&= \frac{98}{\sqrt{2}}\left(\frac{10}{\sqrt{2}}\right)\vec{i} + \left(\frac{98}{\sqrt{2}}\left(\frac{10}{\sqrt{2}}\right) - 4.9\left(\frac{10}{\sqrt{2}}\right)_1^2\right)\vec{j} &\\
&= \frac{980}{2}\vec{i} + \left(\frac{980}{2} - \frac{490}{2}\right)\vec{j} &\\
&= 490\vec{i} + \frac{490}{2}\vec{j} &\\
&= 490\vec{i} + 245\vec{j} & \text{... (5m)}
\end{aligned}
$$

4. **(a)** The diagram shows a light inextensible string having one end fixed, passing under a smooth movable pulley P of mass 8 kg and then over a fixed smooth light pulley Q. A particle of mass 10 kg is attached to the free end of the string.

The system is released from rest.

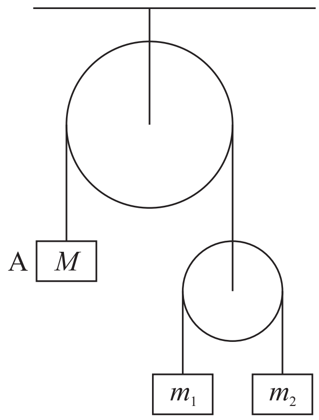
Find the acceleration of P and the particle, in terms of g. **(20)**





❶	$2T - 8g$	$=$	$8a$	... (5m)
❷	$10g - T$	$=$	$10(2a)$	... (5m)
$\Rightarrow$	$2T - 8g$	$=$	$8a$	
	$10g - T$	$=$	$20a$ $(\times 2)$	
$\Rightarrow$	$2T - 8g$	$=$	$8a$	
	$-2T + 20g$	$=$	$40a$	
$\Rightarrow$	$12g$	$=$	$48a$	
$\Rightarrow$	a	$=$	$\frac{12}{48}g$	
		$=$	$\frac{g}{4}$ ms^{-2}	
$\Rightarrow$	a_{P}	$=$	$\frac{g}{4}$ ms^{-2}	... (5m)
$\Rightarrow$	a_{particle}	$=$	$2(\frac{g}{4})$	
		$=$	$\frac{g}{2}$ ms^{-2}	... (5m)

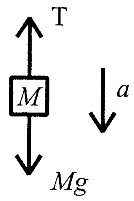
4(b) A light inextensible string passes over a small smooth fixed pulley.
A particle A, of mass M, is attached to one end of the string and a light smooth movable pulley is attached to the other end.
Two particles of masses of m_1 and m_2 are connected by a light inextensible string which passes over this pulley.



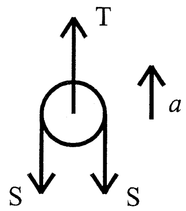
The system is released from rest.

(i) Find, in terms of g, M, m_1 and m_2, the acceleration of the particle A. **(25)**

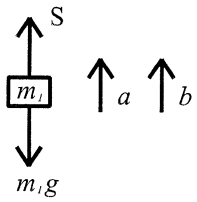
❶ $Mg - T = Ma$ ... (5m)



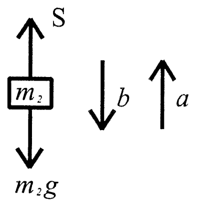
❷ $T - 2S = 0$ ... (5m)



❸ $S - m_1 g = m_1(a + b)$ ... (5m)



❹ $m_2 g - S = m_2(b - a)$ ... (5m)



❶ $Mg - T = Ma$
$\Rightarrow T = Mg - Ma$

❷ $T - 2S = 0$
$\Rightarrow 2S = T$
$= Mg - Ma$
$\Rightarrow S = \frac{1}{2}Mg - \frac{1}{2}Ma$

❸ $S - m_1 g = m_1(a + b)$
$\Rightarrow \frac{1}{2}Mg - \frac{1}{2}Ma - m_1 g = m_1 a + m_1 b$
$\Rightarrow Mg - Ma - 2m_1 g = 2m_1 a + 2m_1 b$
$\Rightarrow m_2 Mg - m_2 Ma - 2m_1 m_2 g = 2m_1 m_2 a + 2m_1 m_2 b$

4(b) (Continued)

❹

$$m_2g - S = m_2(b-a)$$
$$\Rightarrow m_2g - \frac{1}{2}Mg + \frac{1}{2}Ma = m_2b - m_2a$$
$$\Rightarrow 2m_2g - Mg + Ma = 2m_2b - 2m_2a$$
$$\Rightarrow 2m_1m_2g - m_1Mg + m_1Ma = 2m_1m_2b - 2m_1m_2a$$
$$\Rightarrow -2m_1m_2g + m_1Mg - m_1Ma = -2m_1m_2b + 2m_1m_2a$$

Adding these two equations:

$$m_2Mg - m_2Ma - 2m_1m_2g = 2m_1m_2a + 2m_1m_2b$$
$$\Rightarrow \underline{m_1Mg - m_1Ma - 2m_1m_2g = 2m_1m_2a - 2m_1m_2b}$$
$$\Rightarrow m_2Mg - m_2Ma + m_1Mg - m_1Ma - 4m_1m_2g = 4m_1m_2a$$
$$\Rightarrow g(m_2M + m_1M - 4m_1m_2) = a(4m_1m_2 + m_1M + m_2M)$$
$$\Rightarrow g(M(m_2 + m_1) - 4m_1m_2) = a(4m_1m_2 + M(m_1 + m_2))$$
$$\Rightarrow a = \frac{g(M(m_1 + m_2) - 4m_1m_2)}{4m_1m_2 + M(m_1 + m_2)}$$ ... (5m)

(ii) Show that the system will remain at rest if the mass of particle A is $\frac{4m_1m_2}{m_1 + m_2}$. **(5)**

Let $a = 0$

$$\Rightarrow \frac{g(M(m_1 + m_2) - 4m_1m_2)}{4m_1m_2 + M(m_1 + m_2)} = 0$$
$$\Rightarrow g(M(m_2 + m_1) - 4m_1m_2) = 0$$
$$\Rightarrow M(m_2 + m_1) - 4m_1m_2 = 0$$
$$\Rightarrow M(m_2 + m_1) = 4m_1m_2$$
$$\Rightarrow M = \frac{4m_1m_2}{m_1 + m_2}$$ ... (5m)

5. **(a)** A smooth sphere P, of mass m, moving with speed $2u$, collides directly with an identical smooth sphere Q, which is moving in the same direction with speed u. The coefficient of restitution for the collision is e.

Find, in terms of m, u and e, the loss of kinetic energy due to the collision. **(20)**

Sphere	Mass	Speed before	Speed after
P	m	$2u$	p
Q	m	u	q

<u>COM:</u>

$$m_A u_A + m_B u_B = m_A v_A + m_A v_A$$

$$\Rightarrow m(2u) + m(u) = mp + mq \quad \text{... (5m)}$$

$$\Rightarrow mp + mq = 3mu$$

❶ $$\Rightarrow p + q = 3u$$

<u>NEL:</u>

$$\frac{v_A - v_B}{u_A - u_B} = -e$$

$$\Rightarrow \frac{p - q}{2u - u} = -e \quad \text{... (5m)}$$

❷ $$\Rightarrow p - q = -1ue$$

❶ $$p + q = 3u$$

❷ $$p - q = -1ue$$

$$\Rightarrow 2p = u(3 - e)$$

$$\Rightarrow p = \frac{1}{2}u(3 - e)$$

❶ $$q = 3u - p$$

$$\Rightarrow q = 3u - \frac{1}{2}u(3 - e)$$

$$= \frac{1}{2}u(6 - 3 + e)$$

$$= \frac{1}{2}u(3 + e) \quad \text{... (5m)}$$

<u>Before collision:</u>

$$\text{K.E.} = \frac{1}{2}(m)(2u)^2 + \frac{1}{2}(m)(u)^2$$

$$= \frac{5}{2}mu^2$$

<u>After collision:</u>

$$\text{K.E.} = \frac{1}{2}(m)[u(3 - e)]^2 + \frac{1}{2}(m)[u(3 + e)]^2$$

$$= \frac{1}{2}(m)\left(\frac{1}{2}u(3 - e)\right)^2 + \frac{1}{2}(m)\left(\frac{1}{2}u(3 + e)\right)^2$$

$$= \frac{1}{8}(mu^2)(9 - 6e + e^2) + \frac{1}{8}(mu^2)(9 + 6e + e^2)$$

$$= \frac{1}{8}(mu^2)(18 + 2e^2)$$

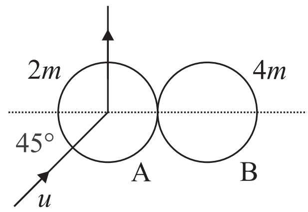
$$= \frac{1}{4}(mu^2)(9 + e^2)$$

$$\Rightarrow \text{Loss in K.E.} = \frac{5}{2}mu^2 - \frac{1}{4}(mu^2)(9 + e^2)$$

$$= mu^2\left(\frac{10}{4} - \frac{9}{4} - \frac{1}{4}e^2\right)$$

$$= \frac{1}{4}mu^2(1 - e^2) \quad \text{... (5m)}$$

5(b) A smooth sphere A, of mass $2m$, moving with speed u, collides with a smooth sphere B, of mass $4m$, which is at rest.
The direction of motion of A, before impact, makes an angle of 45° with the lines of centres of the instant of impact.



After the collision the two spheres move in perpendicular directions.

The coefficient of restitution between the spheres is e.

(i) Show that $e = \frac{1}{2}$. **(20)**

Sphere	Mass	Speed before	Speed after
A	$2m$	$\frac{u}{\sqrt{2}}\vec{i} + \frac{u}{\sqrt{2}}\vec{j}$	$p\vec{i} + \frac{u}{\sqrt{2}}\vec{j}$
B	$4m$	$0\vec{i} + 0\vec{j}$	$q\vec{i} + 0\vec{j}$

<u>After collision, A and B perpendicular:</u>

$$(p\vec{i} + \frac{u}{\sqrt{2}}\vec{j})(q\vec{i} + 0\vec{j}) = 0$$

$$\Rightarrow \quad pq = 0$$

$$\Rightarrow \quad p = 0 \qquad \text{... (5m)}$$

<u>COM:</u>

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

$$\Rightarrow \quad 2m(\frac{u}{\sqrt{2}}) + 4m(0) = 2m(0) + 4mq$$

$$\Rightarrow \quad 4mq = \frac{2mu}{\sqrt{2}}$$

$$\Rightarrow \quad q = \frac{2u}{4\sqrt{2}}$$

❶ $$= \frac{u}{2\sqrt{2}} \qquad \text{... (5m)}$$

<u>NEL:</u>

$$\frac{v_A - v_B}{u_A - u_B} = -e$$

$$\Rightarrow \quad \frac{0 - q}{\frac{u}{\sqrt{2}} - 0} = -e$$

❷ $$\Rightarrow \quad q = \frac{eu}{\sqrt{2}} \qquad \text{... (5m)}$$

❶❷ $$\Rightarrow \quad \frac{eu}{\sqrt{2}} = \frac{u}{2\sqrt{2}}$$

$$\Rightarrow \quad e = \frac{1}{2} \qquad \text{... (5m)}$$

5(b) (ii) Find, in terms of u, the speed of each sphere after the collision. **(5)**

$$v_A = 0\vec{i} + \frac{u}{\sqrt{2}}\vec{j}$$

$$v_B = q\vec{i} + 0\vec{j}$$

$$= \frac{eu}{\sqrt{2}}\vec{i} + 0\vec{j} \qquad \text{... (5m)}$$

(iii) Find the percentage loss in kinetic energy due to the collision. **(5)**

$$\text{K.E.}_{\text{before}} = \frac{1}{2}(2m)(u^2) + \frac{1}{2}(4m)(0^2)$$

$$= mu^2$$

$$\text{K.E.}_{\text{after}} = \frac{1}{2}(2m)(\frac{u}{\sqrt{2}})^2 + \frac{1}{2}(4m)(\frac{u}{2\sqrt{2}})^2$$

$$= m(\frac{u^2}{2} + \frac{2u^2}{4(2)})$$

$$= mu^2(\frac{1}{2} + \frac{1}{4})$$

$$= \frac{3}{4}mu^2$$

$$\Rightarrow \quad \text{K.E.}_{\text{loss}} = mu^2 - \frac{3}{4}mu^2$$

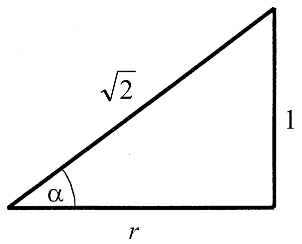
$$= \frac{1}{4}mu^2$$

$$\Rightarrow \quad \text{\% K.E. loss} = \frac{\frac{1}{4}mu^2}{mu^2} \times \frac{100}{1}$$

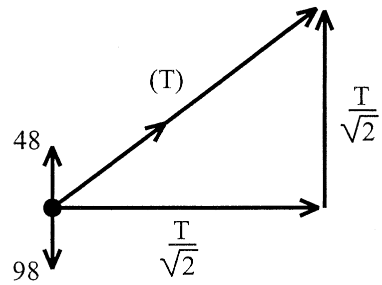
$$= 25\% \qquad \text{... (5m)}$$

6. **(a)** A light inextensible string, of length $\sqrt{2}$ m, is attached at one end to a fixed point P which is 1 m above a smooth horizontal table.
The other end of the string is attached to a particle of mass 10 kg which moves uniformly in a horizontal circle, centre O. O is vertically below P.
The reaction between the particle and the table is 48 N.

(i) Find the tension in the string. **(10)**



$$r^2 + 1^2 = (\sqrt{2})^2$$
$$r^2 + \quad = 2 - 1$$
$$= 1$$
$$\Rightarrow \quad r = 1$$



❶ $mg = R + T\sin\alpha$ ... (5m)

$\Rightarrow \quad 10(9.8) = 48 + T(\frac{1}{\sqrt{2}})$

$\Rightarrow \quad \frac{T}{\sqrt{2}} = 98 - 48$

$= 50$

$\Rightarrow \quad T = 50\sqrt{2}$ ... (5m)

(ii) Find the angular velocity of the particle. **(10)**

❷ $T\cos\alpha = m\omega^2 r$

$\Rightarrow \quad 50\sqrt{2}(\frac{1}{\sqrt{2}}) = (10)(\omega^2)(1)$ ... (5m)

$\Rightarrow \quad 50 = 10\omega^2$

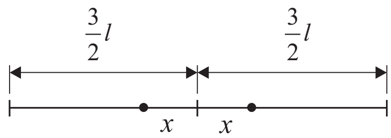
$\Rightarrow \quad \omega^2 = \frac{50}{10}$

$= 5$

$\Rightarrow \quad \omega = \sqrt{5}$ rad/s ... (5m)

6(b) A particle P, of mass m, is attached to the midpoint of an elastic string $[AB]$, of natural length $2l$ and elastic constant k. A and B are attached to fixed points on a smooth horizontal table, a distance of $3l$ apart. Initially the particle is held at rest in a position such that $|AP| = 2l$ and $|PB| = l$, and is then released.

(i) Show that the motion of the particle is simple harmonic. **(20)**



$$F_r = k(l - l_o)$$
$$= k(\tfrac{3}{2}l - x - l)$$
$$= \tfrac{1}{2}ke - kx \qquad \text{... (5m)}$$

$$F_l = k(l - l_o)$$
$$= k(\tfrac{3}{2}l + x - l)$$
$$= k(\tfrac{1}{2}l + x)$$
$$= \tfrac{1}{2}kl + kx \qquad \text{... (5m)}$$

$$F = F_r - F_l$$
$$= (\tfrac{1}{2}kl - kx)(\tfrac{1}{2}lk + kx)$$
$$= \tfrac{1}{2}kl - kx - \tfrac{1}{2}lk - kx$$
$$= -2kx \qquad \text{... (5m)}$$

$$\Rightarrow \quad ma = -2kx$$
$$\Rightarrow \quad a = -\frac{2k}{m}x$$
$$a = -\omega^2 x \qquad \text{... (5m)}$$

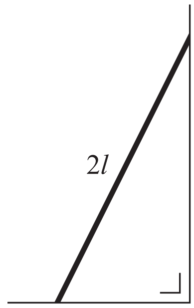
(ii) Find, in terms of m and k, the period of the motion. **(10)**

$$a = -\frac{2k}{m}x$$
$$a = -\omega^2 x$$
$$\Rightarrow \quad \omega = \sqrt{\frac{2k}{m}} \qquad \text{... (5m)}$$
$$T = \frac{2\pi}{\omega}$$
$$= 2\pi\sqrt{\frac{m}{2k}} \qquad \text{... (5m)}$$

7. **(a)** One end of a uniform ladder, of weight W and length $2l$, rests against a rough vertical wall, and the other end rests on rough horizontal ground.

The coefficient of friction at each contact is $\frac{1}{3}$.

The ladder makes an angle of $\tan^{-1} 2$ with the horizontal and is in a vertical plane which is perpendicular to the wall.



Find the distance that a person of weight W can safely climb before the ladder begins to slip. **(20)**

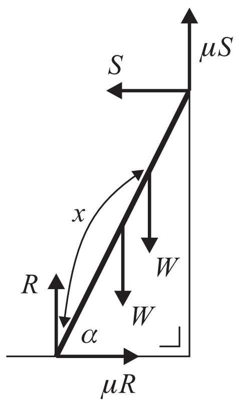
❶ $R + \mu S = W + W$

$\Rightarrow R + \frac{1}{3}S = 2W$ ... (5m)

❷ $S = \mu R$

$= \frac{1}{3}(2W)$

$= \frac{2}{3}W$



$\mu R = S$

$\Rightarrow \frac{1}{3}R = S$

$\Rightarrow \frac{1}{3}(2W - \frac{1}{3}S) = S$

$\Rightarrow \frac{2}{3}W - \frac{1}{9}S = S$

$\Rightarrow 6W - S = 9S$

$\Rightarrow 6W = 10S$

$\Rightarrow S = \frac{3}{5}W$ ... (5m)

Taking moments @ foot of the ladder

❸ $(15)S = (\frac{8}{2})W$ ... (5m)

$W(l\cos\alpha) + W(x\cos\alpha) = S(2l\sin\alpha) + \frac{1}{3}S(2l\cos\alpha)$

$\Rightarrow Wl + Wx = 2lS\tan\alpha + \frac{2}{3}Sl$

$\alpha = \tan^{-1} 2$

$\Rightarrow Wl + Wx = 4lS + \frac{2}{3}Sl$

$\Rightarrow Wl + Wx = 4l(\frac{3}{5}W) + \frac{2}{3}(\frac{3}{5}W)l$

$\Rightarrow Wl + Wx = \frac{12}{5}Wl + \frac{2}{5}Wl$

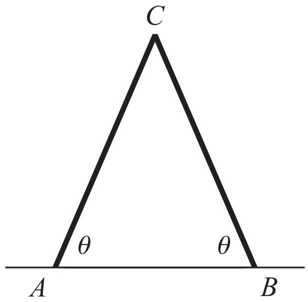
$= \frac{14}{5}Wl$

$\Rightarrow Wx = \frac{14}{5}Wl - Wl$

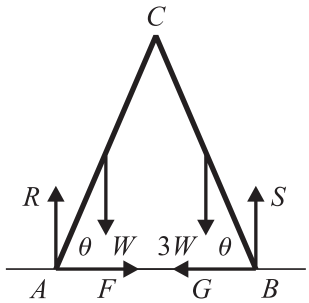
$= \frac{9}{5}Wl$

$\Rightarrow x = \frac{9}{5}l$ ... (5m)

7(b) Two uniform ladders, AC and BC, of equal length $2l$ and weights W and $3W$ respectively, are freely jointed at C. They stand in a vertical plane with A and B on rough horizontal ground. Both ladders make an angle θ with the ground. The coefficient of friction at both A and B is $\frac{1}{2}$.



(i) Find, in terms of W, the normal reactions at A and B. **(15)**



❶ $R + S = W + 3W$
$= 4W$

❷ $F = G$ ... (5m)

Taking moments @ A

❸ $W(l\cos\theta) + 3W(3l\cos\theta) = S(4l\cos\theta)$ ... (5m)

$\Rightarrow W + 9W = 4S$

$\Rightarrow 4S = 10W$

$\Rightarrow S = \frac{5}{2}W$

$\Rightarrow R = 4W - \frac{5}{2}W$

$= \frac{3}{2}W$ ... (5m)

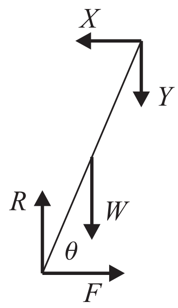
(ii) Show that slipping will occur first at A. **(5)**

$R < S$

$\Rightarrow \mu R < \mu S$

$\Rightarrow$ slipping will occur at A before C ... (5m)

7(b) (iii) Find the value of $\tan\theta$ at which slipping will occur. **(10)**



❶ $X = \mu R$

$= \frac{1}{2}(\frac{3}{2}W)$

$= \frac{3}{4}W$

❷ $Y + W = R$

$= \frac{3}{2}W$

$\Rightarrow Y = \frac{3}{2}W - W$

$= \frac{1}{2}W$ ... (5m)

Taking moments @ A

❸ $W(l\cos\theta) + Y(2l\cos\theta) = X(2l\sin\theta)$

$\Rightarrow W(l\cos\theta) + (\frac{1}{2}W)(2l\cos\theta) = (\frac{3}{4}W)(2l\sin\theta)$

$\Rightarrow Wl\cos\theta + Wl\cos\theta = \frac{3}{2}Wl\sin\theta$

$\Rightarrow \cos\theta + \cos\theta = \frac{3}{2}\sin\theta$

$\Rightarrow 2\cos\theta = \frac{3}{2}\sin\theta$

$\Rightarrow 4\cos\theta = 3\sin\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{4}{3}$

$\Rightarrow \tan\theta = \frac{4}{3}$ ... (5m)

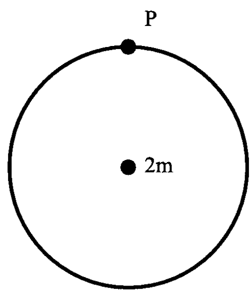
8. **(a)** Prove that the moment of inertia of a uniform square lamina, of mass m and side $2l$, about an axis through its centre parallel to one of the sides is $\frac{1}{3}ml^2$. **(20)**

Bookwork

m	$=$	$2\rho l$	... (5m)
ΔI			... (5m)
set up integral and solve			... (5m)
Deduce			... (5m)

8(b) A uniform circular disc, of mass m and radius r, has a particle, of mass $2m$, attached to its centre. The system performs small oscillations in a vertical plane about a horizontal axis l through a point on its circumference, perpendicular to the plane of the disc.

(i) Find, in terms of g and r, the period of small oscillations. **(15)**



Disc:

$$I_{\text{centre}} = \frac{1}{2}Mr^2$$

$$\Rightarrow \quad I_P = I_c + Mr^2$$

$$= \frac{1}{2}Mr^2 + Mr^2$$

$$= \frac{3}{2}Mr^2 \qquad \text{... (5m)}$$

Point Mass:

$$I_{\text{point mass}} = Md^2$$

$$= (2m)(r)^2$$

$$= 2mr^2$$

$$\Rightarrow \quad I_{\text{total}} = I_P + I_{\text{point mass}}$$

$$= \frac{3}{2}mr^2 + 2mr^2$$

$$= \frac{7}{2}mr^2 \qquad \text{... (5m)}$$

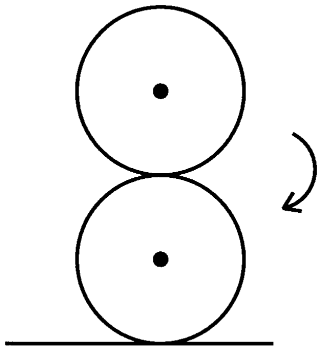
$$T = 2\pi\sqrt{\frac{I}{Mgh}}$$

$$= 2\pi\sqrt{\frac{\frac{7}{2}mr^2}{(3m)g(r)}}$$

$$= 2\pi\sqrt{\frac{7r}{6g}} \qquad \text{... (5m)}$$

8(b) The disc is held with the particle vertically above *l*, and is then released from rest.

(ii) Find the maximum angular velocity in the subsequent motion. **(15)**



$$\underset{\text{Disc}}{Mg\overline{h}} + \underset{\text{Point Mass}}{Mg\overline{h}} + \underset{\text{System}}{\frac{1}{2}I\omega^2} = \underset{\text{Disc}}{Mg\overline{h}} + \underset{\text{Point Mass}}{Mg\overline{h}} + \underset{\text{System}}{\frac{1}{2}I\omega^2}$$

$\Rightarrow \quad mg(3r) + 2mg(3r) + 0 = \text{mg}(r) + 2mg(r) + \frac{1}{2}(\frac{7}{2}mr^2)\omega^2$

$\Rightarrow \quad 3mgr + 6mgr = \text{mg}r + 2mgr + \frac{7}{4}mr^2\omega^2$

... (5m) RHS
... (5m) LHS

$\Rightarrow \quad 9mgr = 3\text{mg}r + \frac{7}{4}mr^2\omega^2$

$\Rightarrow \quad \frac{7}{4}mr^2\omega^2 = 9mgr - 3\text{mg}r$

$= 6mgr$

$\Rightarrow \quad \frac{7}{4}r\omega^2 = 6g$

$\Rightarrow \quad \omega^2 = \frac{(4)(6g)}{7r}$

$\Rightarrow \quad = \frac{24g}{7r}$

$\Rightarrow \quad \omega = \sqrt{\frac{24g}{7r}}$ ... (5m)

9. **(a)** A piece of metal weighs 15 N in air and 12 N in water.
The metal weighs 13 N in a light oil.

Find

(i) the relative density of the metal **(10)**

$$B_w = 15 - 12$$
$$= 3 \quad \text{... (5m)}$$

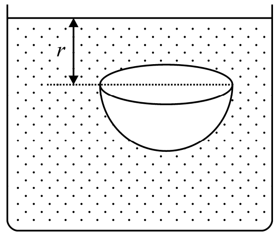
$$B_w = \frac{W}{s_0}$$
$$\Rightarrow \quad 3 = \frac{15}{s_0}$$
$$\Rightarrow \quad s_0 = \frac{15}{3}$$
$$= 5 \quad \text{... (5m)}$$

(ii) the relative density of the oil. **(10)**

$$B_l = 15 - 13$$
$$= 2 \quad \text{... (5m)}$$

$$B_l = \frac{s_l W_0}{s_0}$$
$$\Rightarrow \quad 2 = \frac{s_l (15)}{5}$$
$$\Rightarrow \quad s_l = \frac{(2)(5)}{15}$$
$$= \frac{2}{3} \quad \text{... (5m)}$$

9(b) A solid hemisphere of radius r is held immerged in a tank of liquid of density ρ. Its plane face is in a horizontal position at a distance r below the surface of the liquid.



Find the magnitude and direction of

(i) the downward thrust exerted on the plane face by the liquid **(10)**

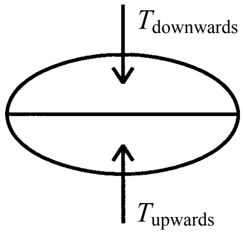
$$\text{Pressure} = h\rho g = r\rho g \quad \text{... (5m)}$$

$$\Rightarrow \text{Thrust} = \text{Pressure} \times \text{area} = (r\rho g)(\pi r^2) = \pi\rho r^3 g \quad \text{... (5m)}$$

(ii) the buoyancy force **(5)**

$$\text{Buoyancy} = V\rho g = \left(\frac{2}{3}\pi r^3\right)\rho g = \frac{2}{3}\pi\rho r^3 g \quad \text{... (5m)}$$

(iii) the total thrust upwards on the curved surface of the hemisphere. **(15)**



$$B = T_{\text{upwards}} - T_{\text{downwards}} \quad \text{... (5m)}$$

$$\Rightarrow \frac{2}{3}\pi\rho r^3 g = T_{\text{upwards}} - \pi\rho r^3 g \quad \text{... (5m)}$$

$$\Rightarrow T_{\text{upwards}} = \frac{2}{3}\pi\rho r^3 g + \pi\rho r^3 g = \frac{5}{3}\pi\rho r^3 g \quad \text{... (5m)}$$

10. **(a)** Solve the differential equation

$$(1 + x^3)\frac{dy}{dx} = x^2y$$

given that $y = 2$ when $x = 1$. **(20)**

$$(1 + x^3)\frac{dy}{dx} = x^2y$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{x^2dx}{1 + x^3}$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int \frac{x^2dx}{1 + x^3} \qquad \text{... (5m)}$$

$$\Rightarrow \quad \ln y = \frac{1}{3}\ln(1 + x^3) + c \qquad \text{... (5m)}$$

@ $y = 2,\ x = 1$

$$\Rightarrow \quad \ln 2 = \frac{1}{3}\ln(1 + 1) + c$$

$$\Rightarrow \quad c = \ln 2 - \frac{1}{3}\ln 2$$

$$= \frac{2}{3}\ln 2 \qquad \text{... (5m)}$$

$$\Rightarrow \quad \ln y = \frac{1}{3}\ln(1 + x^3) + \frac{2}{3}\ln 2$$

$$= \ln(1 + x^3)^{\frac{1}{3}}(2)^{\frac{2}{3}}$$

$$= \ln(1 + x^3)^{\frac{1}{3}}(4)^{\frac{1}{3}}$$

$$= \ln(4 + 4x^3)^{\frac{1}{3}}$$

$$\Rightarrow \quad y = (4 + 4x^3)^{\frac{1}{3}} \ / \ \sqrt[3]{4 + 4x^3} \qquad \text{... (5m)}$$

10(b) A force of magnitude $\dfrac{2m}{x^5}$ thrusts a particle, of mass m, directed away from a fixed point o, where x is the distance of the particle from O.
The particle starts from rest at $x = d$.

(i) Show that the velocity of the particle is $\dfrac{2\sqrt{2}}{3d^2}$ when $x = \sqrt{3}\,d$. **(25)**

$$\begin{aligned}
F &= \frac{2m}{x^3} \\
\Rightarrow \quad ma &= \frac{2m}{x^3} \\
\Rightarrow \quad a &= \frac{2}{x^3} \\
&= 2x^{-5} && \text{... (5m)} \\
\Rightarrow \quad v\frac{dv}{dx} &= 2x^{-5} && \text{... (5m)} \\
\Rightarrow \quad \int_0^v v\,dv &= \int_d^x 2x^{-5}\,dx \\
\Rightarrow \quad \frac{v^2}{2}\bigg|_0^v &= \frac{2x^{-4}}{-4}\bigg|_d^x && \text{... (5m)} \\
\Rightarrow \quad \frac{v^2}{2} - \frac{(0)^2}{2} &= \frac{-1}{2x^4} + \frac{1}{2d^4} \\
\Rightarrow \quad v^2 &= \frac{1}{d^4} - \frac{1}{x^4} && \text{... (5m)}
\end{aligned}$$

when $x = \sqrt{3}\,d$

$$\begin{aligned}
v^2 &= \frac{1}{d^4} - \frac{1}{(\sqrt{3}d)^4} \\
&= \frac{1}{d^4} - \frac{1}{9d^4} \\
&= \frac{8}{9d^4} \\
\Rightarrow \quad v &= \sqrt{\frac{8}{9d^4}} \\
&= \frac{2\sqrt{2}}{3d^2} && \text{... (5m)}
\end{aligned}$$

(ii) Determine the limiting speed, v_1, of the particle.
(that is, $v \to v_1$ as $t \to \infty$). **(5)**

$$v^2 = \frac{1}{d^4} - \frac{1}{x^4}$$

as $t \to \infty$, $x \to \infty$ (as the particle always accelerates to the right)
as $x \to \infty$, $\dfrac{1}{x^4} \to 0$

$$\begin{aligned}
\Rightarrow \quad v^2 &= \frac{1}{d^4} - 0 \\
\Rightarrow \quad v &= \sqrt{\frac{1}{d^4}} \\
&= \frac{1}{d^2} && \text{... (5m)}
\end{aligned}$$