L.41/42

Pre-Leaving Certificate Examination, 2010

Applied Mathematics

Marking Scheme

Ordinary Pg. 2

Higher Pg. 21

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Dublin Examining Board

Pre-Leaving Certificate Examination, 2010

Applied Mathematics

Ordinary Level
Marking Scheme (300 marks)

General Instructions

1. Penalties of three types are applied to students' work as follows:

Slips	- numerical slips	S(–1)
Blunders	- mathematical errors	B(–3)
Misreading	- if not serious	M(–1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

4. Number the grid on each script 1 to 9 in numerical order, not the order of answering.

5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

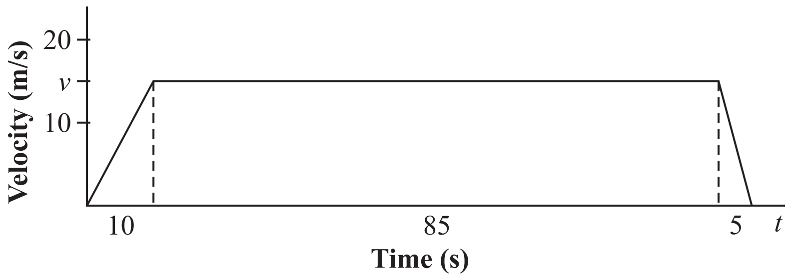
1. **(a)** A car travels along a straight level road.
The car, travelling with uniform acceleration of 2 m/s^2, passes a point a with a speed of 4 m/s and reaches a point b after 10 seconds.
Find the distance from a to b. **(10)**

$$s = ut + \tfrac{1}{2}at^2 \quad \text{... (5m)}$$
$$u = 4$$
$$a = 2$$
$$t = 10$$

$$\Rightarrow |ab| = 4(10) + \frac{1}{2}(2)(10)^2$$
$$= 40 + 100$$
$$= 140 \text{ m} \quad \text{... (5m)}$$

1(b) A bus travels from p to q along a straight level road.
It starts from rest at p and travels with uniform acceleration of 1.5 m/s^2 to its maximum speed in 10 seconds.
The bus then continues at this speed for 85 seconds.
Finally the bus decelerates uniformly to rest at q in a further 5 seconds.

1(b)(i) Draw a speed-time graph of the motion of the bus. **(10)**



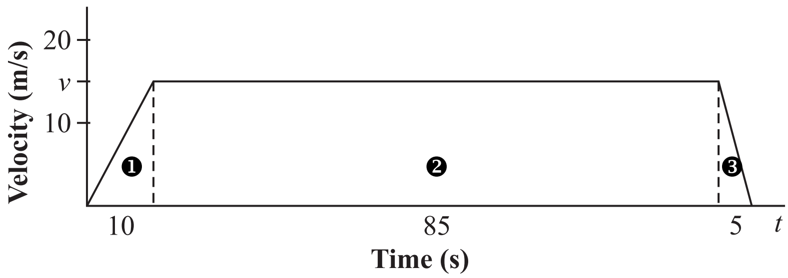
... (10m)

1(b)(ii) Find the maximum speed of the bus. **(10)**

$$v = u + at$$
$$u = 0$$
$$a = 1.5$$
$$t = 10$$

$$\Rightarrow v = 0 + (1.5)(10) \quad \text{... (5m)}$$
$$= 0 + 15$$
$$= 15 \text{ m/s} \quad \text{... (5m)}$$

1(b)(iii) Find $|pq|$, the distance from p to q. **(15)**



$|pq|$ = area 1 + area 2 + area 3 ... (5m)

= $\frac{1}{2}(15)(10) + (15)(85) + \frac{1}{2}(15)(5)$ ... (5m)

= $75 + 1{,}275 + 37.5$

= $1{,}387.5$ m ... (5m)

<u>**or**</u>

$s = \left(\frac{u+v}{2}\right)t$

$u_1 = 0$

$a_1 = 1.5$

$t_1 = 10$

$t_2 = 85$

$t_3 = 5$

<u>Distance accelerating</u>

$s_1 = \left(\frac{0+15}{2}\right)(10)$

$= \frac{1}{2}(15)(10)$

$= 75$ m

<u>Distance at constant speed</u>

$s_2 = \left(\frac{15+15}{2}\right)(85)$

$= \frac{1}{2}(30)(85)$

$= 1{,}275$ m

<u>Distance decelerating</u>

$s_3 = \left(\frac{15+0}{2}\right)(5)$

$= \frac{1}{2}(15)(5)$

$= 37.5$ m

<u>Total distance</u>

$d = s_1 + s_2 + s_3$ ... (5m)

$\Rightarrow \quad d = \frac{1}{2}(15)(10) + \frac{1}{2}(30)(85) + \frac{1}{2}(15)(5)$

... (5m)

$= \frac{1}{2}(150) + \frac{1}{2}(2{,}550) + \frac{1}{2}(75)$

$= 75 + 1{,}275 + 37.5$

$= 1{,}387.5$ m ... (5m)

1(b)(iv) Find the average speed of the bus as it travels from p to q. **(5)**

$$\text{Average speed} = \frac{d}{t}$$

$$s = 1{,}387.5$$

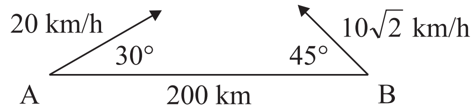
$$t = 100$$

$$\Rightarrow \text{Average speed} = \frac{1{,}387.5}{100}$$

$$= 13.875 \text{ m/s}$$

... (5m)

2. Ship A is travelling in the direction 30° north of east with a constant speed of 20 km/h.
Ship B is travelling north-west with a constant speed of $10\sqrt{2}$ km/h.
At noon, ship A is 200 km due west of ship B.



2(i) Express the velocity of ship A and the velocity of ship B in terms of $\vec{i}$ and $\vec{j}$. **(20)**

$v_A = 20\cos 30°\vec{i} + 20\sin 30°\vec{j}$ ... (5m)

$= 20(0.866025...)\vec{i} + 20(0.5)\vec{j}$

$= 17.320508...\vec{i} + 10\vec{j}$ km/h

$\cong 17.32\vec{i} + 10\vec{j}$ km/h ... (5m)

$v_B = -10\sqrt{2}\cos 45°\vec{i} + 10\sqrt{2}\sin 45°\vec{j}$ ... (5m)

$= -10(1.414213...)(0.707106...)\vec{i} + 10(1.414213...)(0.707106...)\vec{j}$

$= -10\vec{i} + 10\vec{j}$ km/h ... (5m)

2(ii) Find the velocity of ship A relative to ship B in terms of $\vec{i}$ and $\vec{j}$. **(10)**

$v_{AB} = v_A - v_B$

$= (17.32\vec{i} + 10\vec{j}) - (-10\vec{i} + 10\vec{j})$ ... (5m)

$= 17.32\vec{i} + 10\vec{j} + 10\vec{i} - 10\vec{j}$

$= 27.32\vec{i}$ km/h ... (5m)

2(iii) Show that ship A will intercept ship B. **(10)**

P is moving (relative to Q) in an easterly direction ... (5m)
$\Rightarrow$ P is moving directly towards Q ... (5m)
hence they are on collision course

2(iv) Calculate the time at which ship A will intercept ship B. **(10)**

$t = \dfrac{d}{v}$

$= \dfrac{200}{27.32}$ ... (5m)

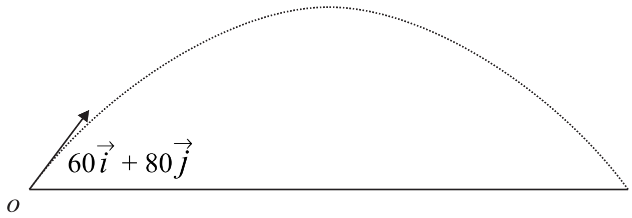
$= 7.320644...$ h

$\cong 7.32$ h

Time $= 12.00 + 7.32$

$= 19.32$ / 7.32 pm ... (5m)

3. A particle is projected from a point o on level horizontal ground with an initial velocity of $60\vec{i} + 80\vec{j}$ m/s.



3(i) Find the height of the particle above ground level after 3 seconds. **(10)**

$s_y = u_y t + ½ a_y t^2$

$u_y = 80$

$a_y = -10$

$t_y = 3$

$\Rightarrow s_y = 80(3) + \frac{1}{2}(-10)(3)^2$ ... (5m)

$= 240 + \frac{1}{2}(-10)(9)$

$= 240 - 45$

$= 195$ m ... (5m)

3(ii) Calculate the time it takes to reach the maximum height. **(10)**

<u>At maximum height</u>

$v_y = 0$

$v_y = u_y + a_y t_y$

$u_y = 80$

$a_y = -10$

$v_y = 0$

$\Rightarrow 0 = 80 + (-10)t_y$ ... (5m)

$\Rightarrow 10t_y = 80$

$\Rightarrow t_y = \frac{80}{10}$

$= 8$ s ... (5m)

3(iii) Calculate the maximum height of the particle above ground level. **(10)**

$s_y = u_y t + ½ a_y t^2$

$u_y = 80$

$a_y = -10$

$t_y = 8$

$\Rightarrow s_y = 80(8) + \frac{1}{2}(-10)(8)^2$ ... (5m)

$= 640 + \frac{1}{2}(-10)(64)$

$= 640 - 320$

$= 320$ m ... (5m)

3(iv) Find the range of the particle. **(10)**

$s_y = 0$

$s_y = u_y t + \frac{1}{2} a_y t^2$

$u_y = 80$

$a_y = -10$

$\Rightarrow 0 = 80t + \frac{1}{2}(-10)t^2$

$= 80t - 5t^2$

$= 5t(16 - t)$

$\Rightarrow 5t = 0$

$\Rightarrow t = 0$

Solution not feasible

$\Rightarrow 16 - t = 0$

$\Rightarrow -t = -16$

$\Rightarrow t = 16$ ... (5m)

$s_x = u_x t + \frac{1}{2} a_x t^2$

$u_x = 60$

$a_x = 0$

$t = 16$

$s_x = 60t$

$= 60(16)$

$= 960$ m ... (5m)

3(v) Find the speed of the particle as it strikes the ground. **(10)**

$u_x = 60$

$\Rightarrow v_x = 60$

$v_y = u_y + a_y t$

$u_y = 80$

$a_y = -10$

$\Rightarrow v_y = 80 - 10t$

Particle strikes the ground at $t = 16$ s

$\Rightarrow v_y = 80 - 10(16)$

$= 80 - 160$

$= -80$ ... (5m)

$\vec{v} = v_x + v_y$

$= 60\vec{i} - 80\vec{j}$

$\Rightarrow |\vec{v}| = \sqrt{60^2 + (-80)^2}$

$= \sqrt{3{,}600 + 6{,}400}$

$= \sqrt{10{,}000}$

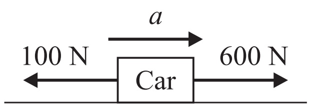
$= 100$ m/sec ... (5m)

4. **(a)** Four people, each of mass 80 kg, get into an empty car of mass 2000 kg.
When the car is started, the engine exerts a force of 600 N.
There is a resistance force of 100 N.

Find, correct to two decimal places,

4(a)(i) the acceleration of the car **(15)**

$$m = 1{,}000 + 4(80)$$
$$= 1{,}000 + 320$$
$$= 1{,}320 \text{ kg} \qquad \text{... (5m)}$$



$$F = ma$$
$$600 - 100 = 1{,}320a \qquad \text{... (5m)}$$
$$\Rightarrow 1{,}320a = 500$$
$$\Rightarrow a = \frac{500}{1{,}320}$$
$$= 0.378787...$$
$$\cong 0.38 \text{ m/s}^2 \qquad \text{... (5m)}$$

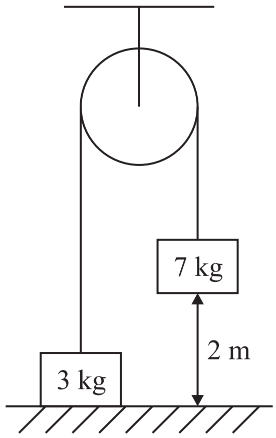
4(a)(ii) the acceleration of the car if only the driver was in the car. **(5)**

$$m = 1{,}000 + 1(80)$$
$$= 1{,}000 + 80$$
$$= 1{,}080 \text{ kg}$$

$$F = ma$$
$$600 - 100 = 1{,}080a$$
$$\Rightarrow 1{,}080a = 500$$
$$\Rightarrow a = \frac{500}{1{,}080}$$
$$= 0.462962...$$
$$\cong 0.46 \text{ m/s}^2 \qquad \text{... (5m)}$$

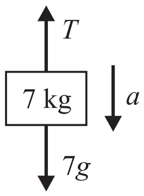
4(b) Two particles of masses 7 kg and 3 kg are connected by a taut, light, inelastic string which passes over a smooth light pulley.

The 3 kg mass rests on a smooth horizontal table and the 7 kg mass is 2 m above the table. The system is released from rest.

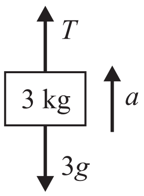


4(b)(i) Show on separate diagrams all the forces acting on each mass. **(10)**

❶ ... (5m)



❷ ... (5m)



4(b)(ii) Find the common acceleration of the particles. **(15)**

❶ $7g - T = 7a$ ... (5m)

❷ $-3g + T = 3a$ ... (5m)

<u>Common acceleration</u>

❶ $7g - T = 7a$

❷ $-3g + T = 3a$

$\Rightarrow 4g = 10a$

$\Rightarrow 4(10) = 10a$

$\Rightarrow a = \dfrac{40}{10}$

$\Rightarrow a = 4 \text{ m/s}^2$ ... (5m)

4(b)(iii) Find the tension in the string. **(5)**

❷ $-3g + T = 3a$

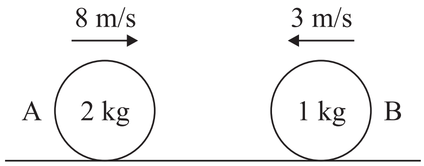
$\Rightarrow T = 3a - 3g$

$= 3(4) + 3(10)$

$= 42 \text{ N}$ ... (5m)

5. A smooth sphere A, of mass 2 kg, collides directly with another smooth sphere B, of mass 1 kg, on a smooth horizontal table.

A and B are moving in opposite directions with speeds of 8 m/s and 3 m/s, respectively.



The coefficient of restitution for the collision is $\frac{4}{11}$.

Find

5(i) the speed of A and the speed of B after the collision **(30)**

Sphere	Mass	Speed before	Speed after
A	2 kg	8	p
B	1 kg	–3	q

<u>COM:</u>

$m_A u_A + m_B u_B = m_A v_A + m_A v_A$

$\Rightarrow \quad 2(8) + 1(-3) = 2(p) + 1(q)$ ... (5m)

❶ $\Rightarrow \quad 13 = 2p + q$ ... (5m)

<u>NEL:</u>

$\frac{v_A - v_B}{u_A - u_B} = -e$

$\frac{p - q}{8 - (-3)} = -\frac{4}{11}$ ... (5m)

$\Rightarrow \quad 11(p - q) = -4(8 + 3)$

$= -4(11)$

❷ $\Rightarrow \quad p - q = -4$ ... (5m)

❶ $2p + q = 13$

❷ $p - q = -4$

$\Rightarrow \quad 3p = 9$

$\Rightarrow \quad p = 3$ m/s ... (5m)

❶ $2p + q = 13$

$\Rightarrow \quad q = 13 - 2p$

$= 13 - (2)(3)$

$= 7$ m/s ... (5m)

5(ii) the loss in kinetic energy due to the collision **(15)**

<u>K.E. before collision</u>

$$\text{K.E.}_{\text{before}} = \frac{1}{2}m_A u_A^2 + \frac{1}{2}m_B u_B^2$$
$$= \frac{1}{2}(2)(8)^2 + \frac{1}{2}(1)(-3)^2$$
$$= \frac{1}{2}(128) + \frac{1}{2}(9)$$
$$= 68.5 \text{ J}$$ ... (5m)

<u>K.E. after collision</u>

$$\text{K.E.}_{\text{after}} = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$
$$= \frac{1}{2}(2)(3)^2 + \frac{1}{2}(1)(7)^2$$
$$= \frac{1}{2}(18) + \frac{1}{2}(49)$$
$$= 33.5 \text{ J}$$ ... (5m)

$$\Rightarrow \quad \text{K.E.}_{\text{loss}} = \text{K.E.}_{\text{before}} - \text{K.E.}_{\text{after}}$$
$$= 68.5 - 33.5$$
$$= 35 \text{ Joules}$$ ... (5m)

5(iii) the magnitude of the impulse imparted to each sphere. **(5)**

$$I_A = m_A v_A - m_A u_A$$
$$= 2(3) - 2(8)$$
$$= 6 - 16$$
$$= -10 \text{ Ns}$$

$$I_B = m_B v_B - m_B u_B$$
$$= 1(7) - 1(-3)$$
$$= 7 + 3$$
$$= 10 \text{ Ns}$$ ... (5m)

6. **(a)** Particles of weight 5 N, 3 N and x N are placed at the points (–1, 1), (4, 7), and (9, 2), respectively.
The co-ordinates of the centre of gravity of the system are (2.5, y).

6(a)(i) Find the value of x. **(10)**

5 N @ (–1, 1), 3 N @ (4, 7), x N @ (9, 2) = (8 + x) N @ (2.5, y)

⇒	$5(-1) + 3(4) + x(9)$	=	$(8 + x)(2.5)$	... (5m)
⇒	$-5 + 12 + 9x$	=	$20 + 2.5x$	
⇒	$9x - 2.5x$	=	13	
⇒	$6.5x$	=	13	
⇒	x	=	2	... (5m)

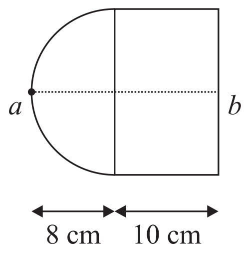
6(a)(ii) Find the value of y. **(10)**

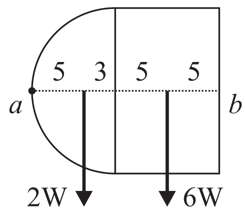
5 N @ (–1, 1), 3 N @ (4, 7), x N @ (9, 2) = (8 + x) N @ (2.5, y)

⇒	$5(1) + 3(7) + 2(2)$	=	$10y$	... (5m)
⇒	$10y$	=	30	
⇒	y	=	3	... (5m)

6(b) A solid consists of a uniform cylinder of length 10 cm and weight 6W, attached to a hemisphere of radius 8 and weight 2W.

Find the distance of the centre of gravity of the solid from the point *b*. **(30)**





... (5m)

Centre of gravity of a solid hemisphere is $\frac{3}{8}$ of the radius from the centre of a solid sphere

$= \frac{3}{8}(8)$

$= 3$ cm ... (5m)

<u>Taking moments @ *a*:</u>

2W @ (5, 0), 6W @ (13, 0) $= 8W$ @ (x, y)

$\Rightarrow$ $2W(5) + 6W(13) = 8W(x)$ ... (10m)

$\Rightarrow$ $88W = 8Wx$

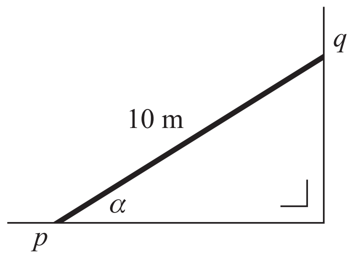
$\Rightarrow$ $x = 11$ ... (5m)

Centre of gravity from *b*

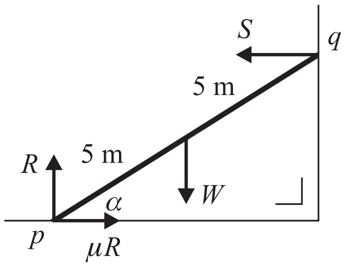
$= 18 - 11$

$= 7$ cm ... (5m)

7. A uniform ladder, $[pq]$, of length 10 m and weight W, rests on rough horizontal ground and leans against a smooth vertical wall.
The coefficient of friction between the ladder and the ground is $\frac{4}{5}$.
The ladder is inclined at an angle α to the ground.



7(i) Show on a diagram all the forces acting on the ladder. **(10)**



... (10m)

7(ii) Write down the two equations that arise from resolving the forces horizontally and vertically. **(10)**

❶ $R = W$ ... (5m)

❷ $S = \frac{4}{5}R$

$= \frac{4}{5}W$ ... (5m)

7(iii) Write down the equation which arises from taking moments about point p. **(5)**

❸ $S(10\sin\alpha) = W(5\cos\alpha)$ ... (5m)

7(iv) Show that the ladder is on the point of slipping when $\tan\alpha = \frac{5}{8}$. **(10)**

❸ $S(10\sin\alpha) = W(5\cos\alpha)$

$\Rightarrow 10(\frac{4}{5}W)S\sin\alpha = 5W\cos\alpha$ ... (5m)

$\Rightarrow 8W\sin\alpha = 5W\cos\alpha$

$\Rightarrow 8\sin\alpha = 5\cos\alpha$

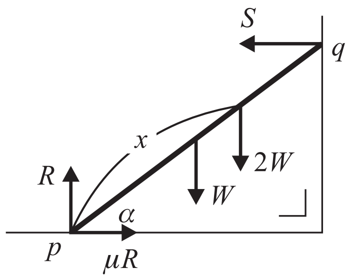
$\Rightarrow \frac{\sin\alpha}{\cos\alpha} = \frac{5}{8}$

$\Rightarrow \tan\alpha = \frac{5}{8}$ ... (5m)

7(v) The same ladder is now placed against the wall where $\tan\alpha = \frac{3}{4}$.

A woman of weight $2W$ begins to climb the ladder.
How far up the ladder can she go before the ladder begins to slip? **(15)**

<u>Diagram</u>



❶ $R = W + 2W$

$= 3W$ ... (5m)

❷ $S = \frac{4}{5}R$

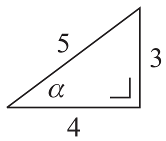
$= \frac{4}{5}(3W)$

$= \frac{12}{5}W$

$= 2.4W$ ... (5m)

$\sin\alpha = \frac{3}{5}$

$= 0.6$



$\cos\alpha = \frac{4}{5}$

$= 0.8$

❸ $S(10\sin\alpha) = W(5\cos\alpha) + 2W(x\cos\alpha)$

$\Rightarrow$ $2.4W(10)(0.6) = W(5)(0.8) + 2W(x)(0.8)$

$\Rightarrow$ $1.6Wx = 14.4W - 4W$

$= 10.4W$

$\Rightarrow$ $1.6x = 10.4$

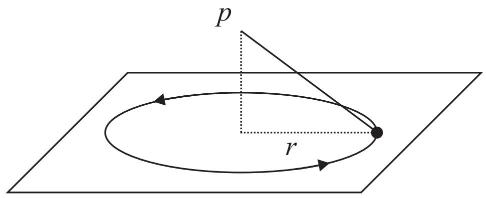
$\Rightarrow$ $x = \frac{10.4}{1.6}$

$= 6.5$ m ... (5m)

8. **(a)** A CD spins at 240 rotations per minute with constant angular velocity ω radians per second. Find the value of ω. **(10)**

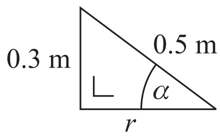
	60 s	::	240 rotations	
$\Rightarrow$	1 s	::	4 rotations	... (5m)
$\Rightarrow$	ω	=	$4(2\pi)$	
		=	8π rad/s <u>or</u> 25.132 rad/s	... (5m)

8(b) A smooth particle of mass 10 kg is attached by a light inelastic string of length 0.5 m to a fixed point p. The particle describes a horizontal circle of radius r cm on the smooth surface of a horizontal table. The centre of the horizontal circle is 0.3 m vertically below the point p.



Find

8(b)(i) the value of r **(10)**



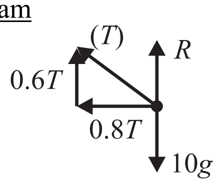
	$r^2 + (0.3)^2$	=	$(0.5)^2$	... (5m)
$\Rightarrow$	$r^2 + 0.09$	=	0.25	
$\Rightarrow$	r^2	=	$0.25 - 0.09$	
		=	0.16	
$\Rightarrow$	r	=	0.4 m	... (5m)

8(b)(ii) the tension in the string **(20)**

$$\sin\alpha = \frac{0.3}{0.5} = 0.6$$

$$\cos\alpha = \frac{0.4}{0.5} = 0.8 \qquad \text{... (5m)}$$

<u>Diagram</u>



... (5m)

	$F_{\text{centripetal}}$	=	$m\omega^2 r$	
		=	$0.8T$	
$\Rightarrow$	$0.8T$	=	$(10)(1.2)^2(0.4)$	... (5m)
		=	5.76	
$\Rightarrow$	T	=	$\frac{5.76}{0.8}$	
		=	7.2 N	... (5m)

8(b)(iii) the reaction force between the particle and the table. **(10)**

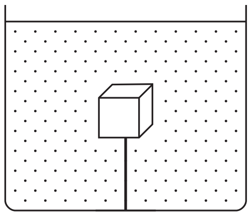
$$R + 0.6T = 100 \quad \text{... (5m)}$$
$$\Rightarrow R + 0.6(7.2) = 100$$
$$\Rightarrow R = 100 - 0.6(7.2)$$
$$= 100 - 4.32$$
$$= 95.68 \text{ N} \quad \text{... (5m)}$$

9. **(a)** A stone of relative density 2.5 weighs 50 N in air.
Find the weight of the stone when it is fully immersed in water. **(10)**

$$B_w = \frac{W}{s}$$
$$= \frac{50}{2.5}$$
$$= 20 \quad \text{... (5m)}$$

$$W_{apparent} = W - B_w$$
$$= 50 - 20$$
$$= 30 \text{ N} \quad \text{... (5m)}$$

9(b) A solid wooden cube, of length 0.5 m and relative density 0.8, is completely immersed in a tank of liquid of relative density 0.9.



The wooden cube is held at rest by a light inelastic vertical string which is attached to the base of the tank.

[Density of water = 1000 kg/m^3]

9(b)(i) Find the volume of the cube in m^3. **(10)**

$$V = (0.5)^3$$
$$= (0.5)(0.5)(0.5)$$
$$= 0.125 \text{ m}^3 \quad \text{... (10m)}$$

9(b)(ii) Find the weight of the cube. **(10)**

$$W = V\rho g$$
$$= 0.125(0.8 \times 1{,}000)(10)$$
$$= 0.125(800)(10)$$
$$= 1{,}000 \text{ N} \quad \text{... (10m)}$$

9(b)(iii) Find the tension in the string. **(20)**

$$B_l = V\rho_l g$$
$$= 0.125(0.9 \times 1{,}000)(10)$$
$$= 0.125(900)(10)$$
$$= 1{,}125 \text{ N} \quad \text{... (10m)}$$

$$T + W = B_l \quad \text{... (5m)}$$
$$\Rightarrow T + 1{,}000 = 1{,}125$$
$$\Rightarrow T = 1{,}125 - 1{,}000$$
$$= 125 \text{ N} \quad \text{... (5m)}$$

Notes:

Dublin Examining Board

Pre-Leaving Certificate Examination, 2010

Applied Mathematics

Higher Level
Marking Scheme (300 marks)

General Instructions

1. Penalties of three types are applied to students' work as follows:

Slips	- numerical slips	S(–1)
Blunders	- mathematical errors	B(–3)
Misreading	- if not serious	M(–1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

4. Number the grid on each script 1 to 10 in numerical order, not the order of answering.

5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

1. **(a)** A train moving in a straight line with uniform deceleration passes three points a, b and c, each 600 m apart. It takes 50 seconds to travel from a to b and 70 seconds to travel from b to c.

Find

1(a)(i) the deceleration of the train **(15)**

<u>Deceleration from a to b:</u>

$$s = ut + \tfrac{1}{2}at^2$$
$$t = 50$$
$$s = 600$$
$$a = a$$
$$u = u$$

$\Rightarrow$ $600 = u(50) + \tfrac{1}{2}a(50)^2$

$\Rightarrow$ $600 = 50u + 1{,}250a$

$\Rightarrow$ $50u + 1{,}250a = 600$

❶ $\Rightarrow$ $u + 25a = 12$ ... (5m)

<u>Deceleration from a to c:</u>

$$s = ut + \tfrac{1}{2}at^2$$
$$t = 120$$
$$s = 1{,}200$$
$$a = a$$
$$u = u$$

$\Rightarrow$ $1{,}200 = u(120) + \tfrac{1}{2}a(120)^2$

$\Rightarrow$ $1{,}200 = 120u + 7{,}200a$

$\Rightarrow$ $120u + 7{,}200a = 1{,}200$

❷ $\Rightarrow$ $u + 60a = 10$ ... (5m)

❶ $u + 25a = 12\ (\times 1)$

❷ $u + 60a = 10\ (\times -1)$

$u + 25a = 12\ (\times 1)$

$-u + -60a = -10\ (\times -1)$

$\Rightarrow$ $-35a = 2$

$\Rightarrow$ $a = -\dfrac{2}{35}$ m/s^2 ... (5m)

❶ $u + 25a = 12$

$\Rightarrow$ $u + 25(-\dfrac{2}{35}) = 12$

$\Rightarrow$ $u = 12 + \dfrac{50}{35}$

$= \dfrac{470}{35}$

$= \dfrac{94}{7}$ m/s^2

1(a)(ii) the distance travelled from c before the train comes to rest. **(10)**

Distance from a to d (point of rest):

$$v^2 = u^2 + 2as$$

$$a = -\frac{2}{35}$$

$$u = \frac{94}{7}$$

$$v = 0$$

$$\Rightarrow \quad 0 = \left(\frac{94}{7}\right)^2 + 2\left(\frac{-2}{35}\right)s \qquad \text{... (5m)}$$

$$\Rightarrow \quad 0 = 180.326530\ldots - 0.114285\ldots s$$

$$\Rightarrow \quad s = \frac{180.326530\ldots}{0.114285\ldots}$$

$$= 1{,}577.857137\ldots$$

$$\cong 1{,}578 \text{ m}$$

Distance travel from c:

$$d = 1{,}578 - 1{,}200$$

$$= 378 \text{ m} \qquad \text{... (5m)}$$

1(b) Two particles, P and Q, moving in the same direction along parallel lines pass a point o with accelerations of 3 m/s^2 and 2 m/s^2 respectively. P passes o with a speed of 5 m/s. One second later, Q passes o with a speed of 35 m/s.

Find the greatest distance between the two particles after Q overtakes P. **(25)**

Time after P passes $o = t$
Time after Q passes $o = t - 1$

Speed of P after passing o:

$$v_P = u_P + a_P t$$

$$u_P = 5$$

$$a_P = 3$$

$$t = t$$

$$\Rightarrow \quad v_P = 5 + 3t \qquad \text{... (5m)}$$

Speed of Q after passing o:

$$v_Q = u_Q + a_Q t$$

$$u_Q = 35$$

$$a_Q = 2$$

$$t = t - 1$$

$$\Rightarrow \quad v_Q = 35 + 2(t - 1) \qquad \text{... (5m)}$$

Greatest distance between particles when $v_P = v_Q$

$$\Rightarrow \quad 5 + 3t = 35 + 2(t - 1)$$

$$= 35 + 2t - 2$$

$$= 33 + 2t$$

$$\Rightarrow \quad 3t - 2t = 33 - 5$$

$$\Rightarrow \quad t = 28 \qquad \text{... (5m)}$$

Distance of P from o:

$s_P = u_P t + \frac{1}{2} a_P t^2$

$u_P = 5$

$t = 28$

$a_P = 3$

$\Rightarrow s_P = 5(28) + \frac{1}{2}(3)(28)^2$

$= 140 + \frac{1}{2}(3)(784)$

$= 140 + 1{,}176$

$= 1{,}316$ m ... (5m)

Distance of Q from o:

$s_Q = u_Q t + \frac{1}{2} a_Q t^2$

$u_Q = 35$

$t = 28 - 1$

$= 27$

$a_Q = 2$

$\Rightarrow s_Q = 35(27) + \frac{1}{2}(2)(27)^2$

$= 945 + \frac{1}{2}(2)(729)$

$= 945 + 729$

$= 1{,}674$m

Greatest distance between particles:

$d = s_Q - s_P$

$= 1{,}674 - 1{,}316$

$= 358$ m ... (5m)

2. **(a)** A woman can row a boat at 5 m/s in still water.
She rows across a river of width 60 m. The river flows with a constant speed of 1.4 m/s parallel to the straight banks.

2(a)(i) Find the direction which the woman should take to cross the river in the shortest time and, hence, calculate the time it takes to cross. **(10)**

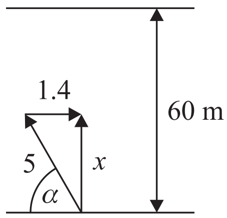
<u>Direction</u>

Answer = straight across ... (5m)

<u>Time</u>

$$t = \frac{d \text{ (across)}}{v \text{ (across)}} = \frac{60}{5} = 12 \text{ sec}$$ ... (5m)

2(a)(ii) Find the direction which the woman should take to cross the river by the shortest path, and, hence calculate the time it takes to cross. **(10)**



$$x^2 + (1.4)^2 = 5^2$$
$$\Rightarrow \quad x^2 + 1.96 = 25$$
$$\Rightarrow \quad x^2 = 25 - 1.96 = 23.04$$
$$\Rightarrow \quad x = \sqrt{23.04} = 4.8$$

<u>Direction</u>

$$\cos\alpha = \frac{1.4}{5}$$
$$\Rightarrow \quad \alpha = \cos^{-1}(0.28) = 73.739795... \cong 73.74° \text{ (to the bank)}$$ ... (5m)

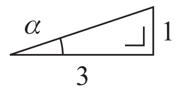
<u>Time</u>

$$t = \frac{d}{v_x} = \frac{60}{4.8} = 12.5 \text{ sec}$$ ... (5m)

2(b) At a certain instant ship B is 900 m west of ship A.
Ship B is travelling with speed $4\sqrt{10}$ m/s in a direction $\tan^{-1}\frac{1}{3}$ North of East
and ship A is travelling with constant velocity.

2(b)(i) Find the magnitude and direction of the minimum velocity of A in order to intercept B. **(15)**

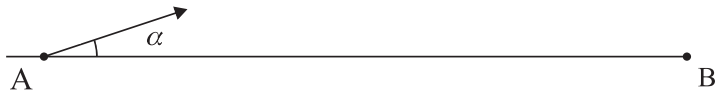
$$\tan\alpha = \frac{1}{3}$$



$$\Rightarrow \quad \sin\alpha = \frac{1}{\sqrt{10}}$$

$$\Rightarrow \quad \cos\alpha = \frac{3}{\sqrt{10}}$$

Diagram



Vector

$$v_B = 4\sqrt{10}\cos\alpha\,\vec{i} + 4\sqrt{10}\sin\alpha\,\vec{j}$$

$$= 4\sqrt{10}\left(\frac{3}{\sqrt{10}}\right)\vec{i} + 4\sqrt{10}\left(\frac{1}{\sqrt{10}}\right)\vec{j}$$

$$= 12\vec{i} + 4\vec{j}$$ ... (5m)

Magnitude

Minimum speed of A is 4 m/s
since it must go at $B\vec{i} + 4\vec{j}$ to keep level with A

$\Rightarrow$ minimum velocity of A is $0\vec{i} + 4\vec{j}$

$\Rightarrow$ magnitude $= |v|$

$= 4$ m/s ... (5m)

Direction

= due North ... (5m)

2(b)(ii) Find the two possible directions in which A could be steered in order to intercept B if the speed of A is 5 m/s.
Calculate the time of interception in each case. **(15)**

$$|B\vec{i} + 4\vec{j}| = 5$$
$$\Rightarrow \sqrt{B^2 + 4^2} = 5$$
$$\Rightarrow B^2 + 16 = 25$$
$$\Rightarrow B^2 = 25 - 16 = 9$$
$$\Rightarrow B = 3 \quad \text{or} \quad -3 \qquad \text{... (5m)}$$

<u>Case 1</u>

$$v_B = 12\vec{i} + 4\vec{j}$$
$$v_A = 3\vec{i} + 4\vec{j}$$
$$v_{BA} = v_B - v_A = 12\vec{i} + 4\vec{j} - (3\vec{i} + 4\vec{j}) = 12\vec{i} - 3\vec{i} + 4\vec{j} - 4\vec{j} = 9\vec{i} - 0\vec{j} = 9\vec{i}$$
$$\Rightarrow |v_{BA}| = 9$$
$$\Rightarrow t = \frac{d}{|v_{BA}|}$$
$$\Rightarrow t = \frac{900}{9} = 100 \text{ sec} \qquad \text{... (5m)}$$

<u>Case 2</u>

$$v_B = 12\vec{i} + 4\vec{j}$$
$$v_A = -3\vec{i} + 4\vec{j}$$
$$v_{BA} = v_B - v_A = 12\vec{i} + 4\vec{j} - (-3\vec{i} + 4\vec{j}) = 12\vec{i} + 3\vec{i} + 4\vec{j} - 4\vec{j} = 15\vec{i} - 0\vec{j} = 15\vec{i}$$
$$\Rightarrow |v_{BA}| = 15$$
$$\Rightarrow t = \frac{d}{|v_{BA}|}$$
$$\Rightarrow t = \frac{900}{15} = 60 \text{ sec} \qquad \text{... (5m)}$$

3. **(a)** A particle is projected from a point o on level horizontal ground with an initial speed of $\sqrt{6g}$ m/s at an angle θ to the horizontal.

It strikes a small target whose position vector relative to o is $3\vec{i} + \frac{11}{12}\vec{j}$ metres.

Find the two possible values for θ. **(20)**

$$s_x = u\cos\theta\, t$$

$$\Rightarrow s_x = \sqrt{6g}\cos\theta\, t = 3$$

$$\Rightarrow \sqrt{6g}\cos\theta\, t = 3$$

$$\Rightarrow t = \frac{3}{\sqrt{6g}\cos\theta} \quad \text{... (5m)}$$

$$s_y = u\sin\theta\, t - \frac{1}{2}gt^2$$

$$\Rightarrow s_y = \sqrt{6g}\sin\theta\, t - \frac{1}{2}gt^2 = \frac{11}{12}$$

$$\Rightarrow \frac{11}{12} = \sqrt{6g}\sin\theta\left(\frac{3}{\sqrt{6g}\cos\theta}\right) - \frac{1}{2}g\left(\frac{9}{6g\cos^2\theta}\right) \quad \text{... (5m)}$$

$$\Rightarrow \quad = 3\tan\theta - \frac{3}{4}\left(\frac{1}{\cos^2\theta}\right)$$

$$\Rightarrow \quad = 3\tan\theta - \frac{3}{4}(1 + \tan^2\theta)$$

$$\Rightarrow 11 = 12[3\tan\theta - \frac{3}{4}(1 + \tan^2\theta)]$$

$$= 36\tan\theta - 9(1 + \tan^2\theta)$$

$$= 36\tan\theta - 9 - 9\tan^2\theta$$

$$\Rightarrow 9\tan^2\theta - 36\tan\theta + 20 = 0 \quad \text{... (5m)}$$

$$\Rightarrow (3\tan\theta - 2)(3\tan\theta - 10) = 0$$

$$\Rightarrow 3\tan\theta - 2 = 0$$

$$\Rightarrow \tan\theta = \frac{2}{3}$$

$$\Rightarrow \theta = \tan^{-1}\frac{2}{3} = 33.690067... \cong 33.69°$$

$$\Rightarrow 3\tan\theta - 10 = 0$$

$$\Rightarrow \tan\theta = \frac{10}{3}$$

$$\Rightarrow \theta = \tan^{-1}\frac{10}{3} = 73.300755... \cong 73.3° \quad \text{... (5m)}$$

3(b) A particle is projected up an inclined plane with initial velocity u m/s.
The line of projection makes an angle α with the inclined plane and the plane is inclined at an angle β to the horizontal.
The plane of projection is vertical and contains the line of greatest slope.

3(b)(i) If the particle is moving horizontally when it strikes the inclined plane, show that $2\tan\alpha\tan^2\beta = \tan\beta - \tan\alpha$. **(25)**

$$v_x = u\cos\alpha - g\sin\beta\, t$$
$$s_x = u\cos\alpha t - \frac{1}{2}g\sin\beta\, t^2$$
$$v_y = u\sin\alpha - g\cos\beta\, t$$
$$s_y = u\sin\alpha t - \frac{1}{2}g\cos\beta\, t^2$$

$\tan\beta = -\dfrac{v_y}{v_x}$ when $s_y = 0$ ... (5m)

$$s_y = u\sin\alpha t - \frac{1}{2}g\cos\beta\, t^2 = 0$$
$$\Rightarrow u\sin\alpha t - \frac{1}{2}g\cos\beta\, t^2 = 0$$
$$\Rightarrow t\left(u\sin\alpha - \frac{1}{2}g\cos\beta\, t\right) = 0$$
$$\Rightarrow t = 0$$
or
$$\Rightarrow u\sin\alpha - \frac{1}{2}g\cos\beta\, t = 0$$
$$\Rightarrow g\cos\beta\, t = 2u\sin\alpha$$
$$\Rightarrow t = \frac{2u\sin\alpha}{g\cos\beta}$$

= time of flight ... (5m)

$$v_x = u\cos\alpha - g\sin\beta\, t$$
$$= u\cos\alpha - g\sin\beta\left(\frac{2u\sin\alpha}{g\cos\beta}\right)$$
$$= u\cos\alpha - 2u\sin\alpha\left(\frac{\sin\beta}{\cos\beta}\right)$$

$= u\cos\alpha - 2u\sin\alpha\tan\beta$ ... (5m)

$$v_y = u\sin\alpha - g\cos\beta\, t$$
$$= u\sin\alpha - g\cos\beta\left(\frac{2u\sin\alpha}{g\cos\beta}\right)$$
$$= u\sin\alpha - 2u\sin\alpha$$

$= -u\sin\alpha$ ... (5m)

$$\tan\beta = -\frac{v_y}{v_x}$$

$$= \frac{u\sin\alpha}{u\cos\alpha - 2u\sin\alpha\tan\beta}$$

$$= \frac{\frac{u\sin\alpha}{\cos\alpha}}{\frac{u\cos\alpha}{\cos\alpha} - \frac{2u\sin\alpha\tan\beta}{\cos\alpha}}$$

$$= \frac{\tan\alpha}{1-2\tan\alpha\tan\beta}$$

$\Rightarrow \quad \tan\beta(1 - 2\tan\alpha\tan\beta) = \tan\alpha$

$\Rightarrow \quad \tan\beta - 2\tan\alpha\tan^2\beta = \tan\alpha$

$\Rightarrow \quad \tan\beta - \tan\alpha = 2\tan\alpha\tan^2\beta$ ... (5m)

3(b)(ii) Hence, find the two possible values for β when $\tan\alpha = \frac{1}{3}$. **(5)**

$\tan\beta - \tan\alpha = 2\tan\alpha\tan^2\beta$

$\tan\alpha = \frac{1}{3}$

$\Rightarrow \quad \tan\beta - \frac{1}{3} = 2\left(\frac{1}{3}\right)\tan^2\beta$

$\Rightarrow \quad 3\tan\beta - 1 = 2\tan^2\beta$

$\Rightarrow \quad 2\tan^2\beta - 3\tan\beta + 1 = 0$

$\Rightarrow \quad (2\tan\beta - 1)(\tan\beta - 1) = 0$

$\Rightarrow \quad 2\tan\beta - 1 = 0$

$\Rightarrow \quad \tan\beta = \frac{1}{2}$

$\Rightarrow \quad \beta = \tan^{-1}\frac{1}{2}$

$= 26.565051...$

$\cong 26.56°$

or

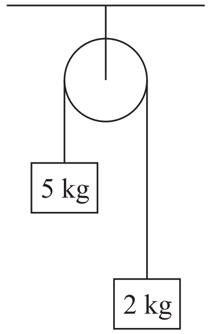
$\Rightarrow \quad \tan\beta - 1 = 0$

$\Rightarrow \quad \tan\beta = 1$

$\Rightarrow \quad \beta = \tan^{-1} 1$

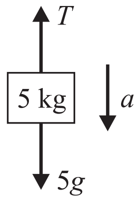
$= 45°$ ... (5m)

4. (a) Two particles, of masses 5 kg and 2 kg, are attached to the ends of a light inextensible string which passes over a smooth light fixed pulley. The system is released from rest.

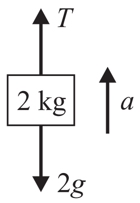


4(a)(i) Find the common acceleration of the particles. **(15)**

❶ $5g - T = 5a$ ... (5m)



❷ $T - 2g = 2a$ ... (5m)



$$
\begin{array}{rrcl}
❶ & 5g - T & = & 5a \\
❷ & \underline{-2g + T} & = & \underline{2a} \\
 & 3g & = & 7a \\
\Rightarrow & a & = & \frac{3}{7}g \\
 & & = & 4.2\ \text{m/s}^2
\end{array}
$$
... (5m)

4(a)(ii) After 2 seconds, the 5 kg particle hits a horizontal table and is brought to rest. How much further will the 2 kg particle rise after the 5 kg particle hits the table? **(5)**

<u>Speed of 5 kg and 2 kg at impact:</u>

$$
\begin{array}{rrcl}
 & v & = & u + at \\
 & u & = & 0 \\
 & a & = & 4.2 \\
 & t & = & 2 \\
\Rightarrow & v & = & 0 + 4.2(2) \\
 & & = & 8.4\ \text{m/s}
\end{array}
$$

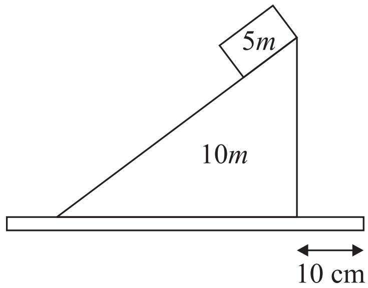
<u>Extra distance travelled by 2 kg particle:</u>

$$
\begin{array}{rrcl}
 & v^2 & = & u^2 + 2as \\
 & u & = & 8.4 \\
 & v & = & 0 \\
 & a & = & -9.8 \text{ (opposed by gravity)} \\
\Rightarrow & 0 & = & (8.4)^2 + 2(-9.8)s \\
\Rightarrow & 19.6s & = & 70.56 \\
\Rightarrow & s & = & 3.6\ \text{m}
\end{array}
$$
... (5m)

4(b) A right-angular wedge of mass $10m$ rests on a rough horizontal table. The coefficient of friction between the wedge and the table is $\frac{1}{10}$.

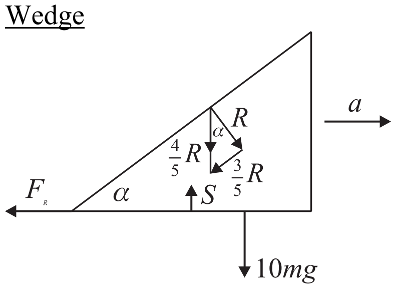
The cross-section of the wedge has height 30 cm and width 40 cm. The wedge is 10 cm from the edge of the table.

A small particle of mass $5m$ is placed at the top of the smooth inclined face of the wedge and the system is released from rest.



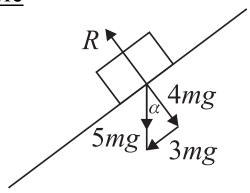
4(b)(i) Show, on separate diagrams, the forces acting on the wedge and the particle. **(10)**

Wedge



... (5m)

Particle



... (5m)

4(b)(ii) Find the acceleration of the wedge. **(15)**

$$l^2 = 40^2 + 30^2 = 1{,}600 + 900 = 2{,}500$$

$$\Rightarrow \quad l = \sqrt{2{,}500} = 50 \text{ cm}$$

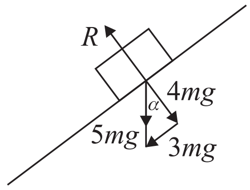
$$\Rightarrow \quad \sin\alpha = \frac{30}{50} = \frac{3}{5}$$

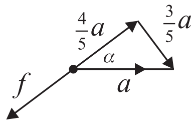
$$\Rightarrow \quad \cos\alpha = \frac{40}{50} = \frac{4}{5}$$

$$\tan\alpha = \frac{30}{40} = \frac{3}{4}$$

Particle

Forces: *Acceleration:*





$$3mg = 5m(f - \frac{4}{5}a)$$

$$\Rightarrow \quad 3mg = 5mf - 4ma$$

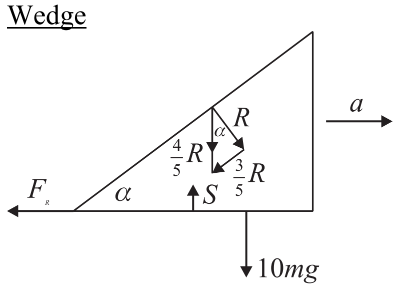
❶ $\Rightarrow \quad 3g = 5f - 4a$

$$4mg - R = 5m(\frac{3}{5}a)$$

$$\Rightarrow \quad 4mg - R = 3ma$$

❷ $\Rightarrow \quad R = 4mg - 3ma$ ... (5m)

Wedge



❸ $S = \frac{4}{5}R + 10mg$

$$\frac{3}{5}R - \mu S = 10ma$$

❹ $\Rightarrow \quad \frac{3}{5}R - \frac{1}{10}S = 10ma$ ... (5m)

Substituting ❸ into ❹

$$\Rightarrow \quad \frac{3}{5}R - \frac{1}{10}(\frac{4}{5}R + 10mg) = 10ma$$

$$\Rightarrow \quad \frac{3}{5}R - \frac{2}{25}R - mg = 10ma$$

❺ $\Rightarrow \quad \frac{13}{25}R - mg = 10ma$

Substituting ❷ into ❺

$$\Rightarrow \quad \frac{13}{25}(4mg - 3ma) - mg = 10ma$$

$$\Rightarrow \quad \frac{52}{25}mg - \frac{39}{25}ma - mg = 10ma$$

$$\Rightarrow \quad 10ma + \frac{39}{25}ma = \frac{52}{25}mg - mg$$

$$\Rightarrow \quad \frac{289}{25}ma = \frac{27}{25}mg$$

$$\Rightarrow \quad a = \frac{27}{289}g$$

$$= \frac{27}{289}(9.8)$$

$$= 0.915570...$$

$= 0.92 \text{ m/s}^2$ ... (5m)

4(b)(iii) Determine whether or not the wedge will reach the edge of the table before the particle reaches the table. **(5)**

<u>Acceleration of 5*m* particle:</u>

❶
$$3g = 5f - 4a$$
$$\Rightarrow \quad 5f = 3g + 4a$$
$$\Rightarrow \quad f = \frac{3g + 4a}{5}$$
$$= \frac{3(9.8) + 4(0.92)}{5}$$
$$= \frac{29.4 + 3.68}{5}$$
$$= \frac{33.08}{5}$$
$$= 6.616 \text{ m/s}^2$$

<u>Time for 5*m* particle to reach bottom of wedge:</u>

$$s = ut + \tfrac{1}{2}at^2$$
$$a = 6.616$$
$$s = 0.5 \text{ (50 cm)}$$
$$u = 0$$

$$\Rightarrow \quad 0.5 = 0 + \frac{1}{2}(6.616)t^2$$
$$\Rightarrow \quad t^2 = \frac{2(0.5)}{6.616}$$
$$= 0.151148...$$
$$\Rightarrow \quad t = 0.388778...$$
$$\cong 0.39 \text{ sec}$$

<u>Distance travelled by 10*m* wedge:</u>

$$s = ut + \tfrac{1}{2}at^2$$
$$a = 0.92$$
$$u = 0$$
$$t = 0.39$$

$$\Rightarrow \quad s = 0 + \frac{1}{2}(0.92)(0.39)^2$$
$$= 0.069966$$
$$\cong 0.07 \text{ m}$$

Wedge only moves 7 cm and therefore does not reach the edge of the table ... (5m)

5. (a) Three smooth spheres, of masses 2 kg, 4 kg and 6 kg, move in the same direction with speeds 10 m/s, 2 m/s and 6 m/s, respectively. The centres of all the spheres lie in a straight line.

If the smaller two masses are first to collide, find the range of values of e, the coefficient of restitution, for which there is no further collisions. **(20)**

Sphere	Mass	Speed before	Speed after
①	2	10	p
②	4	2	q

<u>COM:</u>

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$\Rightarrow 2(10) + 4(2) = 2p + 4q$$
$$\Rightarrow 2p + 4q = 20 + 8 = 28$$
❶ $$\Rightarrow p + 2q = 14$$ ... (5m)

<u>NEL:</u>

$$\frac{v_1 - v_2}{u_1 - u_2} = -e$$
$$\Rightarrow \frac{p - q}{10 - 2} = -e$$
❷ $$\Rightarrow p - q = -8e$$ ... (5m)

❶ $p + 2q = 14 \quad (\times 1)$
❷ $p - q = -8e \quad (\times -1)$

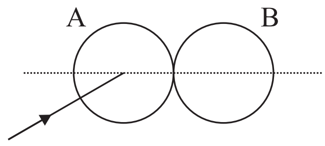
❶ $p + 2q = 14$
❷ $-p + q = 8e$

$$\Rightarrow 3q = 8e + 14$$
$$\Rightarrow q = \frac{8e + 14}{3}$$ ... (5m)

For no further collision to occur, $q \leq 6$ (speed of third sphere)

$$\Rightarrow \frac{8e + 14}{3} \leq 6$$
$$\Rightarrow 8e + 14 \leq 3(6) \leq 18$$
$$\Rightarrow 8e \leq 18 - 14 \leq 4$$
$$\Rightarrow e \leq \frac{4}{8} \leq \frac{1}{2}$$
$$\Rightarrow \text{Range} \quad 0 \leq e \leq \frac{1}{2}$$ ... (5m)

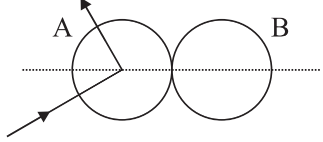
5(b) A smooth sphere A, of mass m, collides obliquely with another smooth sphere B, of mass $2m$, which is at rest.
The velocity of A before the collision is $16\vec{i} + 8\vec{j}$ m/s, where $\vec{i}$ and $\vec{j}$ are unit vectors along and perpendicular to the line of centres at the moment of impact, respectively.
Sphere A is then deflected through an angle of 90° as a result of the collision.



Find

5(b)(i) the speed of each sphere after the collision **(10)**

Sphere	Mass	Speed before	Speed after
A	m	$16\vec{i} + 8\vec{j}$	$p\vec{i} + 8\vec{j}$
B	$2m$	$0\vec{i} + 0\vec{j}$	$q\vec{i} + 0\vec{j}$



Sphere A:

$$(16\vec{i} + 8\vec{j})(p\vec{i} + 8\vec{j}) = 0$$
$$\Rightarrow 16p + 64 = 0$$
$$\Rightarrow p = -4$$ ... (5m)

Sphere A:
COM:

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$
$$\Rightarrow m(16) + 2m(0) = m(-4) + 2mq$$
$$\Rightarrow 2mq = 16m + 4m$$
$$= 20m$$
$$\Rightarrow q = \frac{20m}{2m}$$
$$= 10$$ ... (5m)

5(b)(ii) the coefficient of restitution between the two spheres **(5)**

NEL:

$$\frac{v_1 - v_2}{u_1 - u_2} = -e$$
$$\Rightarrow \frac{-4 - 10}{16 - 0} = -e$$
$$\Rightarrow e = \frac{14}{16}$$
$$= \frac{7}{8}$$ ... (5m)

5(b)(iii) the percentage loss in kinetic energy due to the collision. **(15)**

$$\text{K.E.}_{\text{before}} = \frac{1}{2}(m)(16^2 + 8^2) + \frac{1}{2}(2m)(0^2 + 0^2)$$
$$= \frac{1}{2}(m)(256 + 64)$$
$$= \frac{1}{2}(m)(320)$$
$$= 160m \text{ J}$$ ... (5m)

$$\text{K.E.}_{\text{after}} = \frac{1}{2}(m)((-4)^2 + 8^2) + \frac{1}{2}(2m)(10^2 + 0^2)$$
$$= \frac{1}{2}(m)(16 + 64) + \frac{1}{2}(2m)(100)$$
$$= \frac{1}{2}(m)(80) + \frac{1}{2}(m)(200)$$
$$= \frac{1}{2}(m)(280)$$
$$= 140m \text{ J}$$ ... (5m)

$$\Rightarrow \quad \text{K.E.}_{\text{loss}} = 160m - 140m$$
$$= 20m$$
$$\Rightarrow \quad \%\ \text{K.E. loss} = \frac{20m}{160m} \times \frac{100}{1}$$
$$= 12.5\%$$ ... (5m)

6. **(a)** A particle moves in a straight line such that its displacement x from a fixed point o at time t is given by

$$x = a\cos(\omega t + \varepsilon)$$

where a, ω and ε are positive constants.

6(a)(i) Show that the particle moves with simple harmonic motion. **(10)**

$$\begin{aligned}
\dot{x} &= \frac{dx}{dt} \\
&= \frac{d}{dt}[a\cos(\omega t + \varepsilon)] \\
&= a[-\sin(\omega t + \varepsilon)]\omega \\
&= -a\omega \sin(\omega t + \varepsilon) && \text{... (5m)}
\end{aligned}$$

$$\begin{aligned}
\ddot{x} &= \frac{d\dot{x}}{dt} \\
&= \frac{d}{dt}[-a\omega \sin(\omega t + \varepsilon)] \\
&= -a\omega[\cos(\omega t + \varepsilon)]\omega \\
&= -a\omega^2[\cos(\omega t + \varepsilon)] \\
&= -\omega^2[a\cos(\omega t + \varepsilon)] \\
&= -\omega^2 x && \text{... (5m)}
\end{aligned}$$

$\Rightarrow$ as $\ddot{x} = -\omega^2 x$, the particle moves with SHM

6(a)(ii) At time $t = 0$, the particle's displacement from o is 1.2 m and its speed and acceleration are $6\sqrt{3}$ m/s and 30 m/s^2, respectively.

Find the values of a, ω and ε. **(15)**

$$\begin{aligned}
\ddot{x} &= -\omega^2 x \\
t &= 0 \\
x &= 1.2 \\
\dot{x} &= 6\sqrt{3} \\
\ddot{x} &= -30 && (\ddot{x} \text{ is negative if } x \text{ is positive})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad -30 &= -\omega^2(1.2) \\
\Rightarrow \quad \omega^2 &= \frac{-30}{-1.2} \\
&= 25 \\
\Rightarrow \quad \omega &= \sqrt{25} \\
&= 5 && \text{... (5m)}
\end{aligned}$$

$$\begin{aligned}
\dot{x}^2 &= \omega^2(a^2 - x^2) \\
\Rightarrow \quad (6\sqrt{3})^2 &= (5)^2(a^2 - (1.2)^2) \\
\Rightarrow \quad 108 &= 25(a^2 - 1.44) \\
\Rightarrow \quad a^2 - 1.44 &= \frac{108}{25} \\
&= 4.32 \\
\Rightarrow \quad a^2 &= 4.32 + 1.44 \\
&= 5.76 \\
\Rightarrow \quad a &= \sqrt{5.76} \\
&= 2.4 && \text{... (5m)}
\end{aligned}$$

$$
\begin{aligned}
& x &=\;& a\cos(\omega t + \varepsilon) \\
\Rightarrow\;& 1.2 &=\;& (2.4)\cos(0 + \varepsilon) \\
\Rightarrow\;& \cos\varepsilon &=\;& \frac{1.2}{2.4} \\
\Rightarrow\;& \varepsilon &=\;& \cos^{-1}\left(\frac{1}{2}\right) \\
& &=\;& \frac{\pi}{3} \;\; \underline{\text{or}} \;\; 60^\circ
\end{aligned}
$$

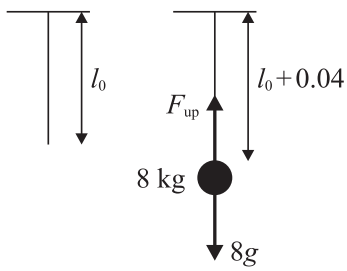
... (5m)

6(b) A particle, of mass 8 kg, is suspended from a fixed point by a light spiral spring which hangs vertically. The spring is extended by 4 cm when the system hangs in equilibrium. A second particle, of mass 4 kg, is attached to the first without moving it and the combined mass is then released from rest.

6(b)(i) Show that the motion of the particle is simple harmonic. **(15)**

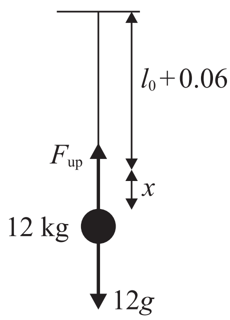
$$
\begin{aligned}
& W_1 &=\;& kx_1 \\
& 8g &=\;& k(0.04) \\
\Rightarrow\;& k &=\;& \frac{8g}{0.04} \\
& &=\;& 200g \text{ N/m}
\end{aligned}
$$

... (5m)



$$
\begin{aligned}
& x_2 &=\;& \frac{W_2}{k} \\
& &=\;& \frac{8g + 4g}{200g} \\
& &=\;& \frac{12g}{200g} \\
& &=\;& 0.06 \text{ m / 6 cm}
\end{aligned}
$$

$\Rightarrow$ centre of oscillation = 0.06 m / 6 cm beyond the natural length of the spring ... (5m)



$$
\begin{aligned}
& F_{\text{down}} &=\;& 12g \\
& F_{\text{up}} &=\;& k(l - l_0) \\
& &=\;& 200g(l_0 + 0.06 + x - l_0) \\
& &=\;& 200g(0.06 + x) \\
& &=\;& 12g + 200gx \\
& F_{\text{resultant}} &=\;& F_{\text{down}} - F_{\text{up}} \\
& &=\;& 12g - (12g + 200gx) \\
& &=\;& 12g - 12g - 200gx \\
& &=\;& -200gx \\
\Rightarrow\;& (m_1 + m_2)a &=\;& -200gx \\
\Rightarrow\;& (8 + 4)a &=\;& -200gx \\
\Rightarrow\;& 12a &=\;& -200gx \\
\Rightarrow\;& a &=\;& -\frac{200g}{12}x \\
& &=\;& -\frac{50g}{3}x
\end{aligned}
$$

since $a \propto x$ <u>and</u> in the opposite direction

$\Rightarrow$ motion of the particle is simple harmonic ... (5m)

6(b)(ii) Find the periodic time and maximum speed of the particle in the subsequent motion. **(10)**

<u>Periodic time</u>

$$a = -\frac{50g}{3}x$$

$$\Rightarrow \quad \omega = \sqrt{\frac{50g}{3}}$$

$$T = \frac{2\pi}{\omega}$$

$$= 2\pi\sqrt{\frac{3}{50g}}$$ ... (5m)

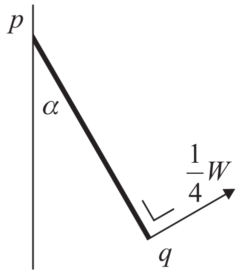
<u>Maximum speed</u>

$$\alpha = 0.06 - 0.04$$
$$= 0.02 \text{ m}$$

$$v_{max} = \omega\alpha$$
$$= \sqrt{\frac{50g}{3}}(0.02)$$
$$= \sqrt{\frac{(50)(9.8)}{3}}(0.02)$$
$$= (\sqrt{163.333333\ldots})(0.02)$$
$$= (12.780193\ldots)(0.02)$$
$$= 0.255603\ldots$$
$$\cong 0.26 \text{ m/s}$$ ... (5m)

7. **(a)** A uniform rod, $[pq]$, of weight W and length $2l$, is free to turn in a vertical plane about a hinge at p. The rod is supported by a light inextensible string at q. When the string is taut, the angle between the rod and the string is 90°.

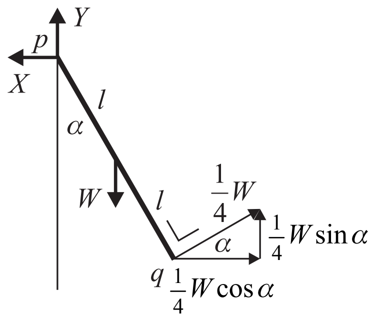
The tension in the string is found to be $\frac{1}{4}W$.



Find

7(a)(i) the angle between the rod makes and the wall **(5)**

<u>Diagram</u>



<u>Taking moments @ p:</u>

$$W(l\sin\alpha) = \frac{1}{4}W(2l)$$

$$Wl\sin\alpha = \frac{Wl}{2}$$

$$\Rightarrow \quad \sin\alpha = \frac{1}{2}$$

$$\Rightarrow \quad \alpha = \sin^{-1}\frac{1}{2}$$

$$= 30° \qquad \text{... (5m)}$$

7(a)(ii) the magnitude of the reaction at p. **(15)**

$$\frac{1}{4}W\sin\alpha + \mathrm{Y} = W$$

$$\Rightarrow \quad \mathrm{Y} = W - \frac{1}{4}W\sin 30°$$

$$= W - \frac{1}{4}W\left(\frac{1}{2}\right)$$

$$= W - \frac{1}{8}W$$

$$= \frac{7}{8}W \qquad \text{... (5m)}$$

$$\mathrm{X} = \frac{1}{4}W\cos\alpha$$

$$= \frac{1}{4}W\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{8}W \qquad \text{... (5m)}$$

$$\vec{R} = -\frac{\sqrt{3}}{8}W\vec{i} + \frac{7}{8}W\vec{j}$$

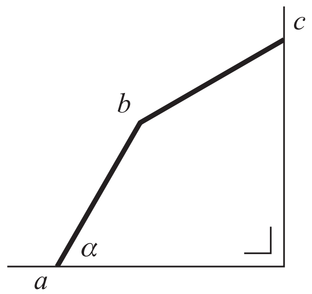
$$\Rightarrow \quad |\vec{R}| = \sqrt{\left(-\frac{\sqrt{3}}{8}W\right)^2 + \left(\frac{7}{8}W\right)^2}$$

$$\Rightarrow \quad |\vec{R}| = \sqrt{\frac{3}{64}W^2 + \frac{49}{64}W^2}$$

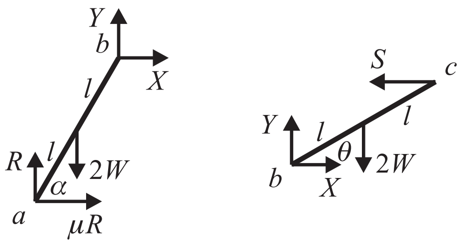
$$= \sqrt{\frac{52}{64}W^2}$$

$$= \frac{\sqrt{13}}{4}W$$ ... (5m)

7(b) Two uniform rods, [*ab*] and [*bc*], of length $2l$ and of weights W and $2W$, respectively, are smoothly jointed at b. a rests on rough horizontal ground and c rests against a smooth vertical wall. ab makes an angle α with the horizontal, where $\tan\alpha = \frac{4}{3}$, and is in a vertical plane which is perpendicular to the wall.
The rods are on the point of slipping.



7(b)(i) Show, on separate diagrams, all the forces acting on each of the rods. **(5)**



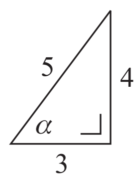
... (5m)

7(b)(ii) Find the coefficient of friction between the rod *ab* and the ground. **(25)**

$$\tan\alpha = \frac{4}{3}$$

$$\Rightarrow \quad \sin\alpha = \frac{4}{5}$$

$$\Rightarrow \quad \cos\alpha = \frac{3}{5}$$



<u>Rod *ab*:</u>

❶ $R = W + Y$

❷ $X = \mu R$ ... (5m)

Taking moments @ b:

$$R(2l\cos\alpha) = W(l\cos\alpha) + \mu R(2l\sin\alpha)$$
$$\Rightarrow\quad 2Rl\cos\alpha = Wl\cos\alpha + 2\mu Rl\sin\alpha$$
$$\Rightarrow\quad 2R = W + 2\mu R\left(\frac{\sin\alpha}{\cos\alpha}\right)$$
$$= W + 2\mu R\tan\alpha$$
$$= W + 2\mu R\left(\frac{4}{3}\right)$$
$$= W + \frac{8}{3}\mu R$$

❸ $\Rightarrow\quad 6R = 3W + 8\mu R$ ... (5m)

Rod bc:

❹ $Y = 2W$

❺ $X = S$ ... (5m)

Taking moments @ b:

$2W(l\cos\theta) = S(2l\sin\theta)$ ... (5m)

❶ $R = W + Y$
$= W + 2W$
$= 3W$

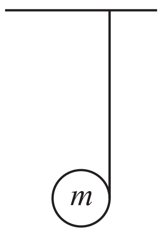
❷ $X = \mu R$
$= \mu(3W)$
$= 3\mu W$

❸
$$\Rightarrow\quad 6R = 3W + 8\mu R$$
$$\Rightarrow\quad 6(3W) = 3W + 8\mu(3W)$$
$$\Rightarrow\quad 18W = 3W + 24\mu W$$
$$\Rightarrow\quad 24\mu W = 18W - 3W$$
$$= 15W$$
$$\Rightarrow\quad 24\mu = 15$$
$$\Rightarrow\quad \mu = \frac{15}{24}$$
$$= \frac{5}{8}$$ ... (5m)

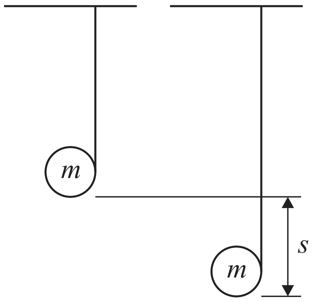
8. **(a)** Prove that the moment of inertia of a uniform circular disc, of mass m and radius r, about an axis through its centre perpendicular to its plane is $\frac{1}{2}mr^2$. **(20)**

Let	ρ	$=$	mass per unit area	
$\Rightarrow$	m	$=$	$\rho(\pi r^2)$	
		$=$	$\pi\rho r^2$	... (5m)
	Δm	$=$	$\rho(2\pi x)\Delta x$	... (5m)
	I	$=$	$\sum \Delta m r^2$	
		$=$	$\int_0^r \rho(2\pi x)\Delta x(x^2)$	
		$=$	$2\pi\rho \int_0^r x^3 dx$	... (5m)
		$=$	$(2\pi\rho)\left.\frac{x^4}{4}\right\|_0^r$	
		$=$	$(2\pi\rho)(\frac{r^4}{4} - 0)$	
		$=$	$\frac{1}{2}\pi\rho r^4$	
		$=$	$\frac{1}{2}(\pi r^2\rho)r^2$	
		$=$	$\frac{1}{2}mr^2$	... (5m)

8(b) A uniform disc, of mass m and radius r, has a light inextensible string wound around its circumference. The other end of the string is attached to a fixed point. When the system is released from rest, the disc moves vertically downwards under gravity. The disc and the string remain in a vertical plane as the disc falls vertically. The disc falls a distance s.



8(b)(i) Show that the disc reaches a linear speed of $\sqrt{\frac{4}{3}gs}$. **(20)**



P.E. + K.E. when released from rest = P.E. + K.E. when disc has fallen a distance s

$$mgs + \frac{1}{2}m(0)^2 + \frac{1}{2}I(0)^2 = mg(0) + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad \text{... (5m)}$$

But $v = r\omega$

$$\Rightarrow \omega = \frac{v}{r} \quad \text{... (5m)}$$

$$\Rightarrow mgs = \frac{1}{2}mv^2 + \frac{1}{2}(\frac{1}{2}mr^2)\left(\frac{v}{r}\right)^2 \quad \text{... (5m)}$$

$$= \frac{1}{2}mv^2 + \frac{1}{4}mr^2(\frac{v^2}{r^2})$$

$$= \frac{1}{2}mv^2 + \frac{1}{4}mv^2$$

$$= \frac{3}{4}mv^2$$

$$\Rightarrow mv^2 = \frac{4}{3}mgs$$

$$\Rightarrow v^2 = \frac{4}{3}gs$$

$$\Rightarrow v = \sqrt{\frac{4}{3}gs} \quad \text{... (5m)}$$

8(b)(ii) Find the linear acceleration of the disc. **(5)**

$$v^2 = u^2 + 2as$$

$$v = \sqrt{\frac{4}{3}gs}$$

$$u = 0$$

$$\Rightarrow \frac{4}{3}gs = 0^2 + 2as$$

$$\Rightarrow 2as = \frac{4}{3}gs$$

$$\Rightarrow a = \frac{4}{6}g$$

$$= \frac{2}{3}g \quad \text{... (5m)}$$

8(b)(iii) Find the tension in the string. **(5)**

$$F = ma$$

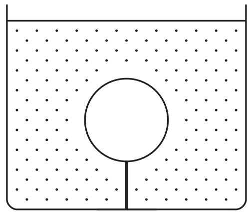
$$\Rightarrow \quad mg - T = m(\frac{2}{3}g)$$

$$\Rightarrow \quad T = mg - \frac{2}{3}mg$$

$$= \frac{1}{3}mg$$

... (5m)

9. **(a)** A uniform solid sphere, of radius 1.5 m and relative density 0.8, is held completely immersed in a liquid, of relative density 0.95, by means of a string tied to the base of the container.



Find, in terms of π and/or g,

9(a)(i) the volume of the sphere **(5)**

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi\left(\frac{3}{2}\right)^3$$
$$= \frac{4}{3}\pi\left(\frac{27}{8}\right)$$
$$= \frac{108}{24}\pi$$
$$= \frac{9}{2}\pi \text{ m}^3 \ / \ 4.5\pi \text{ m}^3$$ ... (5m)

9(a)(ii) the weight of the sphere **(5)**

$$W = \rho Vg$$
$$= (0.8 \times 1{,}000)(4.5\pi)g$$
$$= (800)(4.5)\pi g$$
$$= 3{,}600\pi g \text{ N}$$ ... (5m)

9(a)(iii) the tension in the string. **(10)**

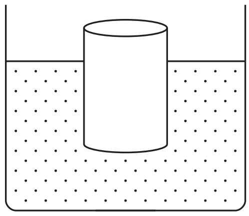
$$B = W + T$$

$$B = \rho_l Vg$$
$$= (0.95 \times 1{,}000)(4.5\pi)g$$
$$= (950)(4.5)\pi g$$
$$= 4{,}275\pi g \text{ N}$$ ... (5m)

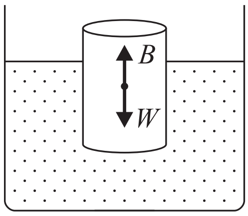
$$\Rightarrow \quad 4{,}275\pi g = 3{,}600\pi g + T$$
$$\Rightarrow \quad T = 4{,}275\pi g - 3{,}600\pi g$$
$$= 675\pi g \text{ N}$$ ... (5m)

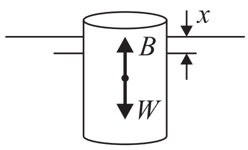
9(b) A uniform solid cylinder, of radius r and height h, floats vertically in water. The cylinder is then depressed vertically a further small distance x and released.

The relative density of the material of the cylinder is s.



9(b)(i) Show that the cylinder will perform simple harmonic motion. **(25)**





... (5m)

$$B = \rho_l V g$$
$$= (1{,}000)(\pi r^2 x)g$$
$$= 1{,}000\pi r^2 xg \quad \text{... (5m)}$$

$$F = -B$$
$$= -1{,}000\pi r^2 xg$$
$$\Rightarrow \quad ma = -1000\pi g r^2 x$$
$$\Rightarrow \quad \pi r^2 h(1{,}000s)a = -1000\pi g r^2 x \quad \text{... (5m)}$$
$$\Rightarrow \quad a = \frac{-1{,}000\pi g r^2 x}{1{,}000\pi r^2 hs}$$
$$= -\frac{g}{hs}x \quad \text{... (5m)}$$

$$a = -\omega^2 x$$

since $a \propto x$ <u>and</u> in the opposite direction

$\Rightarrow$ the cylinder will perform simple harmonic motion ... (5m)

9(b)(ii) Calculate the periodic time of the motion. **(5)**

$$T = \frac{2\pi}{\omega}$$

$$\omega = \sqrt{\frac{g}{hs}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{hs}{g}} \text{ seconds} \quad \text{... (5m)}$$

10. **(a)** If

$$x^2\frac{dy}{dx}+\frac{dy}{dx}=2xy,$$

and $y = 2$ when $x = 3$, find the value of y when $x = 7$. **(20)**

$$x^2\frac{dy}{dx}+\frac{dy}{dx} = 2xy$$

$$\Rightarrow \frac{dy}{dx}(x^2+1) = 2xy$$

$$\Rightarrow \frac{dy}{y} = \frac{2x}{x^2+1}\,dx$$

$$\Rightarrow \int_2^y \frac{dy}{y} = \int_3^7 \frac{2x}{x^2+1}\,dx \qquad \text{... (5m)}$$

$$\Rightarrow \log_e y\Big|_2^y = \log_e(x^2+1)\Big|_3^7 \qquad \text{... (5m)}$$

$$\Rightarrow \log_e y-\log_e 2 = \log_e(7^2+1)-\log_e(3^2+1)$$

$$= \log_e 50-\log_e 10 \qquad \text{... (5m)}$$

$$\Rightarrow \log_e\frac{y}{2} = \log_e\frac{50}{10}$$

$$= \log_e 5$$

$$\Rightarrow \frac{y}{2} = 5$$

$$\Rightarrow y = 10 \qquad \text{... (5m)}$$

10(b) A particle, of mass 2 kg, is projected in a straight line from a fixed point by a force which increases uniformly from zero to 20 N in 80 seconds. If s and v are the displacement and speed of the particle after t seconds,

10(b)(i) prove that the acceleration of the particle is $\frac{t}{8}$ m/s^2 **(10)**

After 80 s, the force reaches 20 N

$\Rightarrow$ After 1 s, the force reaches $\frac{20}{80}$ or $\frac{1}{4}$ N

$\Rightarrow$ After t s, the force reaches $\frac{20t}{80}$ or $\frac{t}{4}$ N ... (5m)

$$F = \frac{t}{4}$$

$$\Rightarrow ma = \frac{t}{4}$$

$$\Rightarrow 2a = \frac{t}{4}$$

$$\Rightarrow a = \frac{t}{8} \qquad \text{... (5m)}$$

10(b)(ii) show that $9s^2 = 16v^3$. **(20)**

$$a = \frac{t}{8}$$

$$\Rightarrow \frac{dv}{dt} = \frac{t}{8} \quad \text{... (5m)}$$

$$\Rightarrow 8\,dv = t\,dt$$

$$\Rightarrow \int_0^v 8\,dv = \int_0^t t\,dt$$

$$\Rightarrow 8v\Big|_0^v = \frac{t^2}{2}\Big|_0^t$$

$$\Rightarrow 8v = \frac{t^2}{2}$$

❶ $$\Rightarrow 16v = t^2 \quad \text{... (5m)}$$

$$\Rightarrow 16\frac{ds}{dt} = t^2$$

$$\Rightarrow 16\,ds = t^2\,dt$$

$$\Rightarrow \int_0^s 16\,ds = \int_0^t t^2\,dt$$

$$\Rightarrow 16s\Big|_0^s = \frac{t^3}{3}\Big|_0^t$$

$$\Rightarrow 16s = \frac{t^3}{3}$$

❷ $$\Rightarrow 48s = t^3 \quad \text{... (5m)}$$

❶ $$(t^2)^3 = (16v)^3$$
$$\Rightarrow t^6 = 4{,}096v^3$$

❷ $$\Rightarrow (t^3)^2 = (48s)^2$$
$$\Rightarrow t^6 = 2{,}304s^2$$

$$\Rightarrow 2{,}304s^2 = 4{,}096v^3$$
$$\Rightarrow 9s^2 = 16v^3 \quad \text{... (5m)}$$