M.41/42

Pre-Leaving Certificate Examination, 2009

Applied Mathematics

Marking Scheme

Ordinary	Pg. 2
Higher	Pg. 19

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Dublin Examining Board

Pre-Leaving Certificate Examination, 2009

Applied Mathematics

Ordinary Level
Marking Scheme (300 marks)

General Instructions

1. Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	S(–1)
Blunders	- mathematical errors	B(–3)
Misreading	- if not serious	M(–1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

4. Number the grid on each script 1 to 9 in numerical order, not the order of answering.

5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

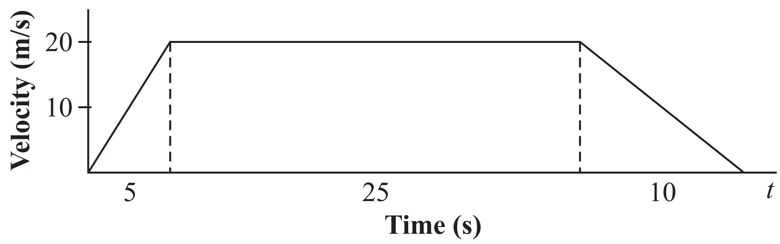
1. (a) A train decelerates uniformly from 40 m/s to 30 m/s over a distance of 1.4 km.
Find the constant deceleration of the train. **(15)**

$$v^2 = u^2 + 2as \quad \text{... (5m)}$$
$$u = 40$$
$$v = 30$$
$$s = 1{,}400$$

$$\Rightarrow (30)^2 = (40)^2 + 2a(1{,}400) \quad \text{... (5m)}$$
$$\Rightarrow 900 = 1{,}600 + 2{,}800a$$
$$\Rightarrow 2{,}800a = 900 - 1{,}600$$
$$= -700$$
$$\Rightarrow a = -\frac{700}{2{,}800}$$
$$= -\frac{1}{4} \text{ m/s}^2 \quad \text{... (5m)}$$
or $$d = \frac{1}{4} \text{ m/s}^2$$

1(b) A car travels from p to q in a straight line.
It starts from rest at p and accelerates uniformly for 5 seconds to its maximum speed of 20 m/s.
The car then moves at this constant speed for 25 seconds.
Finally it decelerates uniformly to rest at q in a further 10 seconds

(i) Draw a speed-time graph of the motion of the car from p to q. **(10)**



... (10m)

(ii) Find the uniform acceleration of the car. **(10)**

$$v = u + at$$
$$u = 0$$
$$v = 20$$
$$t = 5$$

$$\Rightarrow 20 = 0 + a(5) \quad \text{... (5m)}$$
$$\Rightarrow 5a = 20$$
$$\Rightarrow a = 4 \text{ m/s}^2 \quad \text{... (5m)}$$

1(b) (iii) Find the distance from p to q. **(10)**

$$s = \left(\frac{u+v}{2}\right)t$$

$$u_1 = 0$$
$$a_1 = 4$$
$$t_1 = 5$$
$$t_2 = 25$$
$$t_3 = 10$$

Distance accelerating

$$s_1 = \left(\frac{0+20}{2}\right)(5) = \frac{1}{2}(20)(5) = 50 \text{ m}$$

Distance at constant speed

$$s_2 = \left(\frac{20+20}{2}\right)(25) = \frac{1}{2}(40)(25) = 500 \text{ m}$$

Distance decelerating

$$s_3 = \left(\frac{20+0}{2}\right)(10) = \frac{1}{2}(20)(10) = 100 \text{ m}$$

Total distance

$$d = s_1 + s_2 + s_3$$

$$\Rightarrow d = \frac{1}{2}(20)(5) + \frac{1}{2}(40)(25) + \frac{1}{2}(20)(10)$$ ... (5m)

$$= \frac{1}{2}(100) + \frac{1}{2}(1{,}000) + \frac{1}{2}(200)$$
$$= 50 + 500 + 100$$
$$= 650 \text{ m}$$ ... (5m)

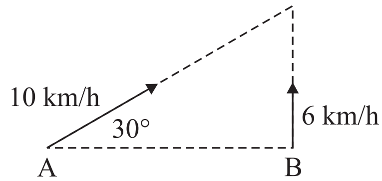
(iv) Find the average speed of the car as it travels from p to q. **(5)**

$$v = \frac{d}{t}$$

$$s = 650$$
$$t = 40$$

$$\Rightarrow v = \frac{650}{40} = 16.25 \text{ m/s}$$ ... (5m)

2. Ship A is 10 km due west of ship B.
Ship B is travelling due north at a constant speed of 6 km/h.
Ship A is travelling at a constant speed of 10 km/h in the direction 30° north of east.



(i) Express the velocity of B in terms of $\vec{i}$ and $\vec{j}$. **(5)**

$v_B = 0\vec{i} + 6\vec{j}$ … (5m)

(ii) Express the velocity of A in terms of $\vec{i}$ and $\vec{j}$. **(10)**

$v_A = 10\cos 30°\,\vec{i} + 10\sin 30°\,\vec{j}$ ... (5m)

$v_A = 10(0.866025...)\vec{i} + 10(0.5)\vec{j}$

$= 8.66\vec{i} + 5\vec{j}$ ... (5m)

(iii) Find the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$. **(10)**

$v_{AB} = v_A - v_B$

$= (8.66\vec{i} + 5\vec{j}) - (6\vec{j})$ ... (5m)

$= 8.66\vec{i} + 1\vec{j}$ ... (5m)

(iv) Find the magnitude and direction of the velocity of A relative to B. **(10)**

Magnitude

$|v_{AB}| = |8.66\vec{i} + 1\vec{j}|$

$= \sqrt{8.66^2 + 1^2}$

$= \sqrt{74.9956 + 1}$

$= \sqrt{75.9956}$

$= 8.717545...$

$= 8.72$ m/s ... (5m)

Direction

$\tan\theta = \dfrac{1}{8.66}$

$\Rightarrow \theta = \tan^{-1}(\dfrac{1}{8.66})$

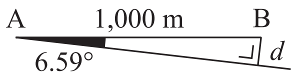
$= \tan^{-1}(0.115473...)$

$= 6.586967...°$

$\Rightarrow$ direction $= 6.59°$ south of west ... (5m)

2(v) Find the shortest distance between the ships. **(15)**

Diagram



... (5m)

Closest distance

d	=	1,000(sin 6.59°)	... (5m)
	=	1,000(0.114763...)	
	=	114.763...	
	=	114.7 m	... (5m)

3. A particle is fired horizontally with an initial speed of 50 m/s from the top of a vertical cliff of 180 m

(i) What is the initial velocity of the particle in terms of $\vec{i}$ and $\vec{j}$? **(5)**

$$\vec{v} = 50\vec{i} + 0\vec{j} \qquad \text{… (5m)}$$

(ii) Find the time it takes the particle to strike the ground. **(15)**

$$s_y = u_y t + \tfrac{1}{2}a_y t^2$$

$$s_y = -180$$

$$u_y = 0$$

$$a_y = g = -10$$

$$\Rightarrow -180 = 0(t) + \frac{1}{2}(-10)t^2 \qquad \text{... (5 + 5m)}$$

$$\Rightarrow -180 = -5t^2$$

$$\Rightarrow 5t^2 = 180$$

$$\Rightarrow t^2 = 36$$

$$\Rightarrow t = 6 \text{ s} \qquad \text{... (5m)}$$

(iii) How far from the foot of the cliff does the particle strike the ground? **(10)**

$$s_x = u_x t$$

$$u_x = 50$$

$$t = 6$$

$$\Rightarrow s_x = 50(6) \qquad \text{... (5m)}$$

$$= 300 \text{ m} \qquad \text{... (5m)}$$

(iv) Find the speed at which the particle strikes the ground. **(20)**

$$v_x = 50 \qquad \text{... (5m)}$$

$$v_y = u_y + a_y t$$

$$u_y = 0$$

$$a_y = -10$$

$$\Rightarrow v_y = 0 - 10t$$

$$= -10t \qquad \text{... (5m)}$$

Particle strikes the ground at $t = 6$ s

$$\Rightarrow v_y = -10(6)$$

$$= -60 \qquad \text{... (5m)}$$

$$\vec{v} = v_x + v_y$$

$$= 50\vec{i} - 60\vec{j}$$

$$\Rightarrow |\vec{v}| = \sqrt{50^2 + (-60)^2}$$

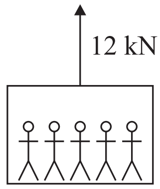
$$= \sqrt{2{,}500 + 3{,}600}$$

$$= \sqrt{6{,}100}$$

$$= 78.102496...$$

$$\cong 78.1 \text{ m/s} \qquad \text{... (5m)}$$

4. **(a)** A lift has a mass of 600 kg. In the lift, there are 5 people, each of mass 80 kg. When the lift is raised, the tension in the cable is 12 kN.



Find: **(i)** the acceleration of the lift **(10)**

$$W = mg$$
$$= (600 + 5(80))(10)$$
$$= (1{,}000)(10)$$
$$= 10{,}000 \text{ N} \qquad \text{... (5m)}$$

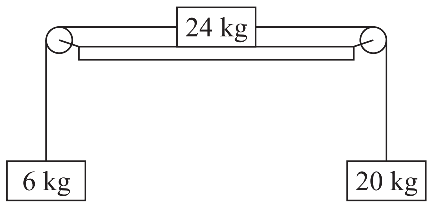
$$T = 12{,}000 \text{ N}$$

$$T - W = ma$$
$$\Rightarrow 12{,}000 - 10{,}000 = 1{,}000(a)$$
$$\Rightarrow 1{,}000a = 2{,}000$$
$$\Rightarrow a = 2 \text{ m/s}^2 \qquad \text{... (5m)}$$

(ii) the reaction between each person and the floor of the lift. **(10)**

$$F = ma \qquad \text{... (5m)}$$
$$\Rightarrow R - W = ma$$
$$\Rightarrow R - 80g = 80(2)$$
$$\Rightarrow R - 800 = 160$$
$$\Rightarrow R = 160 + 800$$
$$\Rightarrow R = 960 \text{ N} \qquad \text{... (5m)}$$

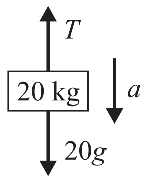
4(b) Two particles, of masses 6 kg and 20 kg, are connected by light, taut, inextensible strings passing over smooth light pulleys at the edges of a rough horizontal table to a third particle of mass 24 kg resting on the table.
The coefficient of friction between the 24 kg mass and the table is μ.



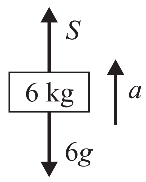
The system is released from rest.

(i) Show that if $\mu = \frac{1}{4}$, the common acceleration of the masses is 1.6 m/s^2. **(20)**

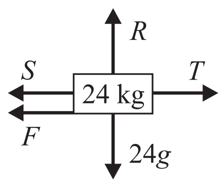
$$200 - T = 20a$$ ... (5m)



$$S - 60 = 6a$$ ... (5m)



$$\begin{aligned} R &= 24g \\ &= 24(10) \\ &= 240 \text{ N} \end{aligned}$$



$$\begin{aligned} T - S - F &= ma \\ \Rightarrow \quad T - S - \mu R &= 24a \\ \Rightarrow \quad T - S - 240\mu &= 24a \\ \Rightarrow \quad T - S - 240(\tfrac{1}{4}) &= 24a \\ \Rightarrow \quad T - S - 60 &= 24a \end{aligned}$$ ... (5m)

<u>Common acceleration</u>

$$\begin{aligned} 200 - T &= 20a \\ -60 + \quad + S &= 6a \\ -60 + T - S &= 24a \\ \hline \Rightarrow \quad 200 - 60 - 60 &= 50a \\ \Rightarrow \quad 50a &= 80 \\ \Rightarrow \quad a &= 1.6 \text{ m/s}^2 \end{aligned}$$ ... (5m)

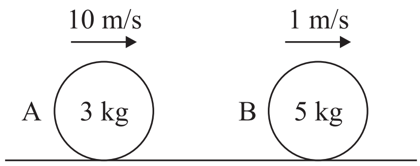
(ii) What is the least value of μ for which the masses will not move? **(10)**

$$\text{Let } a = 0$$

$$\begin{aligned} 200 - T &= 20(0) \\ -60 + \quad + S &= 6(0) \\ T - S - 240\mu &= 24(0) \\ \hline \Rightarrow \quad 140 \quad -240\mu &= 0 \end{aligned}$$ ... (5m)

$$\begin{aligned} \Rightarrow \quad -240\mu &= -140 \\ \Rightarrow \quad 240\mu &= 140 \\ \Rightarrow \quad \mu &= \frac{140}{240} \text{ or } \frac{7}{12} \end{aligned}$$ ... (5m)

5. A smooth sphere A, of mass 3 kg, collides directly with another smooth sphere B, of mass 5 kg, on a smooth horizontal table.



A and B are moving in the same direction with speeds of 10 m/s and 1 m/s, respectively.

The coefficient of restitution for the collision is e. As a result of the collision, A is brought to rest.

Find:

(i) the speed of B after the collision **(15)**

Sphere	Mass	Speed before	Speed after
A	3 kg	10	0
B	5 kg	1	q

... (5m)

$m_A u_A + m_B u_B = m_A v_A + m_A v_A$

$\Rightarrow 3(10) + 5(1) = 3(0) + 5(q)$ ... (5m)

$\Rightarrow 35 = 5q$

$\Rightarrow q = 7$ m/s … (5m)

(ii) the value of e **(10)**

$-e = \dfrac{v_1 - v_2}{u_1 - u_2}$

$\Rightarrow -e = \dfrac{0 - q}{10 - 1}$ ... (5m)

$= \dfrac{0 - 7}{10 - 1}$

$= \dfrac{7}{9}$ ... (5m)

(iii) the loss in kinetic energy due to the collision **(15)**

<u>KE before collision</u>

$K.E._{before} = \frac{1}{2}(3)(10)^2 + \frac{1}{2}(5)(1)^2$

$= \frac{1}{2}(300) + \frac{1}{2}(5)$

$= 152.5$ J ... (5m)

<u>KE after collision</u>

$K.E._{after} = \frac{1}{2}(3)(0)^2 + \frac{1}{2}(5)(7)^2$

$= \frac{1}{2}(5)(49)$

$= 122.5$ J ... (5m)

$\Rightarrow K.E._{loss} = K.E._{before} - K.E._{after}$

$= 152.5 - 122.5$

$= 30$ J ... (5m)

5(iv) the magnitude of the impulse imparted to A and B due to the collision. **(10)**

$$I_A = m_A v_A + m_A u_A$$
$$= 3(0) - 3(10)$$
$$= -30 \text{ N s} \quad \text{... (5m)}$$

$$I_B = m_B v_B + m_B u_B$$
$$= 5(7) - 5(1)$$
$$= 35 - 5$$
$$= 30 \text{ N s} \quad \text{... (5m)}$$

6. **(a)** Particles of weight 4 N, 3 N, 2 N and x N are placed at the points (4, 1), (2, 2), (11, −11) and (6, 2), respectively.
The co-ordinates of the centre of gravity of the system are (5.5, y).

Find **(i)** the value of x **(10)**

4 N @ (4, 1), 3 N @ (2, 2), 2 N @ (11, –11), x N @ (6, 2)

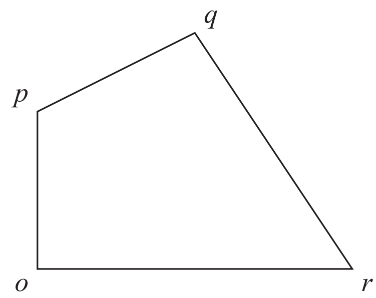
$\Rightarrow$ $4(4) + 3(2) + 2(11) + x(6) = (4 + 3 + 2 + x)(5.5)$ ... (5m)
$\Rightarrow$ $16 + 6 + 22 + 6x = 49.5 + 5.5x$
$\Rightarrow$ $0.5x = 5.5$
$\Rightarrow$ $x = 11$ ... (5m)

(ii) the value of y. **(10)**

$\Rightarrow$ $4(1)+3(2)+2(–11)+11(2) = (4 + 3 + 2 + 11)(y)$ ... (5m)
$\Rightarrow$ $4 + 6 + (–22) + 22 = 20y$
$\Rightarrow$ $10 = 20y$
$\Rightarrow$ $y = \frac{1}{2}$ ... (5m)

6(b) A quadrilateral lamina has vertices o, p, q and r.

The co-ordinates of the vertices are $o(0, 0)$, $p(0, 12)$, $q(18, 18)$ and $r(30, 0)$.



Find **(i)** the areas of triangles opq and oqr **(10)**

Area $\Delta opq = \frac{1}{2}|(0)(18) – (12)(18)|$
$= \frac{1}{2}|216|$
$= 108$ units^2 ... (5m)

Area $\Delta oqr = \frac{1}{2}|(18)(0) – (30)(18)|$
$= \frac{1}{2}|540|$
$= 270$ units^2 ... (5m)

(ii) the centres of gravity of triangles opq and oqr **(10)**

$g_1 = \left(\frac{0 + 0 + 18}{3}, \frac{0 + 12 + 18}{3}\right)$
$\Rightarrow$ $cog\ \Delta opq = (6, 10)$ ... (5m)

$g_2 = \left(\frac{0 + 18 + 30}{3}, \frac{0 + 18 + 0}{3}\right)$
$\Rightarrow$ $cog\ \Delta oqr = (16, 6)$ ... (5m)

6(b) (iii) the co-ordinates of the centre of gravity of the lamina. **(10)**

	108 @ (6, 10) + 270 @ (16, 6)	=	(108 + 270) @ (x, y)	
$\Rightarrow$	108(6) + 270(16)	=	378(x)	
$\Rightarrow$	640 + 4,320	=	378x	
$\Rightarrow$	4,968	=	378x	
$\Rightarrow$	x	=	$\frac{4{,}968}{378}$	
		=	13.142857...	
		$\cong$	13.14	... (5m)
$\Rightarrow$	108(10) + 270(6)	=	378(y)	
$\Rightarrow$	1,080 + 1,620	=	378y	
$\Rightarrow$	2,700	=	378y	
$\Rightarrow$	y	=	$\frac{2{,}700}{378}$	
		=	7.142857...	
		$\cong$	7.14	... (5m)
$\Rightarrow$	*cog* Δ*opqr*	=	(13.14, 7.14)	

7. **(a)** In your answer book, write down the missing word in **each** of the following sentences: **(15)**

(i) **Friction** is the force between two rough surfaces in contact which tends to oppose sliding. … (5m)

(ii) The ratio of the limiting friction and the normal reaction is called the **coefficient** of friction. … (5m)

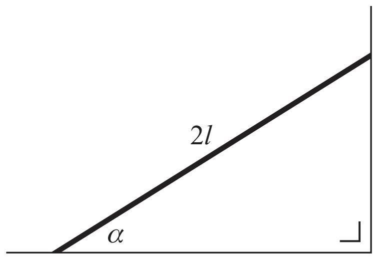
(iii) The measure of the turning effect of the force is called the **moment** of the force. ... (5m)

7(b) A uniform ladder, of mass 25 kg and length $2l$, rests on rough horizontal ground and leans against a smooth vertical wall.

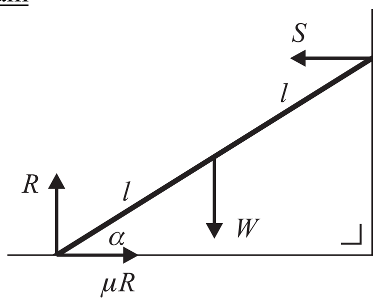
The coefficient of friction between the ladder and the ground is 0.8.

The ladder makes an angle of α with the floor, as shown.

If the ladder is on the point of slipping, find the value of $\tan\alpha$. **(35)**



Diagram



... (5m + 5m)

$$R = W = mg = (25)(10) = 250 \text{ N}$$ ... (5m)

$$S = F = \mu R = (0.8)(250) = 200$$ ... (5m)

Taking moments about the point of contact with the horizontal ground:

$$250l\cos\alpha = S(2l\sin\alpha)$$

$$\Rightarrow 250\cos\alpha = (200)(2)\sin\alpha = 400\sin\alpha$$ ... (5m + 5m)

$$\Rightarrow \frac{\sin\alpha}{\cos\alpha} = \frac{250}{400}$$

$$\Rightarrow \tan\alpha = \frac{5}{8}$$ ... (5m)

8. A smooth particle of mass 4 kg is attached to one end of a light inextensible string, of length 50 cm.
The particle describes a circle with constant angular velocity of 3 radians per second on a smooth horizontal table.
The other end of the string is connected to a fixed point on the table.

Find **(i)** the tension in the string. **(10)**

$$T = m\omega^2 r$$
$$= 4(3)^2(\tfrac{1}{2}) \quad \text{... (5m)}$$
$$= 18 \text{ N} \quad \text{... (5m)}$$

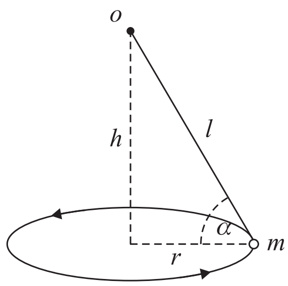
(ii) the number of revolutions the particle completes every minute. **(10)**

$$\omega = 3 \text{ rad / sec}$$
$$= 180 \text{ rad / minute} \quad \text{... (5m)}$$

2π rads in a full circle

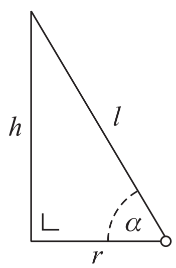
$$\Rightarrow \text{revolutions} = \frac{180}{2\pi}$$
$$= \frac{90}{\pi}$$
$$= 28.647889...$$
$$= 28.6 \text{ revolutions} \quad \text{... (5m)}$$

8(b) A smooth particle of mass m kg describes a horizontal circle of radius r m on a smooth horizontal table with constant angular velocity ω radians per second.
The particle is connected by means of a light inelastic string to a fixed point o which is h m vertically above the centre of the circle.
The length of the string is l m and it makes an angle α with the horizontal.

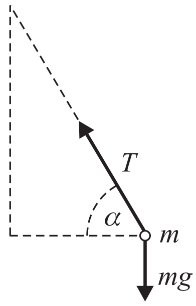


(i) Find $\sin\alpha$ and $\cos\alpha$ in terms of l, h and r. **(5)**

$$\sin\alpha = \frac{h}{l}$$
$$\cos\alpha = \frac{r}{l} \quad \text{... (5m)}$$



8(b) (ii) Show on a diagram all the forces acting on the particle. **(5)**



... (5m)

(iii) Show that $h = \dfrac{10}{\omega^2}$. **(15)**

$$T\sin\alpha = 10m$$
$$\Rightarrow T\left(\frac{h}{l}\right) = 10m$$
$$\Rightarrow T = \frac{10\ ml}{h} \quad \text{... (5m)}$$

$$T\cos\alpha = m\omega^2 r \quad \text{... (5m)}$$
$$\Rightarrow T\left(\frac{r}{l}\right) = m\omega^2 r$$
$$\Rightarrow \left(\frac{10ml}{h}\right)\left(\frac{r}{l}\right) = m\omega^2 r$$
$$\Rightarrow \frac{10}{h} = \omega^2$$
$$\Rightarrow 10 = \omega^2 h$$
$$\Rightarrow h = \frac{10}{\omega^2} \quad \text{... (5m)}$$

(iv) If the horizontal circle must remain at least 1.6 metres below *o*, find the maximum value of ω. **(5)**

$$h \geq 1.6$$
$$\Rightarrow \frac{10}{\omega^2} \geq 1.6$$
$$\Rightarrow 10 \geq 1.6\omega^2$$
$$\Rightarrow \omega^2 \leq \frac{10}{1.6}$$
$$\leq 6.25$$
$$\Rightarrow \omega \leq \sqrt{6.25}$$
$$\leq 2.5$$

$$\Rightarrow \text{max. } \omega = 2.5 \text{ rad/s} \quad \text{... (5m)}$$

9. **(a)** State the Law of Archimedes. **(10)**

– when a body is wholly or partly immersed in a liquid, ... (5m)
– it suffers an upthrust or buoyancy equal in magnitude to the weight of the liquid displaced ... (5m)

A piece of metal which has a relative density of 6 weighs 30 N in air.

(i) How much does it weigh in water? **(10)**

$$B_w = \frac{W}{s} = \frac{30}{6} = 5 \quad \text{... (5m)}$$

Apparent weight

$$= 30 - 5 = 25 \text{ N} \quad \text{... (5m)}$$

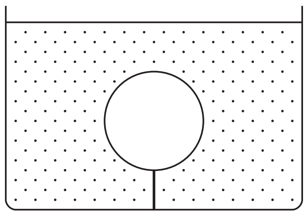
(ii) How much does it weigh in a liquid which has a relative density of 0.9? **(10)**

$$B_l = \frac{s_l W}{s} = \frac{0.9(30)}{6} = 4.5 \quad \text{... (5m)}$$

Apparent weight

$$= 30 - 4.5 = 25.5 \text{ N} \quad \text{... (5m)}$$

9(b) A sphere has a radius of 1.5 m. The relative density of the sphere is 0.8 and it is completely immersed in a liquid of relative density 0.85. The sphere is held at rest by a light inelastic string which is attached to the base of the tank.



(i) Find the weight of the sphere. **(10)**

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}(3.14...)(1.5)^3 = 14.137166... \cong 14.14 \text{ m}^3 \quad \text{... (5m)}$$

$$W = \rho V g = (0.80)(14.14)(10) = 113.12 \text{ N} \quad \text{... (5m)}$$

9(b) (ii) Find the tension in the string. **(10)**

$$B = W + T$$

$$B = \rho V g$$
$$= (0.85)(14.14)(10)$$
$$= 120.19 \text{ N}$$ ... (5m)

$$\Rightarrow 120.19 = 113.12 + T$$
$$\Rightarrow T = 120.19 - 113.12$$
$$= 7.07 \text{ N}$$ ... (5m)

Dublin Examining Board

Pre-Leaving Certificate Examination, 2009

Applied Mathematics

Higher Level
Marking Scheme (300 marks)

General Instructions

1. Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	S(–1)
Blunders	- mathematical errors	B(–3)
Misreading	- if not serious	M(–1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.

4. Number the grid on each script 1 to 10 in numerical order, not the order of answering.

5. Scrutinise **all** pages of the answer book.

6. The marking scheme shows one correct solution to each question. In many cases there are other equally valid methods.

Six questions to be answered. All questions carry equal marks. (6 × 50m)

1. **(a)** A particle is projected vertically upwards with an initial speed 14.7 m/s from level ground.

(i) Find the two times at which the particle is 9.8 m above ground level. **(10)**

$$s = ut + \tfrac{1}{2}at^2$$
$$u = 14.7$$
$$a = -9.8$$
$$s = 9.8$$

$\Rightarrow \quad 9.8 = (14.7)t + \tfrac{1}{2}(-9.8)t^2$ ... (5m)

$= 14.7t - 4.9t^2$

$\Rightarrow \quad 4.9t^2 - 14.7t + 9.8 = 0$

$\Rightarrow \quad t^2 - 3t + 2 = 0$

$\Rightarrow \quad (t-2)(t-1) = 0$

$\Rightarrow \quad t = 2,\ 1$ ... (5m)

(ii) How far does the particle travel between these two times? **(10)**

particle reaches maximum height @ $t = 1.5$

$$s = ut + \tfrac{1}{2}at^2$$
$$u = 14.7$$
$$a = -9.8$$
$$t = 1.5$$

$\Rightarrow \quad s = (14.7)(1.5) + \tfrac{1}{2}(-9.8)(1.5)^2$

$= 22.05 - 4.9(2.25)$

$= 22.05 - 11.025$

$= 11.025$ m/s ... (5m)

<u>Distance that the particle rises above 9.8 m</u>

$d_1 = 11.025 - 9.8$

$= 1.225$ m

<u>Distance that the particle falls from above 9.8 m</u>

$d_2 = 11.025 - 9.8$

$= 1.225$ m

<u>Total distance travelled</u>

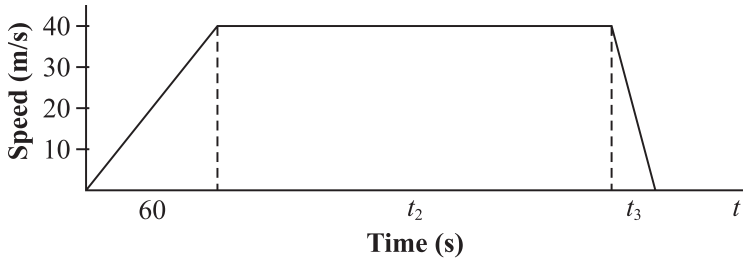
$d_{\text{total}} = d_1 + d_2$

$= 1.225 + 1.225$

$= 2.45$ m ... (5m)

1(b) A time trial in a rally race covers a distance of 8.8 km along a horizontal straight road. The winning car accelerates uniformly in the first minute to reach its maximum speed of 40 m/s. It continues at this constant speed and then brakes uniformly to rest, the magnitude of the deceleration being three times that of the initial acceleration.

(i) Draw a speed-time graph and find the distance travelled by the rally car at its maximum speed. **(15)**



... (5m)

<u>1st stage</u>

$v_1 = u_1 + a_1 t_1$

$v_1 = 40$

$u_1 = 0$

$t_1 = 60$

$\Rightarrow 40 = 0 + 60(a)$

$\Rightarrow a = \dfrac{40}{60}$

$= \dfrac{2}{3}$ m/s^2

<u>3rd stage</u>

$a_3 = 3|a_1|$

$\Rightarrow a_3 = 3(\frac{2}{3})$

$= 2$ m/s^2

$v_3 = u_3 + a_3 t_3$

$v_3 = 0$

$u_3 = 40$

$a_3 = -2$

$\Rightarrow 0 = 40 + (-2)t_3$

$\Rightarrow 2t_3 = 40$

$\Rightarrow t_3 = 20$ s ... (5m)

<u>Total distance travelled</u>

8.8 km = 8800 m

$\Rightarrow$ Area = 8,800

$\Rightarrow \frac{1}{2}(60)(40) + 40t_2 + \frac{1}{2}(20)(40) = 8{,}800$

$\Rightarrow 1{,}200 + 40t_2 + 400 = 8{,}800$

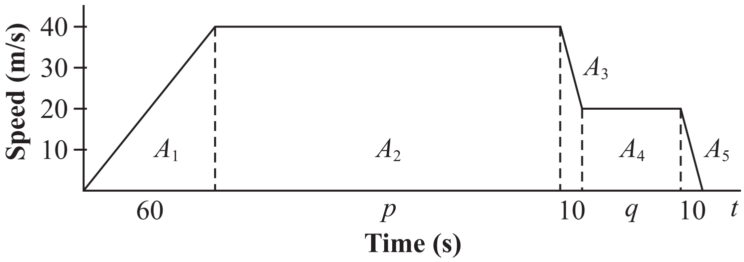
$\Rightarrow 40t_2 = 8800 - 1{,}200 - 400$

$\Rightarrow 40t_2 = 7{,}200$

$\Rightarrow t_2 = \dfrac{7{,}200}{40}$

$= 180$ s ... (5m)

(ii) If the maximum speed over the final kilometre of the race is restricted to 20 m/s, show that the time taken would be 22.5 seconds longer, assuming that the rates of acceleration and deceleration are unchanged. **(15)**



... (5m)

$$A_1 = \frac{1}{2}(60)(40) = 1200$$

$$A_3 = \frac{1}{2}(40+20)(20) = 300$$

$$A_5 = \frac{1}{2}(10)(20) = 100$$

$$\Rightarrow A_4 = 1{,}000 - 100 = 900$$

$$\Rightarrow A_2 = 8{,}800 - 1{,}200 - 300 - 900 - 100 = 8{,}800 - 2{,}500 = 6{,}300$$ ... (5m)

$$\Rightarrow 20q = A_4 = 900$$

$$\Rightarrow q = \frac{900}{20} = 45 \text{ s}$$

$$\Rightarrow 40p = A_2 = 6{,}300$$

$$\Rightarrow p = \frac{6{,}300}{40} = 157.5 \text{ s}$$

Original time

$$t_{\text{original}} = t_1 + t_2 + t_3 = 60 + 180 + 20 = 260 \text{ s}$$

New time

$$t_{\text{new}} = t_1 + t_2 + t_3 + t_4 + t_5 = 60 + 157.5 + 10 + 45 + 10 = 282.5 \text{ s}$$

Additional time taken

$$t_{\text{additional}} = t_{\text{new}} - t_{\text{original}} = 282.5 - 260 = 22.5 \text{ s}$$ ... (5m)

2. **(a)** A man is cycling in the direction 30° north of east at 10 m/s. A woman is running north at 3 m/s. At a certain instant, the man is 100 m west of the woman.

(i) Find the velocity of the man relative to the woman. **(5)**

$$v_m = 10\cos 30°\vec{i} + 10\sin 30°\vec{j}$$
$$= 8.66\vec{i} + 5\vec{j} \text{ m/s } \underline{\text{or}}\ 5\sqrt{3}\,\vec{i} + 5\vec{j}$$

$$v_w = 3\vec{j}$$

$$\Rightarrow \quad v_{mw} = v_m - v_w$$
$$= 8.66\vec{i} + 5\vec{j} - 3\vec{j}$$
$$= 8.66\vec{i} + 2\vec{j} \text{ m/s} \qquad \text{... (5m)}$$

(ii) Find the magnitude and direction of the velocity of the man relative to the woman.
Give your answer to the nearest m/s and the nearest degree, respectively. **(10)**

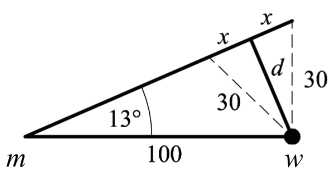
<u>Magnitude</u>

$$|v_{mw}| = |8.66\vec{i} + 2\vec{j}|$$
$$= \sqrt{8.66^2 + 2^2}$$
$$= \sqrt{75 + 4}$$
$$= \sqrt{79}$$
$$= 8.888194$$
$$\cong 8.89 \text{ m/s} \qquad \text{... (5m)}$$

<u>Direction</u>

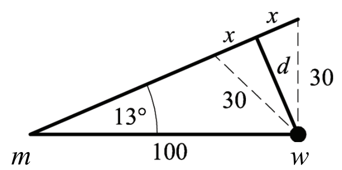
$$\tan\theta = \frac{2}{8.66}$$
$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2}{8.66}\right)$$
$$= \tan^{-1}(0.230946...)$$
$$= 13.004280...°$$
$$\cong 13° \qquad \text{... (5m)}$$

(iii) Find the shortest distance between the man and woman in the subsequent motion. **(5)**



$$\Rightarrow \quad d_{\text{shortest}} = 100\sin 13°$$
$$= 22.495105...$$
$$\cong 22.5 \text{ m} \qquad \text{... (5m)}$$

(iv) Calculate the length of time for which the man and woman are within 30 m of one another. **(5)**

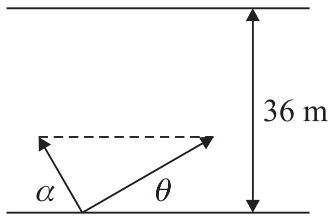


$$
\begin{aligned}
& x^2 + (22.5)^2 &=& \quad 30^2 \\
\Rightarrow\quad & x^2 + 506.25 &=& \quad 900 \\
\Rightarrow\quad & x^2 + &=& \quad 900 - 506.25 \\
& &=& \quad 393.75 \\
\Rightarrow\quad & x &=& \quad \sqrt{393.75} \\
& &=& \quad 19.843134... \\
& &\cong& \quad 19.84
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & 2x &=& \quad 2(19.84) \\
& &=& \quad 39.68 \\
\Rightarrow\quad & \text{Time} &=& \quad \frac{\text{Distance}}{\text{Speed}} \\
& &=& \quad \frac{39.68}{8.89} \\
& &=& \quad 4.463442... \\
& &=& \quad 4.5 \text{ s}
\end{aligned}
$$

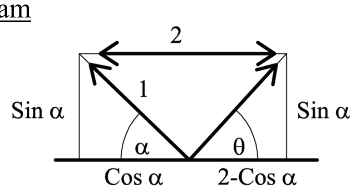
... (5m)

2(b) A woman can swim at 1 m/s in still water. She swims across a river of width 36 m. She sets out at an angle of α to the bank. The river flows with a constant speed of 2 m/s parallel to the straight banks. If the direction of her actual velocity is θ to the bank, show that



$$\tan\theta = \frac{\sin\alpha}{2 - \cos\alpha}$$ **(10)**

Diagram



... (5m)

$$
\begin{aligned}
\tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\
&= \frac{\sin\alpha}{2 - \cos\alpha}
\end{aligned}
$$

... (5m)

2(b) Prove that tan θ has a maximum value when α is 60°. **(10)**

Maximum value of $\tan\theta$

$$\Rightarrow \quad \frac{d(\tan\theta)}{d\alpha} = \frac{(2-\cos\alpha)(\cos\alpha)-(\sin\alpha)(\sin\alpha)}{(2-\cos\alpha)^2}$$

$$= 0 \qquad \text{... (5m)}$$

$$\Rightarrow \quad (2-\cos\alpha)(\cos\alpha)-(\sin\alpha)(\sin\alpha) = 0$$

$$\Rightarrow \quad 2\cos\alpha - \cos^2\alpha - \sin^2\alpha = 0$$

$$\Rightarrow \quad 2\cos\alpha = \sin^2\alpha + \cos^2\alpha$$

$$= 1$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2}$$

$$\Rightarrow \quad \alpha = 60° \qquad \text{... (5m)}$$

Hence, show that the time taken for the woman to cross the river by the shortest path is $24\sqrt{3}$ seconds. **(5)**

Shortest path

$$\Rightarrow \quad \theta = \text{max.}$$

$$\Rightarrow \quad \tan\theta = \text{max.}$$

$$\Rightarrow \quad \alpha = 60°$$

Each second the woman crosses a distance of

$$= \sin\alpha$$

$$= \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \text{Time} = \frac{36}{\frac{\sqrt{3}}{2}}$$

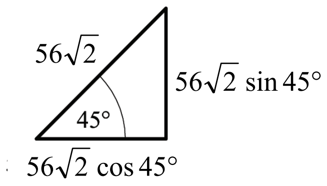
$$= \frac{(36)(2)}{\sqrt{3}}$$

$$= \left(\frac{72}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$= 24\sqrt{3}\ \text{s} \qquad \text{... (5m)}$$

3. **(a)** A particle is projected with a speed of $56\sqrt{2}$ m/s at an angle of 45° to the horizontal plane.

Find **(i)** the maximum height of the particle above the plane; **(15)**



$$\vec{u} = 56\sqrt{2}\cos 45°\vec{i} + 56\sqrt{2}\sin 45°\vec{j}$$
$$= 56(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right)\vec{i} + 56(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right)\vec{j}$$
$$= 56\vec{i} + 56\vec{j} \quad \text{... (5m)}$$

In the $\vec{j}$ direction

$$v^2 = u^2 + 2as$$
$$u = 56$$
$$a = -9.8$$
$$v = 0$$

$$\Rightarrow 0 = (56)^2 + 2(-9.8)s$$
$$= 3{,}136 - 19.6s$$
$$\Rightarrow 19.6s = 3{,}136$$
$$\Rightarrow s = \frac{3{,}136}{19.6}$$
$$= 160 \text{ m} \quad \text{... (10m)}$$

(ii) the speed of the particle after 4 seconds. **(10)**

$$v_x = 56\vec{i}$$
$$\Rightarrow v_y = (u_y + a_y t)\vec{j}$$
$$= (56 - 9.8(4))\vec{j}$$
$$= (56 - 39.2)\vec{j}$$
$$= 16.8\vec{j}$$

$$\Rightarrow \text{Speed} = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{56^2 + 16.8^2}$$
$$= \sqrt{3{,}136 + 282.24}$$
$$= \sqrt{3{,}418.24}$$
$$= 58.465716\ldots$$
$$\cong 58.5 \text{ m/s} \quad \text{... (10m)}$$

3(b) A particle is projected down an inclined plane with initial speed u m/s. The line of projection makes an angle of α with the inclined plane and the plane is inclined at θ to the horizontal.

(i) Find, in terms of u, g, α and θ, the range of the particle on the inclined plane. **(15)**

$$s_x = u\cos\alpha t - \frac{1}{2}g\sin\theta\, t^2$$

$$s_y = u\sin\alpha t - \frac{1}{2}g\cos\theta\, t^2 \quad \text{... (5m)}$$

$$s_y = 0$$

$$\Rightarrow \quad t(u\sin\alpha - \frac{1}{2}gt\cos\theta) = 0$$

$$\Rightarrow \quad u\sin\alpha - \frac{1}{2}gt\cos\theta = 0$$

$$\Rightarrow \quad \frac{1}{2}gt\cos\theta = u\sin\alpha$$

$$\Rightarrow \quad t = \frac{2u\sin\alpha}{g\cos\theta} \quad \text{... (5m)}$$

$$\text{Range} = s_x$$

$$= u\cos\alpha\left(\frac{2u\sin\alpha}{g\cos\theta}\right) - \frac{1}{2}g\sin\theta\left(\frac{4u^2\sin^2\alpha}{g^2\cos^2\theta}\right)$$

$$= \frac{2u^2\sin\alpha\cos\alpha}{g\cos\theta} - \frac{2u^2\sin\theta\sin^2\alpha}{g\cos^2\theta}$$

$$= \frac{2u^2\sin\alpha\cos\alpha\cos\theta - 2u^2\sin\theta\sin^2\alpha}{g\cos^2\theta}$$

$$= \frac{2u^2\sin\alpha(\cos\alpha\cos\theta - \sin\theta\sin\alpha)}{g\cos^2\theta}$$

$$= \frac{2u^2\sin\alpha(\cos(\alpha - \theta))}{g\cos^2\theta}$$

$$= \frac{2u^2\sin\alpha\cos(\alpha - \theta)}{g\cos^2\theta} \quad \text{... (5m)}$$

(ii) Hence, find the maximum range and α of the particle when θ is 10°. **(10)**

$$R = \frac{2u^2 \sin\alpha \cos(\alpha - \theta)}{g\cos^2\theta}$$

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$\Rightarrow \quad R = \frac{u^2}{g\cos^2\theta}[\sin(\alpha + \alpha - \theta) + \sin(\alpha - \alpha + \theta)]$$

$$= \frac{u^2}{g\cos^2\theta}[\sin(2\alpha - \theta) + \sin\theta]$$

$$R_{max} = \frac{u^2}{g\cos^2\theta}[1 + \sin\theta]$$

$$= \frac{u^2(1 + \sin\theta)}{g(1 - \sin^2\theta)}$$

$$= \frac{u^2(1 + \sin\theta)}{g(1 - \sin\theta)(1 + \sin\theta)}$$

$$= \frac{u^2}{g(1 - \sin\theta)} \qquad \text{... (5m)}$$

Max. range occurs when:

$$\sin(2\alpha - \theta) = 1$$

$$\Rightarrow \quad 2\alpha - \theta = \sin^{-1} 1$$

$$= 90°$$

If $\theta = 10°$,

then $2\alpha - 10° = 90°$

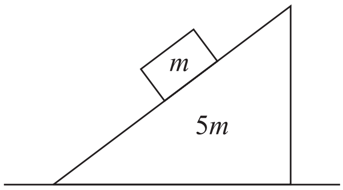
$$2\alpha = 90° + 10°$$

$$= 100°$$

$$\Rightarrow \quad \alpha = 50° \qquad \text{... (5m)}$$

4. **(a)** A smooth wedge of mass $5m$ rests on a smooth horizontal table. The inclined face has a slope of $\frac{3}{4}$ to the table.

A particle of mass m is placed on the smooth inclined face of the wedge and the system is released from rest.

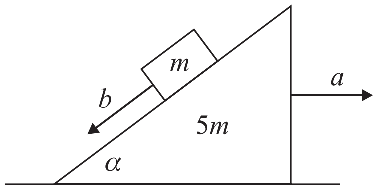


Find the acceleration of the wedge. **(30)**

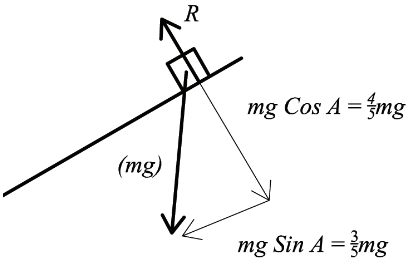
$$\tan\alpha = \frac{3}{4}$$

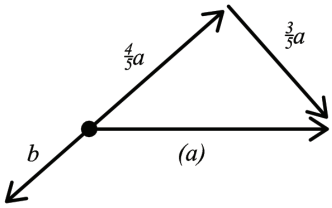
$$\Rightarrow \quad \sin\alpha = \frac{3}{5}$$

$$\Rightarrow \quad \cos\alpha = \frac{4}{5}$$



<u>Particle</u>



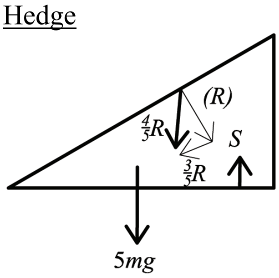


... (5m)

☩ $\frac{3}{5}mg = m(b - \frac{4}{5}a)$

☩ $\frac{4}{5}mg - R = m(\frac{3}{5}a)$ ... (5m)

<u>Hedge</u>



... (5m)

$$\Rightarrow \quad F = ma$$

⊙-∘ $\frac{3}{5}R = 5ma$ ... (5m)

☩ $\frac{4}{5}mg - R = m(\frac{3}{5}a)$ ... (5m)

$$\Rightarrow \quad R = \frac{4}{5}mg - \frac{3}{5}ma$$

Substituting into ⊙-∘

$$\frac{3}{5}(\frac{4}{5}mg - \frac{3}{5}ma) = 5ma$$

$$\Rightarrow \quad \frac{12}{25}mg - \frac{9}{25}ma = 5ma$$

$$\Rightarrow \quad 12mg - 9ma = 125ma$$

$$\Rightarrow \quad 134ma = 12mg$$

$$\Rightarrow \quad a = \frac{12}{134}g \text{ or } \frac{6g}{67}$$ ... (5m)

(b) Two particles of masses, 1 kg and 2 kg, are attached to the ends of a light inextensible string 6 m long which passes over a fixed smooth light pulley. The pulley is 3.5 metres directly above the 2 kg mass which rests on a horizontal table. The 1 kg mass is held next to the pulley and is released from rest. It falls vertically 2.5 m before the string becomes taut

(i) Show that the 2 kg mass rises from the table with a speed of $\frac{7}{3}$ m/s **(10)**

1 kg particle

$$v^2 = u^2 + 2as$$

$$u = 0$$

$$a = 9.8$$

$$s = 2.5$$

$$v^2 = u^2 + 2as$$

$$= 0 + 2(9.8)(2.5)$$

$$= 49$$

$$\Rightarrow \quad v = 7$$ ... (5m)

Conservation of momentum

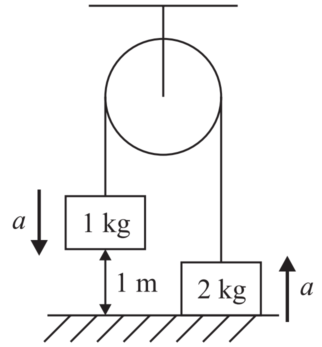
$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

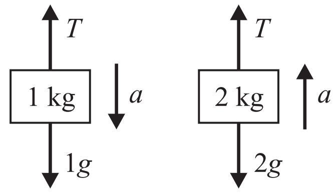
$$\Rightarrow \quad 1(7) + 2(0) = 1v + 2v$$

$$\Rightarrow \quad 3v = 7$$

$$\Rightarrow \quad v = \frac{7}{3} \text{ m/s}$$ ... (5m)

(ii) Investigate if the 1 kg mass will reach the table. **(10)**





$$
\begin{aligned}
1g - T &= 1a \\
T - 2g &= 2a \\
-g &= 3a \\
\Rightarrow \quad a &= -\frac{1}{3}g \\
&= -\frac{9.8}{3} \\
&= -\frac{49}{15}
\end{aligned}
$$

... (5m)

$$
\begin{aligned}
v^2 &= u^2 + 2as \\
u &= \frac{7}{3} \\
a &= -\frac{49}{15} \\
v &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad 0 &= \left(\frac{7}{3}\right)^2 + 2\left(-\frac{49}{15}\right)s \\
\Rightarrow \quad 0 &= \frac{49}{9} - \frac{98}{15}s \\
\Rightarrow \quad \frac{98}{15}s &= \frac{49}{9} \\
\Rightarrow \quad s &= \left(\frac{49}{9}\right)\left(\frac{15}{98}\right) \\
&= \frac{5}{6} \text{ m}
\end{aligned}
$$

$\Rightarrow$ it will not reach the table, as $\frac{5}{6} < 1$ ... (5m)

5. **(a)** A ball is dropped from a height of h metres onto a horizontal surface. The coefficient of restitution between the table and the ball is e.

Find, in terms of g, h and e,

(i) the speed at which the ball hits the ground for the first time; **(5)**

$$v^2 = u^2 + 2as$$
$$u = 0$$
$$a = g$$
$$s = h$$
$$\Rightarrow v^2 = 0 + 2gh$$
$$\Rightarrow v = \sqrt{2gh}$$
... (5m)

(ii) the speed at which the balls rises from the ground for the first time; **(5)**

$$\text{New speed} = ev$$
$$= e\sqrt{2gh}$$
... (5m)

(iii) the height the ball reaches after the first bounce; **(5)**

$$v^2 = u^2 + 2as$$
$$v = 0$$
$$u = e\sqrt{2gh}$$
$$a = -g$$
$$\Rightarrow 0 = (e\sqrt{2gh})^2 + 2(-g)s$$
$$= e^2(2gh) - 2gs$$
$$\Rightarrow 2gs = 2e^2gh$$
$$\Rightarrow s = e^2h$$
... (5m)

(iv) the height the ball reaches after the second bounce. **(5)**

$$v^2 = u^2 + 2as$$
$$v = 0$$
$$u = e^2\sqrt{2gh}$$
$$a = -g$$
$$\Rightarrow 0 = (e^2\sqrt{2gh})^2 + 2(-g)s$$
$$= e^4(2gh) - 2gs$$
$$\Rightarrow 2gs = 2e^4gh$$
$$\Rightarrow s = e^4h$$
... (5m)

5(b) Two smooth spheres of masses 6 kg and 3 kg moving in different directions impinge obliquely. The 3 kg mass is brought to rest by the impact. The coefficient of restitution between the spheres is $\frac{1}{2}$.

(i) Show that the spheres were moving in directions perpendicular to one another before the collision. **(15)**

Before	Mass	After
$x\vec{i} + y\vec{j}$	6	$p\vec{i} + y\vec{j}$
$q\vec{i} + 0\vec{j}$	3	$0\vec{i} + 0\vec{j}$

<u>Conservation of momentum</u>

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

$\Rightarrow \quad 6x + 3q = 6p + 3(0)$

$\Rightarrow \quad 2x + q = 2p$ ... (5m)

<u>NEL</u>

$\frac{v_1 - v_2}{u_1 - u_2} = -e$

$\Rightarrow \quad \frac{p - 0}{x - q} = -\frac{1}{2}$

$\Rightarrow \quad 2(p - 0) = -1(x - q)$

$\Rightarrow \quad 2p = -x + q$ ... (5m)

$\Rightarrow \quad 2x + q = -x + q$

$\Rightarrow \quad 2x + x = q - q$

$\Rightarrow \quad 3x = 0$

$\Rightarrow \quad x = 0$

$2x + q = 2p$

$\Rightarrow \quad 2(0) + q = 2p$

$\Rightarrow \quad q = 2p$

6 kg: $0\vec{i} + y\vec{j}$

3 kg: $q\vec{i} + 0\vec{j}$

as: $(0\vec{i} + y\vec{j})(q\vec{i} + 0\vec{j})$

$= 0 + 0$

$= 0$

$\Rightarrow$ 2 masses are moving in a ⊥ direction to each other ... (5m)

(ii) Show that the loss in kinetic energy due to the collision by the 3 kg mass is twice that gained by the 6 kg mass. **(15)**

<u>Kinetic energy of 3 kg mass:</u>

$\Rightarrow$ $K.E._{before} = \frac{1}{2}(3)q^2$

$= \frac{3}{2}q^2$

$\Rightarrow$ $K.E._{after} = \frac{1}{2}(3)(0)$

$= 0$

$\Rightarrow$ $K.E._{loss} = \frac{3}{2}q^2$ ... (5m)

<u>Kinetic energy of 6 kg mass:</u>

$\Rightarrow$ $K.E._{before} = \frac{1}{2}(6)y^2$

$= 3y^2$

$\Rightarrow$ $K.E._{after} = \frac{1}{2}(6)(p^2 + y^2)$

$= 3p^2 + 3y^2$

$\Rightarrow$ $K.E._{gain} = 3p^2 + 3y^2 - 3y^2$

$= 3p^2$ ... (5m)

Loss $= \frac{3}{2}q^2$

$q = 2p$

$\Rightarrow$ Loss $= \frac{3}{2}(2p)^2$

$= 6p^2$

$= 2(\text{Gain})$ ... (5m)

6. **(a)** A particle moves in a straight line such that its displacement from a fixed point *o* at time *t* is given by

$$x = 8\cos 5t + 15\sin 5t.$$

(i) Show that the particle moves with simple harmonic motion. **(5)**

$$\frac{dx}{dt} = (8)(-5)\sin 5t + (15)(5)\cos 5t$$

$$= -40\sin 5t + 75\cos 5t$$

$$\frac{d^2x}{dt^2} = (-40)(5)\cos 5t + (75)(-5)\sin 5t$$

$$= -200\cos 5t - 375\sin 5t$$

$$\Rightarrow \quad a = -25(8\cos 5t + 15\sin 5t)$$

$$\Rightarrow \quad a = -25x$$

$$\Rightarrow \quad a = -(5)^2 x$$

$\Rightarrow$ particle moves with SHM as $a = -\omega^2 x$ ... (5m)

(ii) Calculate the amplitude and period of the motion. **(10)**

<u>Amplitude</u>

$$A = \sqrt{8^2 + 15^2}$$

$$= \sqrt{64 + 225}$$

$$= \sqrt{289}$$

$$= 17$$ ... (5m)

<u>Period</u>

$$T = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{5}\text{ s}$$ ... (5m)

(iii) Find the time at which the particle is at $x = 0$. **(5)**

$$x = 8\cos 5t + 15\sin 5t$$

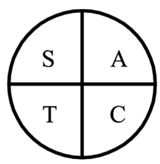
$$\Rightarrow \quad 0 = 8\cos 5t + 15\sin 5t$$

$$\Rightarrow \quad 15\sin 5t = -8\cos 5t$$

$$\Rightarrow \quad \frac{\sin 5t}{\cos 5t} = \frac{-8}{15}$$

$$\Rightarrow \quad \tan 5t = \frac{-8}{15}$$

$$\Rightarrow \quad 5t = \tan^{-1}\left(\frac{-8}{15}\right)$$



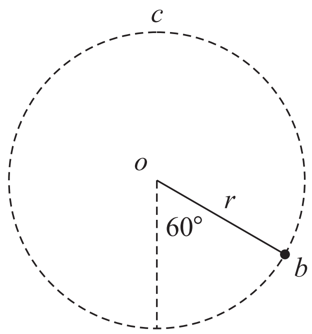
$$\Rightarrow \quad 5t = -0.489957\ldots \quad \underline{\text{or}} \quad 5t = 2.651635\ldots$$

$$\Rightarrow \quad t = -0.097991\ldots \qquad t = 0.530327\ldots$$

$$\cong -0.098\text{ s} \qquad \cong 0.53\text{ s}$$

$$\Rightarrow \quad t = 0.53\text{ sec}$$ ... (5m)

(b) A bead b is attached to a fixed point o by an inextensible wire of length r. The bead is free to rotate in a vertical plane. The bead is projected with speed u from the highest point c. The speed is just sufficient to enable the bead to perform a full rotation.



(i) Show that $u^2 = 4gr$. **(10)**

speed = 0 when the bead reaches the top

<u>Conservation of Energy</u>

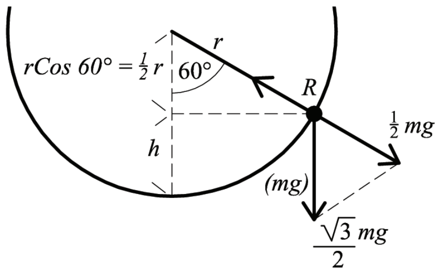
$$mgh_1 + \frac{1}{2}mv_1^2 = mgh_2 + \frac{1}{2}mv_2^2$$

$$mg(0) + \frac{1}{2}mu^2 = mg(2r) + 0 \quad \text{... (5m)}$$

$$\frac{1}{2}mu^2 = 2mgr$$

$$\Rightarrow \quad u^2 = 4gr \quad \text{... (5m)}$$

(ii) Find the reaction between the bead and the wire when the wire makes an angle of 60° with the downward vertical. **(20)**



$$\text{Let } v = \text{speed at } 60°$$

$$h = r - r\cos 60°$$

$$h = r - \frac{1}{2}r$$

$$= \frac{1}{2}r \quad \text{... (5m)}$$

$$R - \frac{1}{2}mg = \frac{mv^2}{r}$$

$$\Rightarrow \quad rR - \frac{1}{2}mgr = mv^2 \quad \text{... (5m)}$$

$$mgh_1 + \frac{1}{2}mv_1^2 = mgh_2 + \frac{1}{2}mv_2^2$$

$$\Rightarrow \quad 0 + \frac{1}{2}mu^2 = mg(\frac{1}{2}r) + \frac{1}{2}mv^2$$

$$\Rightarrow \quad mu^2 = mgr + mv^2$$

$$u^2 = 4gr$$

$$\Rightarrow \quad m(4gr) = mgr + mv^2$$

$$\Rightarrow \quad 4mgr - mgr = mv^2$$

$$\Rightarrow \quad 3mgr = mv^2 \quad \text{... (5m)}$$

$$\Rightarrow \quad rR - \frac{1}{2}mgr = 3mgr$$

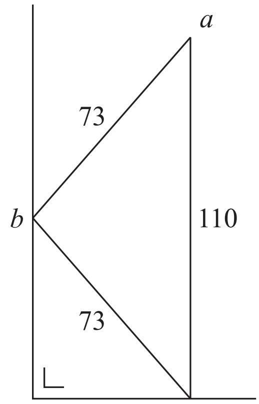
$$\Rightarrow \quad rR = 3mgr + \frac{1}{2}mgr$$

$$\Rightarrow \quad rR = 3.5mgr$$

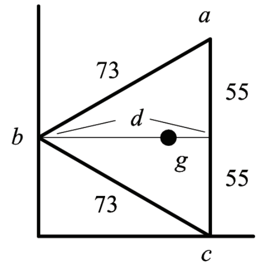
$$\Rightarrow \quad R = 3.5mg \quad \text{... (5m)}$$

7. **(a)** A uniform lamina is in the shape of an equilateral triangle *abc*.
$|ab| = |bc| = 73$ cm and $|ac| = 110$ cm.
The lamina rests in a vertical plane as shown, with vertex *b* against a smooth vertical wall and vertex *c* on a rough horizontal ground and side [*ac*] in a vertical position.

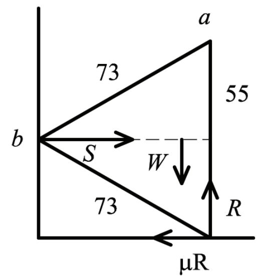
Find the least possible value of μ, the coefficient of friction between the lamina and the ground, if slipping is not to occur.



(25)



$$73^2 = 55^2 + d^2$$
$$d^2 = 73^2 - 55^2$$
$$= 5{,}329 - 3{,}025$$
$$= 2{,}304$$
$$\Rightarrow \quad d = 48$$
$$\Rightarrow \quad |bg| = \frac{2}{3}(48)$$
$$= 32$$ ... (5m)



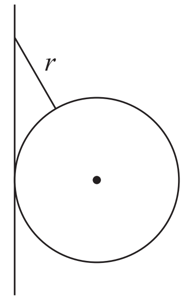
$$R = W$$ ... (5m)
$$S = \mu R$$ ... (5m)

Taking moments @ point of contact on ground:

$$W(16) = S(55)$$
$$\Rightarrow \quad S = \frac{16}{55}W$$ ... (5m)

$$\Rightarrow \quad \frac{16}{55}W = \mu W$$
$$\Rightarrow \quad \mu = \frac{16}{55}$$ ... (5m)

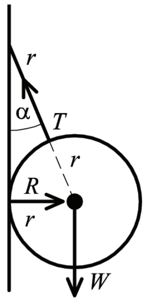
(b) A sphere of weight W and radius r has a light inextensible string, of length r, attached to its surface. The other end of the string is attached to a smooth vertical wall and the sphere is allowed to hang freely, resting against the wall.



Find **(i)** the angle of inclination of the string; **(15)**

Forces R, W and T are collinear ... 3-force theorem ... (5m)

<u>Diagram</u>



... (5m)

$$\sin\alpha = \frac{r}{2r} = \frac{1}{2}$$

$$\Rightarrow \quad \alpha = \sin^{-1}\frac{1}{2} = 30°$$ ... (5m)

(ii) the tension in the string. **(5)**

$$W = T\cos 30°$$

$$\Rightarrow \quad W = \frac{\sqrt{3}}{2}T$$

$$\Rightarrow \quad T = \frac{2W}{\sqrt{3}}$$ ... (5m)

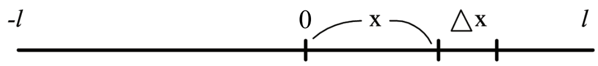
(iii) the reaction between the sphere and the wall; **(5)**

$$R = T\sin 30° = \frac{1}{2}T = \frac{1}{2}\left(\frac{2W}{\sqrt{3}}\right) = \frac{W}{\sqrt{3}}$$ ... (5m)

8. **(a)** Prove that the moment of inertia of a uniform rod, of mass m and length $2l$, about an axis through its centre perpendicular to its length is $\frac{1}{3}ml^2$. **(20)**

$$\text{Let } \rho = \text{mass per unit length}$$

$$\Rightarrow M = 2l\rho \qquad \text{... (5m)}$$



$$\Delta M = \rho\Delta x$$

$$I = \sum \Delta M r^2$$

$$= \sum \rho\Delta x(x^2) \qquad \text{... (5m)}$$

$$= \int_{-l}^{l} \rho x^2\, dx \qquad \text{... (5m)}$$

$$= \left.\frac{\rho x^3}{3}\right|_{-l}^{l}$$

$$= \frac{\rho l^3}{3} + \frac{\rho l^3}{3}$$

$$= \frac{2\rho l^3}{3}$$

$$= \frac{1}{3}(2\rho l)l^2$$

$$= \frac{1}{3}\text{m}l^2 \qquad \text{... (5m)}$$

(b) A uniform rod of mass $5m$ and length $2l$, has a particle of mass $5m$ attached to its centre

(i) If the system is free to rotate in a vertical plane about a horizontal axis through one endpoint of the rod, find the period of small oscillations. **(15)**

<u>Rod</u>

$$I_{\text{endpoint}} = \frac{4}{3}ml^2$$

<u>Point mass</u>

$$I_{\text{endpoint}} = 5m(l)^2$$

$$= 5ml^2$$

<u>System</u>

$$I_{\text{endpoint}} = \frac{4}{3}ml^2 + 5ml^2$$

$$= \frac{19}{3}ml^2 \qquad \text{... (10m)}$$

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

$$= 2\pi\sqrt{\frac{\frac{19}{3}ml^2}{6mgl}}$$

$$= 2\pi\sqrt{\frac{19l}{18g}} \qquad \text{... (5m)}$$

(ii) If the system is rotated about an axis parallel to L but a distance x from the midpoint of the rod, find the value of x for which the period of small oscillations is a minimum. **(15)**

Rod

$$I = \frac{1}{3}ml^2 + mx^2$$

Point mass

$$I = 5mx^2$$

System

$$I = \frac{1}{3}ml^2 + mx^2 + 5mx^2$$

$$= \frac{1}{3}ml^2 + 6mx^2 \qquad \text{... (10m)}$$

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ml^2 + 6mx^2}{6mgx}}$$

$$= 2\pi\sqrt{\frac{l^2 + 18x^2}{18gx}}$$

$$\Rightarrow \quad T^2 = \frac{4\pi^2}{18g}\left(\frac{l^2 + 18x^2}{x}\right)$$

$$\frac{dT^2}{dx} = 0$$

$$\Rightarrow \quad \frac{4\pi^2}{18g}\left[\frac{x(36x) - (l^2 + 18x^2)}{x^2}\right] = 0$$

$$\Rightarrow \quad 36x^2 - l^2 - 18x^2 = 0$$

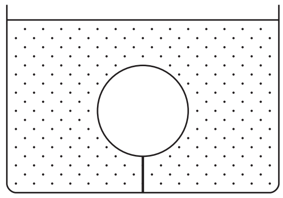
$$\Rightarrow \quad 36x^2 - 18x^2 = l^2$$

$$\Rightarrow \quad 18x^2 = l^2$$

$$\Rightarrow \quad x^2 = \frac{l^2}{18}$$

$$\Rightarrow \quad x = \frac{l}{\sqrt{18}} \text{ or } \frac{l}{3\sqrt{2}} \qquad \text{... (5m)}$$

9. **(a)** A uniform sphere, of radius 1.5 m and relative density 0.8, is held completely immersed in a liquid, of relative density 0.85, by means of a string tied to a point on the base of the tank.



Find, in terms of π,

(i) the volume of the sphere **(5)**

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi\left(\frac{3}{2}\right)^3$$
$$= \frac{4}{3}\pi\left(\frac{27}{8}\right)$$
$$= \frac{9}{2}\pi \text{ m}^2 \qquad \text{... (5m)}$$

(ii) the weight of the sphere; **(5)**

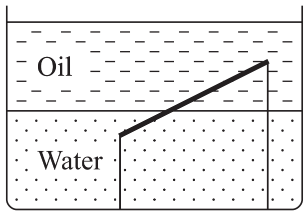
$$W = \rho V g$$
$$= (0.8)(\frac{9}{2}\pi)g$$
$$= 3.6\pi g \qquad \text{... (5m)}$$

(iii) the tension in the string. **(10)**

$$B = \rho_l V g$$
$$= (0.85)(\frac{9}{2}\pi)g$$
$$= 3.825\pi g \qquad \text{... (5m)}$$

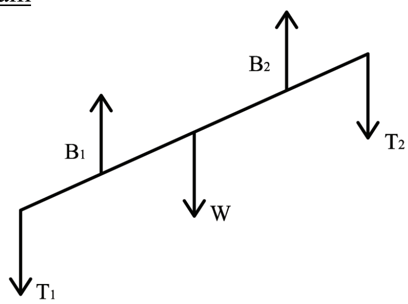
$$T + W = B$$
$$T = B - W$$
$$= 3.825\pi g - 3.6\pi g$$
$$= 0.225\pi g \text{ N} \qquad \text{... (5m)}$$

(b) A tank contains a layer of water and a layer of oil of relative density 0.8. A uniform rod, of weight W and relative density $\frac{7}{9}$ is totally immersed with one-third of its volume in the water and two-thirds in the oil. It is maintained in that position by two vertical strings attached to the ends of the rod and the base of the tank.



Show, in a diagram, the forces acting on the rod and calculate, in terms of W, the tension in the strings. **(30)**

Diagram



... (5m)

Tension in the strings

$$B_1 = \text{buoyancy in water} = \frac{\frac{1}{3}W}{\frac{7}{9}} = \frac{3}{7}W$$ ... (5m)

$$B_2 = \text{buoyancy in oil} = \frac{\frac{2}{3}W}{\frac{7}{9}}(0.8) = \frac{24}{35}W$$ ... (5m)

$$B_1 + B_2 = T_1 + W + T_2$$

$$\Rightarrow \quad \frac{3}{7}W + \frac{24}{35}W = T_1 + W + T_2$$

$$\Rightarrow \quad T_1 + T_2 = \frac{3}{7}W + \frac{24}{35}W - W = \frac{4}{35}W$$ ... (5m)

Taking moments about endpoint

$$\alpha = \text{angle of inclination of the rod}$$

$$B_1(\frac{l}{6})\sin\alpha + B_2(\frac{4l}{6})\sin\alpha = W(\frac{3l}{6})\sin\alpha + T_2(l)\sin\alpha \quad \text{... (5m)}$$

$$(\frac{l}{6})\sin\alpha[B_1(1) + B_2(4)] = (\frac{l}{6})\sin\alpha[W(3) + T_2(6)]$$

$$[B_1(1) + B_2(4)] = [W(3) + T_2(6)]$$

$$\Rightarrow \quad \frac{3}{7}W(1) + \frac{24}{35}W(4) = 3W + 6T_2$$

$$\Rightarrow \quad 6T_2 = \frac{3}{7}W + \frac{96}{35}W - 3W$$

$$= \frac{6}{35}W$$

$$\Rightarrow \quad T_2 = \frac{1}{35}W$$

$$\Rightarrow \quad T_1 + T_2 = \frac{4}{35}W$$

$$\Rightarrow \quad T_1 = \frac{4}{35}W - T_2$$

$$= \frac{4}{35}W - \frac{1}{35}W$$

$$= \frac{3}{35}W \quad \text{... (5m)}$$

10. **(a)** Solve the differential equation

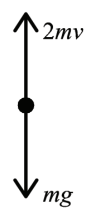
$$\cos x \frac{dy}{dx} = y \sin x$$

given that $y = e^3$ when $x = \frac{\pi}{3}$. **(20)**

$$
\begin{aligned}
\cos x \frac{dy}{dx} &= y \sin x \\
\Rightarrow \quad \frac{dy}{y} &= \frac{\sin x}{\cos x}\, dx && \text{... (5m)} \\
\Rightarrow \quad \int_{e^3}^{y} \frac{1}{y}\, dy &= \int_{\frac{\pi}{3}}^{x} \tan x \, dx \\
\Rightarrow \quad \log_e y \Big|_{e^3}^{y} &= \log_e (\sec x) \Big|_{\frac{\pi}{3}}^{x} && \text{... (5m)} \\
\Rightarrow \quad \log_e y - 3 &= \log_e (\sec x) - \log_e 2 && \text{... (5m)} \\
\Rightarrow \quad \log_e (y + 2 - \sec x) &= 3 \\
\Rightarrow \quad \log_e \left(\frac{2y}{\sec x} \right) &= 3 \\
\Rightarrow \quad \frac{2y}{\sec x} &= e^3 \\
\Rightarrow \quad y &= \frac{e^3 \sec x}{2} && \text{... (5m)}
\end{aligned}
$$

10(b) A parachutist falls from rest under gravity. The mass of the parachutist and his parachute is m. Air resistance is equal to twice the momentum of the parachutist

(i) Show that $5\frac{dv}{dt} = 49 - 10v$. **(10)**



$$
\begin{aligned}
F &= mg - 2mv && \text{... (5m)} \\
\Rightarrow \quad ma &= mg - 2mv \\
\Rightarrow \quad a &= g - 2v \\
\Rightarrow \quad a &= 9.8 - 2v \\
\Rightarrow \quad \frac{dv}{dt} &= 9.8 - 2v \\
\Rightarrow \quad 5\frac{dv}{dt} &= 4.9 - 10v && \text{... (5m)}
\end{aligned}
$$

(ii) Show that, as time elapses, the speed of the parachutist tends to a constant value of 4.9 m/s called the 'terminal velocity'. **(15)**

$$5\frac{dv}{dt} = 49 - 10v$$

$$\frac{5dv}{49-10v} = dt$$

$$\Rightarrow \int_0^t dt = \int_0^v \frac{5dv}{49-10v}$$

$$\Rightarrow t\Big|_0^t = -\frac{5}{10}\log_e(49-10v)\Big|_0^v$$

$$\Rightarrow t = \frac{1}{2}\log_e(49-10v) + \frac{1}{2}\log_e(49)$$

$$\Rightarrow t = \frac{1}{2}\log_e\left(\frac{49}{49-10v}\right) \quad \text{... (5m)}$$

$$\Rightarrow 2t = \log_e\left(\frac{49}{49-10v}\right)$$

$$\Rightarrow e^{2t} = \frac{49}{49-10v}$$

$$\Rightarrow 49 - 10v = \frac{49}{e^{2t}}$$

$$\Rightarrow 10v = 49 - \frac{49}{e^{2t}}$$

$$\Rightarrow v = \frac{1}{10}(49 - \frac{49}{e^{2t}})$$

$$\text{as } t \to \infty,\ e^{2t} \to \infty \Rightarrow \frac{49}{e^{2t}} \to 0 \quad \text{... (5m)}$$

$$\Rightarrow v = \frac{1}{10}(49-0)$$

$$= 4.9 \text{ m/s} \quad \text{... (5m)}$$

(iii) Show that the time taken to reach half the terminal velocity is $\ln\sqrt{2}$ seconds. **(5)**

$$t = \frac{1}{2}\log_e\left(\frac{49}{49-10v}\right)$$

$$\text{Let } v = \frac{4.9}{2}$$

$$= 2.45 \text{ m/s}$$

$$\Rightarrow t = \frac{1}{2}\log_e\left(\frac{49}{49-10(2.45)}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{49}{24.5}\right)$$

$$= \frac{1}{2}\log_e 2$$

$$= \log_e 2^{\frac{1}{2}}$$

$$= \log_e\sqrt{2} \quad \text{... (5m)}$$