M32

Coimisiún na Scrúduithe Stáit
State Examinations Commission

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Marking Scheme	*Leaving Certificate Examination, 2004*
Applied Mathematics	*Higher Level*

General Guidelines

1 Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	S(-1)
Blunders	- mathematical errors	B(-3)
Misreading	- if not serious	M(-1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (att 2).

2 Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3 Mark scripts in red unless candidate uses red. If a candidate uses red, mark the script in blue or black.

4 Number the grid on each script 1 to 10 in numerical order, not the order of answering.

5 Scrutinise **all** pages of the answer book.

6 The marking scheme shows one correct solution to each question. In many cases there are other equally valid methods.

1. **(a)** A ball is thrown vertically upwards with an initial velocity of 20 m/s. One second later, another ball is thrown vertically upwards from the same point with an initial velocity of u m/s. The balls collide after a further 2 seconds.

(i) Show that $u = 17.75$.

(ii) Find the **distance** travelled by each ball before the collision, giving your answers correct to the nearest metre.

(i)

$$s = ut + \tfrac{1}{2}at^2$$

$$s_1 = 20(3) - \tfrac{1}{2}(9.8)3^2 \qquad 5$$

$$s_2 = u(2) - \tfrac{1}{2}(9.8)2^2 \qquad 5$$

$$s_1 = s_2$$

$$60 - 44.1 = 2u - 19.6$$

$$2u = 35.5$$

$$u = 17.75 \qquad 5$$

(ii) Second ball

$$v^2 = u^2 + 2as$$

$$0 = 17.75^2 - 2gs$$

$$s = 16.07$$

$$\text{distance} = 16.07 + (16.07 - 15.9)$$

$$= 16.24$$

$$= 16 \qquad 5$$

First ball

$$v^2 = u^2 + 2as$$

$$0 = 20^2 - 2gs$$

$$s = 20.4$$

$$\text{distance} = 20.4 + (20.4 - 15.9)$$

$$= 24.9$$

$$= 25 \qquad 5$$

25

(b) A car of mass 1200 kg tows a caravan of mass 900 kg first along a horizontal road with acceleration f and then up an incline α to the horizontal road at uniform speed.
The force exerted by the engine is 2700 N. Friction and air resistance amount to 150 N on the car and 240 N on the caravan.

Calculate

(i) the acceleration, f, of the car along the horizontal road

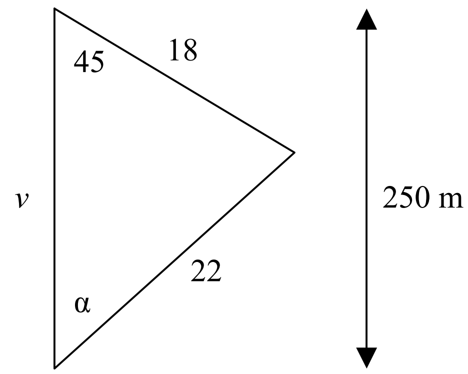
(ii) the value of α , to the nearest degree.

(i)	Car	$2700 - T - 150 = 1200f$	5
	Caravan	$T - 240 = 900f$	5
		$2700 - 240 - 150 = 2100f$	
		$f = \dfrac{2310}{2100}$ or 1.1	5
(ii)	Car	$2700 - 1200g\sin\alpha - T - 150 = 0$	5
	Caravan	$T - 900g\sin\alpha - 240 = 0$	
		$2700 - 2100g\sin\alpha - 240 - 150 = 0$	
		$2100g\sin\alpha = 2310$	
		$\sin\alpha = \dfrac{2310}{2100g} = 0.1122$	
		$\alpha = 6.4°$	
		$= 6°$	5
			25

2. **(a)** A bird flies at a uniform speed of 22 m/s. It wishes to fly to its nest which is 250 m due north of its present position. There is a wind blowing from the southeast at 18 m/s.

Find **(i)** the direction, to the nearest degree, in which the bird must fly to reach its nest

(ii) the time taken to reach the nest, correct to two decimal places.



(i) $\dfrac{\sin\alpha}{18} = \dfrac{\sin 45}{22}$ or $18\sin 45 = 22\sin\alpha$	5	
$\sin\alpha = \dfrac{18\sin 45}{22}$ $= 0.5785$ $\alpha = 35.3451$ $= 35°$	5	
(ii) $\dfrac{v}{\sin 100} = \dfrac{22}{\sin 45}$ or $v = 18\cos 45 + 22\cos 35$ $v = 30.6400$	5	
$\text{time} = \dfrac{250}{30.64}$ $= 8.16$	5	20

(b) At time t = 0, two particles P and Q are set in motion. At time t = 0, Q has position vector $20\vec{i} + 40\vec{j}$ metres relative to P.

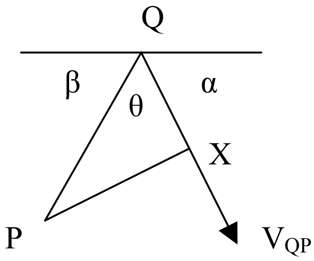
P has a constant velocity of $3\vec{i} + 5\vec{j}$ m/s and Q has a constant velocity of $4\vec{i} - 3\vec{j}$ m/s.

Find

(i) the velocity of Q relative to P

(ii) the shortest distance between P and Q, to the nearest metre

(iii) the time when P and Q are closest together, correct to one decimal place.



(i)

$$\vec{V}_{QP} = \vec{V}_Q - \vec{V}_P \qquad 5$$

$$= (4\vec{i} - 3\vec{j}) - (3\vec{i} + 5\vec{j})$$

$$= \vec{i} - 8\vec{j} \qquad 5$$

$$\left.\begin{array}{l} \text{magnitude} = \sqrt{(1)^2 + (8)^2} = 8.06 \\ \text{direction } \alpha = \tan^{-1}(8) \text{ or } 82.87^\circ \text{ S. of E.} \end{array}\right\} \qquad 5$$

(ii)

$$\alpha = \tan^{-1}(8) \text{ or } 82.87^\circ$$

$$\beta = \tan^{-1}\left(\tfrac{40}{20}\right) \text{ or } 63.43^\circ$$

$$\theta = 180 - 82.87 - 63.43$$

$$= 33.70^\circ$$

$$|PQ| = \sqrt{20^2 + 40^2} = 44.72$$

$$|PX| = |PQ| \sin 33.70^\circ$$

$$= 24.8$$

$$= 25 \qquad 5$$

(iii)

$$|QX| = |PQ| \cos 33.70^\circ = 37.21 \qquad 5$$

$$\text{time} = \frac{|QX|}{V_{QP}} = \frac{37.21}{8.06} = 4.6 \qquad 5 \qquad \boxed{30}$$

3. **(a)** A particle is projected from a point on the horizontal floor of a tunnel with maximum height of 8 m. The particle is projected with an initial speed of 20 m/s inclined at an angle α to the horizontal floor.

Find, to the nearest metre, the greatest range which can be attained in the tunnel.

At highest point $\quad \vec{r}\,\vec{j} = 8 \quad \Rightarrow 20\sin\alpha(t) - \frac{1}{2}gt^2 = 8$ $\quad$ 5

$\vec{v}\,\vec{j} = 0 \quad \Rightarrow \quad 20\sin\alpha - gt = 0$ $\quad$ 5

$\Rightarrow \quad t = \dfrac{4}{\sqrt{g}} \quad$ or $\quad 1.28$

$\alpha = \sin^{-1}\left(\dfrac{\sqrt{g}}{5}\right) \quad$ or $\quad 38.76°$ $\quad$ 5

time $= 2t = 2.56$ $\quad$ 5

$$\begin{aligned} \text{Range } &= 20\cos\alpha\{2t\} \\ &= 20\{0.7798\}\{2.56\} \\ &= 39.9 \\ &= 40 \end{aligned}$$ 5 $\quad$ 25

$$\left.\begin{aligned} \text{Max Range } &= \frac{u^2}{g} = \frac{20^2}{9.8} = 40.8 \\ &= 41 \text{ m} \end{aligned}\right\} \quad \text{Worthless}$$

$$\begin{aligned} \text{Greatest height } &= \frac{u^2\sin^2\alpha}{2g} = 8 \\ \alpha &= 38.76° \\ \text{Range } &= \frac{u^2\sin 2\alpha}{g} = 39.85 = 40 \end{aligned}$$

5

5

5 $\quad$ 15

(b) A particle is projected up an inclined plane with initial velocity u m/s. The line of projection makes an angle α **with the horizontal** and the inclined plane makes an angle θ with the horizontal. (The plane of projection is vertical and contains the line of greatest slope.)

If the particle strikes the inclined plane at right angles, show that

$$\tan\alpha = \frac{1+2\tan^2\theta}{\tan\theta}.$$

$$r_j = 0$$

$$\text{usin}(\alpha-\theta).\text{t} - \tfrac{1}{2}g\cos\theta.\text{t}^2 = 0 \qquad 5$$

$$\Rightarrow \quad \text{t} = \frac{2\text{usin}(\alpha-\theta)}{\text{gcos}\theta}$$

$$v_i = 0$$

$$\text{ucos}(\alpha-\theta) - g\sin\theta.\text{t} = 0 \qquad 5$$

$$\Rightarrow \quad \text{t} = \frac{\text{ucos}(\alpha-\theta)}{\text{gsin}\theta}$$

$$\frac{2\text{usin}(\alpha-\theta)}{\text{gcos}\theta} = \frac{\text{ucos}(\alpha-\theta)}{\text{gsin}\theta} \qquad 5$$

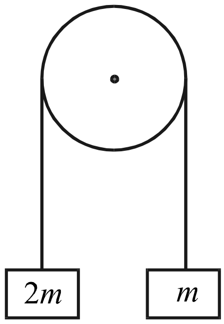
$$2\tan(\alpha-\theta)\tan\theta = 1$$

$$2\left\{\frac{\tan\alpha-\tan\theta}{1+\tan\alpha\tan\theta}\right\}\tan\theta = 1 \qquad 5$$

$$2\tan\alpha\tan\theta - 2\tan^2\theta = 1+\tan\alpha\tan\theta$$

$$\tan\alpha = \frac{1+2\tan^2\theta}{\tan\theta} \qquad 5 \qquad 25$$

4. **(a)** Two particles, of masses $2m$ and m, are attached to the ends of a light inextensible string which passes over a fixed smooth light pulley.



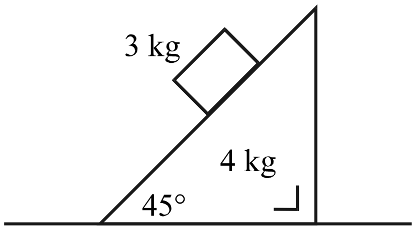
The system is released from rest with both particles at the same horizontal level.

(i) Find the acceleration of the system, in terms of g.

(ii) The string breaks when the speed of each particle is v.
Find, in terms of v, the vertical distance between the particles when the string breaks.

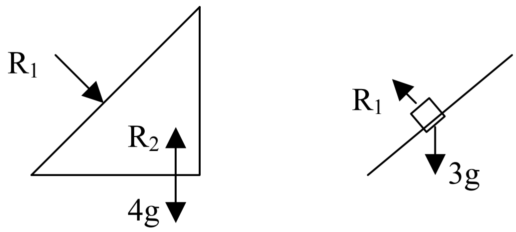
(i)	$2mg - T = 2mf$	5
	$T - mg = mf$	5
	$mg = 3mf$	
	$f = \frac{1}{3}g \ \ ms^{-2}$	5
(ii)	$v^2 = u^2 + 2as$	
	$v^2 = 0 + 2\left(\frac{g}{3}\right)s$	
	$s = \frac{3v^2}{2g}$	
	vertical distance $= 2s = \frac{3v^2}{g}$	5
		20

(b) A smooth wedge of mass 4 kg and slope 45° rests on a smooth horizontal surface.
A particle of mass 3 kg is placed on the smooth inclined face of the wedge.
The system is released from rest.



(i) Show, on separate diagrams, the forces acting on the wedge and on the particle.
(ii) Find the acceleration of the particle relative to the wedge.
(iii) Find how far the wedge has travelled when the particle has moved a distance of 1 m down the inclined face of the wedge.

(i)



5

(ii)
$$R_1 \sin 45 = 4f$$
$$3g \sin 45 = 3(p - f\cos 45)$$
5

$$3g\cos 45 - R_1 = 3f \sin 45$$
5

$$f = \frac{3g}{11} \quad \text{and} \quad p = \frac{14g}{11\sqrt{2}}$$
5

(iii) particle
$$s = ut + \tfrac{1}{2}pt^2$$
$$1 = 0 + \tfrac{1}{2}\left(\frac{14g}{11\sqrt{2}}\right)(t^2)$$
$$t^2 = \frac{11\sqrt{2}}{7g} \quad \text{or} \quad 0.23$$
5

wedge
$$s = ut + \tfrac{1}{2}ft^2$$
$$= 0 + \tfrac{1}{2}\left(\frac{3g}{11}\right)\left(\frac{11\sqrt{2}}{7g}\right)$$
$$= \frac{3\sqrt{2}}{14} \quad \text{or} \quad 0.3$$
5

30

5. **(a)** A smooth sphere P, of mass $3m$, moving with speed u, collides directly with a smooth sphere Q, of mass $5m$, which is at rest.
The coefficient of restitution for the collision is e.

Find

(i) the speed, in terms of u and e, of each sphere after the collision

(ii) the condition to be satisified by e in order that the spheres move in opposite directions after the collision.

(i) PCM $\quad 3m(u) + 5m(0) = 3mv_1 + 5mv_2$ — 5

NEL $\quad v_1 - v_2 = -e(u - 0)$ — 5

$$\left.\begin{aligned} v_1 &= \frac{u}{8}(3-5e) \\ v_2 &= \frac{3u}{8}(1+e) \end{aligned}\right\}$$ — 5

(ii)

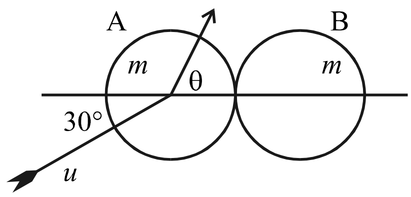
$$\left.\begin{aligned} &v_2 > 0 \quad \forall\ e \\ &\Rightarrow \quad v_1 < 0 \end{aligned}\right\}$$ — 5

$3 - 5e < 0$

$\frac{3}{5} < e$ — 5

25

(b) A smooth sphere A, of mass m, moving with speed u, collides with an identical smooth sphere B which is at rest.
The direction of motion of A, before impact, makes an angle 30º with the line of centres at impact.
After impact the direction of A makes an angle θ with the line of centres, where $0^\circ \leq \theta < 90^\circ$.
The coefficient of restitution between the spheres is e.
The **speeds** of A and B immediately after impact are equal.



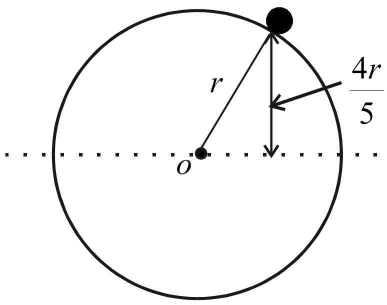
(i) Calculate the value of θ.

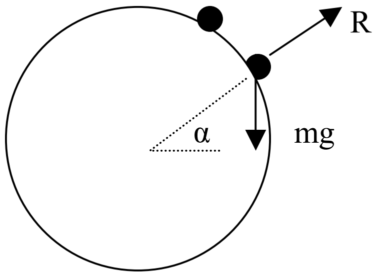
(ii) Find e.

PCM	$mu\cos 30 \;+\; m(0) \;=\; mv_1 + mv_2$	5
NEL	$v_1 - v_2 = -e\,(u\cos 30 - 0)$	5
	$v_1 = \dfrac{u\sqrt{3}\{1-e\}}{4}$ $v_2 = \dfrac{u\sqrt{3}\{1+e\}}{4}$	5
(ii)	$(v_1)^2 + (u\sin 30)^2 = (v_2)^2$ $\dfrac{3u^2}{16}\left(1-2e+e^2\right)+\dfrac{u_2}{4}=\dfrac{3u^2}{16}\left(1+2e+e^2\right)$ $\Rightarrow \quad e=\frac{1}{3}$	5
(i)	$\tan\theta = \dfrac{\frac{u}{2}}{\frac{u\sqrt{3}\{1-e\}}{4}}$ $=\dfrac{2}{\sqrt{3}\{1-e\}}$ $=\sqrt{3}$ $\Rightarrow \quad \theta = 60^\circ$	5
		25

6. **(a)** A particle can move on the smooth outer surface of a fixed sphere of radius r. The particle is released from rest on the smooth surface of the sphere at a height $\frac{4r}{5}$ above the horizontal plane through the centre o of the sphere.

Find, in terms of r, the height above this plane at which the particle leaves the sphere.





Gain in KE = Loss in PE

$$\tfrac{1}{2}mv^2 = mg\left(\frac{4r}{5}\right) - mgh \qquad 5$$

$$mgh = \frac{4mgr}{5} - \frac{mv^2}{2}$$

$$\text{Force towards the centre} = \frac{mv^2}{r}$$

$$mg\sin\alpha - R = \frac{mv^2}{r} \qquad 5$$

$$R = 0 \qquad 5$$

$$mgr\sin\alpha = mv^2 \Rightarrow mv^2 = mgh$$

$$mgh = \frac{4mgr}{5} - \frac{mv^2}{2}$$

$$mgh = \frac{4mgr}{5} - \frac{mgh}{2} \qquad 5$$

$$\text{Height} \quad h = \frac{8r}{15} \qquad 5 \qquad 25$$

(b) A particle moves in a straight line such that its displacement from a fixed point o at time t is given by

$$x = a\cos(\omega t - \beta)$$

where a, ω and β are positive constants.

(i) Show that the motion of the particle is simple harmonic motion.

The period of the motion is 16 seconds. At time $t = 4$ s, the particle is 12 m from o and 4 s later the particle is on the other side of o and at a distance of 5 m from o.

(ii) Find a, ω and β.

(i)

$$x = a\cos(\omega t - \beta)$$

$$\dot{x} = -a\omega\sin(\omega t - \beta)$$

$$\ddot{x} = -a\omega^2\cos(\omega t - \beta)$$
$$= -\omega^2 x$$

$\therefore$ SHM 5

(ii)

$$T = \frac{2\pi}{\omega}$$

$$16 = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{\pi}{8}$$ 5

$$x = a\cos(\omega t - \beta)$$

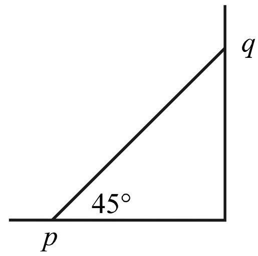
$$12 = a\cos\left(\frac{\pi}{2} - \beta\right)$$ 5

$$-5 = a\cos(\pi - \beta)$$ 5

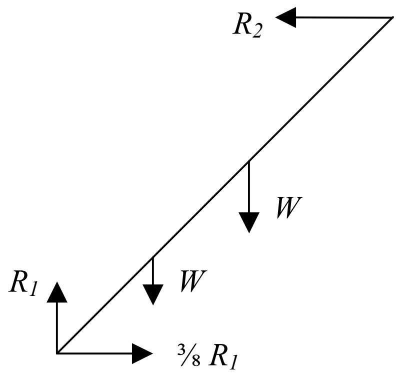
$$\left.\begin{aligned} \beta &= \tan^{-1}\left(\frac{12}{5}\right) = 1.176 \text{ rad} \\ a &= 13 \end{aligned}\right\}$$ 5

25

7. **(a)** A uniform rod [pq], of length $2l$ and weight W is in equilibrium with the end p on a rough horizontal floor and the end q against a smooth vertical wall. The rod makes an angle 45° with the horizontal and is in a vertical plane which is perpendicular to the wall. The coefficient of friction between the floor and the rod is $\frac{3}{8}$.



Find the distance from p of the highest point of the rod, in terms of l, from which a particle of weight W can be attached without disturbing equilibrium.



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Resolve forces acting on rod :

vertical : $R_1 = 2W$ 5

horizontal; $R_2 = \frac{3}{8}R_1 \quad \Rightarrow \quad R_2 = \frac{3}{4}W$

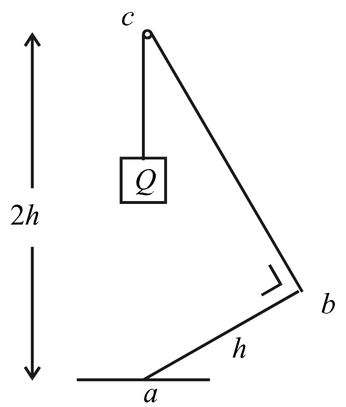
Take moments about p for rod :

$$R_2\,(2\ell\sin 45) = W(\ell\cos 45) + W(x\cos 45)$$ 5,5

$$\frac{3}{4}W\,(2\ell) = W\,\ell + W\,x$$

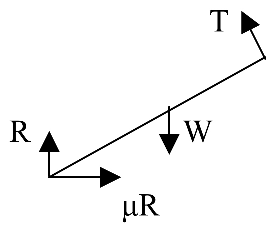
$$x = \tfrac{1}{2}\ell$$ 5 25

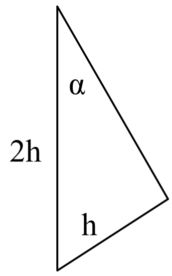
(b) A uniform rod $[ab]$, of length h and weight W is in equilibrium with the end a resting on a rough horizontal plane. The rod is maintained in equilibrium by means of a light inextensible string which passes over a small smooth peg, of negligible diameter, at c, with one end of the string attached to b and with a weight Q attached to the other end of the string. The peg at c is at a height $2h$ vertically above a and $|\angle abc| = 90^\circ$. Find



(i) Q in terms of W

(ii) the magnitude of the force acting on the peg at c, in terms of W and correct to two decimal places.





(i) Take moments about a for rod:

$$T\left(h\right) = W\left(\tfrac{1}{2}h\cos\alpha\right)$$ 5,5

$$\Rightarrow \quad T = \tfrac{1}{2}W\cos\alpha$$

$$\sin\alpha = \frac{h}{2h} = \frac{1}{2} \quad \Rightarrow \quad \alpha = 30^\circ$$ 5

$$\left.\begin{aligned} Q &= T \\ &= \tfrac{1}{2}W\cos\alpha \\ &= \frac{W\sqrt{3}}{4} \end{aligned}\right\}$$ 5

(ii)
$$R = \sqrt{\left(T\sin 30\right)^2 + \left(T + T\cos 30\right)^2}$$

$$= \sqrt{\left(\frac{W\sqrt{3}}{8}\right)^2 + \left(\frac{W\sqrt{3}}{4} + \frac{3W}{8}\right)^2}$$

$$= 0.84\,W$$ 5

25

8. **(a)** Prove that the moment of inertia of a uniform circular disc, of mass m and radius r, about an axis through its centre perpendicular to its plane is $\frac{1}{2}mr^2$.

Let M = mass per unit area

mass of element $= M\{2\pi\, x.dx\}$

moment of inertia of the element $= M\{2\pi\, x.dx\}x^2$ 5

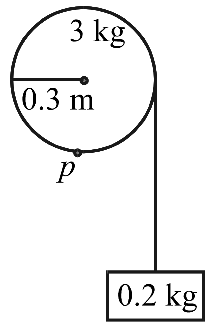
moment of inertia of the disc $= 2\pi M \int_0^r x^3\, dx$ 5

$= 2\pi M \left[\frac{x^4}{4}\right]_0^r$ 5

$= \frac{1}{2}\pi M r^4$

$= \frac{1}{2} m r^2$ 5 20

(b) A smooth pulley wheel has a mass of 3 kg and a radius of 0.3 m. One end of a light inextensible rope is attached to a point p on the rim of the wheel. A particle of mass 0.2 kg attached to the other end of the rope hangs freely.
The axis of rotation of the wheel is horizontal, perpendicular to the wheel, and passes through the centre of the wheel.
The particle is released from rest and moves vertically downwards.



When the particle has acquired a speed of 1.2 m/s, find

(i) the kinetic energy gained by the wheel
(ii) the distance descended by the particle, correct to two decimal places.

(i)

$$I = \tfrac{1}{2}mr^2$$
$$= \tfrac{1}{2}(3)(0.3)^2 \qquad 5$$
$$= 0.135$$

$$\omega = \frac{v}{r} = \frac{1.2}{0.3} = 4$$

$$\text{kinetic energy of wheel} = \tfrac{1}{2}I\omega^2 \qquad 5$$
$$= \tfrac{1}{2}(0.135)(4)^2$$
$$= 1.08 \qquad 5$$

(ii)

$$\text{Gain in KE} = \text{Loss in PE}$$

$$\tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2 = mgh \qquad 5$$

$$1.08 + \tfrac{1}{2}(0.2)(1.2)^2 = (0.2)gh \qquad 5$$

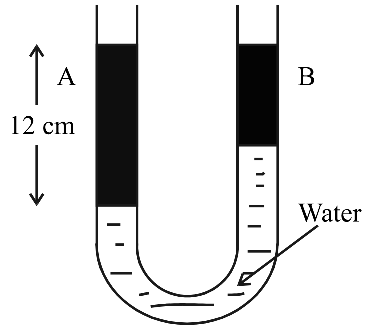
$$1.08 + 0.144 = (0.2)gh$$

$$h = 0.62 \qquad 5$$

30

9. **(a)** A U-tube whose limbs are vertical contains water.
Liquid A of relative density 0.95 is poured into one limb and liquid B of relative density 0.925 is poured into the other limb until the two levels are equal.
The height of the column of liquid A is 12 cm.

Find the height of the column of liquid B.



$$(\rho g h)_A = 950(9.8)(0.12)$$ 5

$$(\rho g h)_B = 925(9.8)h$$ 5

$$(\rho g h)_{Water} = 1000(9.8)(0.12 - h)$$ 5

$$(\rho g h)_A = (\rho g h)_B + (\rho g h)_{Water}$$ 5

$$950(9.8)(0.12) = 925(9.8)h + 1000(9.8)(0.12 - h)$$

$$1117.2 = 9065h + 1176 - 9800h$$

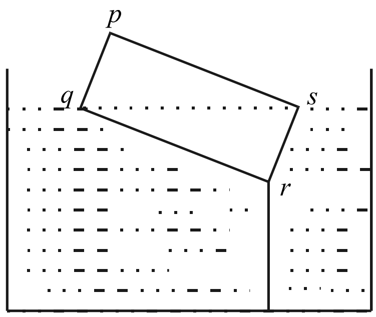
$$735h = 58.8$$

$$h = 0.08$$ 5

height of column = 8 cm

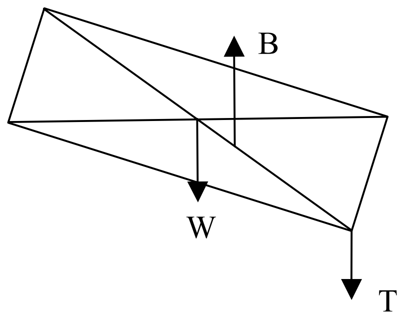
25

(b) A uniform rectangular lamina *pqrs* , of weight *W*, floats vertically in a tank of water with diagonal [*qs*] on the surface. The lamina is held in this position by a light inelastic vertical string tied to the rectangle at *r* and tied to the bottom of the tank.



(i) Calculate the relative density of the lamina.

(ii) Find, in terms of *W*, the tension in the string.



(i) Buoyancy $B = \dfrac{W_l \, s_L}{s}$

$$= \frac{\frac{1}{2}W(1)}{s} \quad \text{or} \quad \frac{W}{2s}$$ 5

Take moments about r :

$$B\left(\frac{2}{6}|rp|\cos\theta\right) = W\left(\frac{3}{6}|rp|\cos\theta\right)$$ 5,5

$$2B = 3W$$

$$2\left(\frac{W}{2s}\right) = 3W$$

$$s = \frac{1}{3}$$ 5

(ii) $$B = W + T$$

$$\frac{W}{2s} = W + T$$

$$\frac{3W}{2} = W + T$$

$$\Rightarrow \quad T = \frac{W}{2}$$ 5 25

10. **(a)** Solve the differential equation

$$x^2 \frac{dy}{dx} - y + 4 = 0$$

given that $y = 5$ when $x = 1$.

$$x^2 \frac{dy}{dx} - y + 4 = 0$$

$$\frac{dy}{dx} = \frac{y-4}{x^2}$$

$$\int \frac{dy}{y-4} = \int \frac{dx}{x^2}$$ 5

$$\ln(y - 4) = -\frac{1}{x} + C$$ 5

$$y = 5,\ x = 1 \Rightarrow C = 1$$ 5

$$\ln(y - 4) = -\frac{1}{x} + 1$$

$$y = e^{-\frac{1}{x}+1} + 4$$ 5 20

(b) A particle is projected vertically upwards with an initial speed of $2g$ m/s in a medium in which there is a resistance kv^2 N per unit mass where v is the speed of the particle and k is a constant, where $k > 0$.

(i) Prove that the maximum height reached is

$$\frac{1}{2k}\ln(1+4kg).$$

(ii) If the speed of the particle is g m/s when it has reached half its maximum height, find the value of k.

(i) Force = Mass × Acceleration

$$-mg - mkv^2 = mv\,\frac{dv}{dx}$$

$$-\,dx = \frac{v\,dv}{g+kv^2}$$

$$\left[-x\right]_0^h = \left[\frac{1}{2k}\ln\left(g+kv^2\right)\right]_{2g}^{0} \qquad 5$$

$$-h = \frac{1}{2k}\ln(g) - \frac{1}{2k}\ln\left(g+4kg^2\right)$$

$$h = \frac{1}{2k}\ln\left(\frac{g+4kg^2}{g}\right)$$

$$h = \frac{1}{2k}\ln(1+4kg) \qquad 5 \qquad 10$$

(ii)

$$\left[-x\right]_0^{\frac{h}{2}} = \left[\frac{1}{2k}\ln\left(g+kv^2\right)\right]_{2g}^{g}$$

$$-\tfrac{1}{2}h = \frac{1}{2k}\ln\left(g+kg^2\right) - \frac{1}{2k}\ln\left(g+4kg^2\right) \qquad 5$$

$$h = \frac{1}{k}\ln\left(\frac{g+4kg^2}{g+kg^2}\right)$$

$$\frac{1}{2k}\ln(1+4kg) = \frac{1}{k}\ln\left(\frac{g+4kg^2}{g+kg^2}\right)$$

$$(1+4kg)^{\frac{1}{2}} = \frac{g+4kg^2}{g+kg^2} \qquad 5$$

$$(1+4kg)^{\frac{1}{2}} = \frac{1+4kg}{1+kg}$$

$$1+kg = (1+4kg)^{\frac{1}{2}}$$

$$1+2kg+k^2\mathrm{g}^2 = 1+4kg \qquad 5$$

$$k^2\mathrm{g}^2 = 2kg$$

$$\Rightarrow \quad k = \frac{2}{g} \qquad 5$$

20

Some Alternative Solutions

1. **(a)** A ball is thrown vertically upwards with an initial velocity of 20 m/s. One second later, another ball is thrown vertically upwards from the same point with an initial velocity of u m/s. The balls collide after a further 2 seconds.

(i) Show that $u = 17.75$.

(ii) Find the **distance** travelled by each ball before the collision, giving your answers correct to the nearest metre.

(i) First ball, after 1 s:

$$v = u + at = 20 - 9.8 = 10.2$$

$$s = ut + \tfrac{1}{2} ft^2 = 20 - 4.9 = 15.1$$

First ball, 2 s later: 5

$$s = ut + \tfrac{1}{2} ft^2 = 20.4 - 19.6 = 0.8$$

$$s_1 = 15.1 + 0.8 = 15.9$$

$$s_2 = u(2) - \tfrac{1}{2}(9.8)2^2$$ 5

$$s_1 = s_2$$

$$15.9 = 2u - 19.6$$ 5

$$2u = 35.5$$

$$u = 17.75$$

15

1. **(a)** A ball is thrown vertically upwards with an initial velocity of 20 m/s.
One second later, another ball is thrown vertically upwards from the same point with an initial velocity of u m/s.
The balls collide after a further 2 seconds.

(i) Show that $u = 17.75$.

(ii) Find the **distance** travelled by each ball before the collision, giving your answers correct to the nearest metre.

(i) $s = ut + \tfrac{1}{2}at^2$

$s_1 = 20(3) - \tfrac{1}{2}(9.8)3^2$ 5

$s_2 = (17.75)(2) - \tfrac{1}{2}(9.8)2^2$ 5

$s_1 = s_2 = 15.9$

$\Rightarrow$ The balls collide 5 15

1. **(a)** A ball is thrown vertically upwards with an initial velocity of 20 m/s.
One second later, another ball is thrown vertically upwards from the same point with an initial velocity of u m/s.
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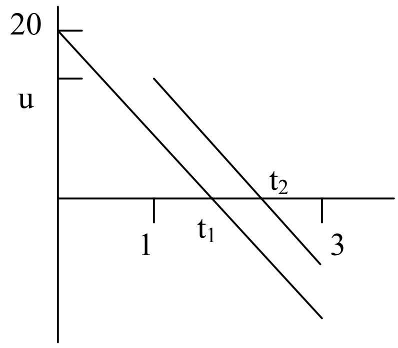
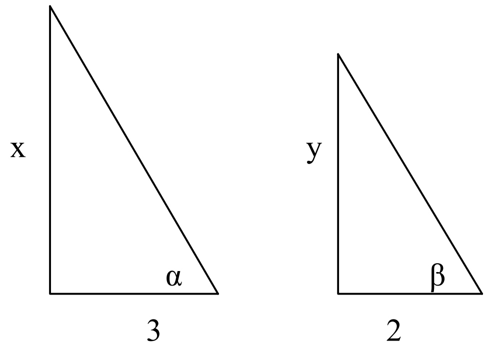




(i)

$$\tan\alpha = \frac{x}{3} = g = \frac{20}{t_1}$$

$$x = 29.4 \quad \text{and} \quad t_1 = \frac{60}{29.4}$$

$$s_1 = \tfrac{1}{2}\left(\frac{60}{29.4}\right)(20) - \tfrac{1}{2}\left(3 - \frac{60}{29.4}\right)(9.4)$$

$$= 20.41 - 4.51$$

$$= 15.9$$

5

$$\tan\beta = g = \frac{u}{t_2 - 1} = \frac{y}{2}$$

$$y = 19.6 \quad \text{and} \quad t_2 - 1 = \frac{u}{9.8}$$

$$s_2 = \tfrac{1}{2}\left(\frac{u}{9.8}\right)(u) - \tfrac{1}{2}\left(3 - \frac{u}{9.8} - 1\right)(19.6 - u)$$

$$= 2u - 19.6$$

5

$$s_1 = s_2$$

$$15.9 = 2\mathrm{u} - 19.6$$

$$2\mathrm{u} = 35.5$$

$$\mathrm{u} = 17.75$$

5

15

2(b) At time t = 0, two particles P and Q are set in motion. At time t = 0, Q has position vector $20\vec{i} + 40\vec{j}$ metres relative to P.

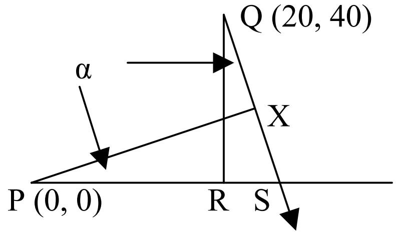
P has a constant velocity of $3\vec{i} + 5\vec{j}$ m/s and Q has a constant velocity of $4\vec{i} - 3\vec{j}$ m/s.

Find

(i) the velocity of Q relative to P

(ii) the shortest distance between P and Q, to the nearest metre

(iii) the time when P and Q are closest together, correct to one decimal place.



(i)

$$\vec{V}_{QP} = \vec{V}_Q - \vec{V}_P$$
$$= (4\vec{i} - 3\vec{j}) - (3\vec{i} + 5\vec{j})$$
$$= \vec{i} - 8\vec{j}$$
$$\text{magnitude} = \sqrt{(1)^2 + (8)^2} = 8.06.$$

5, 5

(ii)

$$\alpha = \tan^{-1}\left(\tfrac{1}{8}\right) \text{ or } 7.13°$$
$$|QS|\cos\alpha = 40 \Rightarrow |QS| = 40.31$$
$$|RS| = |QS|\sin\alpha \Rightarrow |RS| = 5$$

5

$$\cos\alpha = \frac{|PX|}{|PS|} \Rightarrow |PX| = 25\cos(7.13)$$
$$|PX| = 24.8$$
$$= 25$$

5

(iii)

$$\sin\alpha = \frac{|XS|}{|PS|} \Rightarrow |XS| = 25\sin 7.13 = 3.10$$
$$|QX| = 40.31 - 3.10 = 37.21$$

5

$$\text{time} = \frac{|QX|}{V_{QP}} = \frac{37.21}{8.06} = 4.6$$

5

30

2(b) At time t = 0, two particles P and Q are set in motion. At time t = 0, Q has position vector $20\vec{i} + 40\vec{j}$ metres relative to P.

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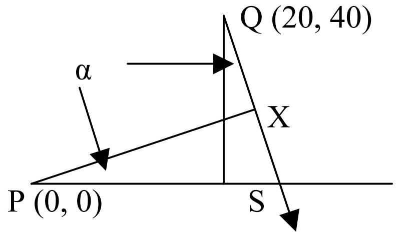


(i) $\vec{V}_{QP} = \vec{V}_Q - \vec{V}_P$ 5

$= \left(4\vec{i} - 3\vec{j}\right) - \left(3\vec{i} + 5\vec{j}\right)$

$= \vec{i} - 8\vec{j}$ 5

magnitude $= \sqrt{(1)^2 + (8)^2} = 8.06.$

(ii) slope of $[QS] = -8$ 5

equation of $[QS]$: $y - 40 = -8(x - 20)$

$8x + y - 200 = 0$

shortest distance $= \left|\dfrac{8(0) + 1(0) - 200}{\sqrt{(8)^2 + (1)^2}}\right| = 24.8$

$= 25$ 5

(iii) equation of $[PX]$: $y - 0 = -\frac{1}{8}(x - 0)$

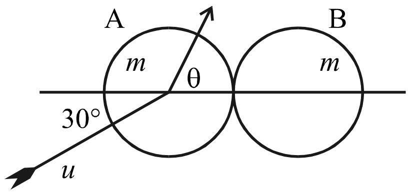
$x - 8y = 0$

coords of X: $\left(\dfrac{320}{13}, \dfrac{40}{13}\right)$

$|QX| = \sqrt{\left(\frac{320}{13} - 20\right)^2 + \left(\frac{40}{13} - 40\right)^2} = 37.21$ 5

$\text{time} = \dfrac{|QX|}{V_{QP}} = \dfrac{37.21}{8.06} = 4.6$ 5 30

5. **(b)** A smooth sphere A, of mass m, moving with speed u, collides with an identical smooth sphere B which is at rest.
The direction of motion of A, before impact, makes an angle 30º with the line of centres at impact.
After impact the direction of A makes an angle θ with the line of centres, where $0^\circ \le \theta < 90^\circ$.
The coefficient of restitution between the spheres is e.
The **speeds** of A and B immediately after impact are equal.



(i) Calculate the value of θ.
(ii) Find e.

PCM $\quad mu\cos 30 + m(0) = mv\cos\theta + mv$ | 5

NEL $\quad v\cos\theta - v = -e(u\cos 30 - 0)$ | 5

$$v\cos\theta = \frac{u\sqrt{3}\{1-e\}}{4}$$
$$v = \frac{u\sqrt{3}\{1+e\}}{4}$$
5

(i)
$$v\sin\theta = u\sin 30 = u\left(\frac{1}{2}\right)$$
$$v\cos\theta + v = u\frac{\sqrt{3}}{2}$$
$$v\cos\theta + v = \sqrt{3}\,v\sin\theta$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$\Rightarrow \quad \theta = 60^\circ$$
5

(ii)
$$\tan\theta = \tan 60 = \frac{\frac{u}{2}}{\frac{u\sqrt{3}\{1-e\}}{4}}$$
$$\Rightarrow \quad e = \frac{1}{3}$$
5

25

5. **(b)** A smooth sphere A, of mass m, moving with speed u, collides with an identical smooth sphere B which is at rest.
The direction of motion of A, before impact, makes an angle 30º with the line of centres at impact.
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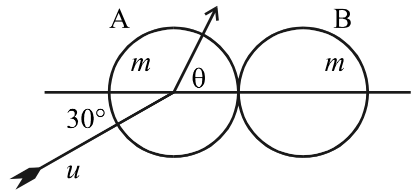


PCM $$mu\cos 30 + m(0) = mv_1 + mv_2$$ 5

$$v_1 + v_2 = \frac{u\sqrt{3}}{2}$$

NEL $$v_1 - v_2 = -e(u\cos 30 - 0)$$ 5

$$v_1 - v_2 = -\frac{eu\sqrt{3}}{2}$$

$$(v_1)^2 + (u\sin 30)^2 = (v_2)^2$$

$$\left(\frac{u\sqrt{3}}{2} - v_2\right)^2 + \frac{u^2}{4} = v_2^2$$

$$\Rightarrow v_2 = \frac{u}{\sqrt{3}} \text{ and } v_1 = \frac{u}{2\sqrt{3}}$$ 5

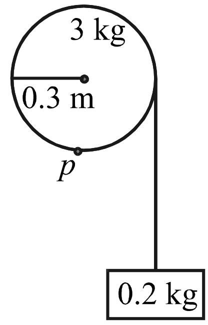
(i) $$\tan\theta = \frac{\frac{u}{2}}{\frac{u}{2\sqrt{3}}} \Rightarrow \theta = 60^\circ$$ 5

(ii) $$v_1 - v_2 = -\frac{eu\sqrt{3}}{2}$$

$$\Rightarrow e = \tfrac{1}{3}$$ 5

25

8 (b) A smooth pulley wheel has a mass of 3 kg and a radius of 0.3 m. One end of a light inextensible rope is attached to a point p on the rim of the wheel. A particle of mass 0.2 kg attached to the other end of the rope hangs freely.
The axis of rotation of the wheel is horizontal, perpendicular to the wheel, and passes through the centre of the wheel.
The particle is released from rest and moves vertically downwards.



When the particle has acquired a speed of 1.2 m/s, find

(i) the kinetic energy gained by the wheel
(ii) the distance descended by the particle, correct to two decimal places.

(i)

$$I = \tfrac{1}{2}mr^2 = \tfrac{1}{2}(3)(0.3)^2 = 0.135$$ 5

$$\omega = \frac{v}{r} = \frac{1.2}{0.3} = 4$$

kinetic energy of wheel $= \tfrac{1}{2}I\omega^2$ 5

$= \tfrac{1}{2}(0.135)(4)^2$

$= 1.08$ 5

(ii)

$$\left.\begin{aligned} mr\frac{d\omega}{dt} &= mg - T \\ T(r) &= \mathrm{I}\frac{d\omega}{dt} \end{aligned}\right\}$$ 5

$$a = r\frac{d\omega}{dt} = \frac{mgr^2}{mr^2 + I}$$

$= 1.153$ 5

$$v^2 = u^2 + 2as$$

$$1.2^2 = 0 + 2(1.153)h$$

$$h = 0.62$$ 5

30